Numerical Analysis of Low Reynolds Number Flow around
a Body Immersed in a Uniform Stream *

By Kohzaburo NAKABAYASHI,** Yutaka MIYAKE*** and Tyusei AOI†

In this report, the factors that are responsible for the accuracy of numerical calculations of two-dimensional low Reynolds number flows around cylindrical bodies immersed in a uniform stream are discussed on the basis of the Navier-Stokes equations.

The finite difference calculations have been carried out for a circular cylinder and it has been confirmed that the difference of the outer boundary condition affects little the accuracy, if the outer boundary is sufficiently far from the body, but if a comparatively near outer boundary is assumed, Oseen flow is recommended as the outer flow on the boundary. The results can be extended to the other geometries of cylinders and Imai's asymptotic solution can be used in place of the Oseen flow.

Some aspects of the flow, such as streamlines, length of twin-vortices, angle of flow separation, etc. are also discussed.

Key words : Fluid Mechanics, Low Reynolds Number Flow, Numerical Simulation, Finite Difference Method, Full Navier-Stokes Equation, Oseen Approximation

1. Introduction

Many researchers have long been interested in obtaining an accurate solution to the problem for laminar flows around a body up to as high Reynolds numbers as possible. Because the flow governed by the elliptic equations, the outer boundary conditions at infinity r_∞ must be given. Since the conditions are determined not by the geometry of the body but by the force exerted to the flow by the body, a circular cylinder has been taken up as a typical body for a two-dimensional flow. In order to numerically solve the flow around a circular cylinder, many attempts ^{(1)~(8)} have been made. Among them are the pioneering papers by Thom ⁽¹⁾ and Kawaguti ⁽²⁾ and the recent report by Fornberg ⁽³⁾ who dealt with the problem in a wide range of the Reynolds numbers Re .

In numerical solution for flows around a body by finite difference calculations, the accuracy of the solution is affected considerably by both the choice of the boundary conditions at r_∞ and the distance between the body and the outer boundary which is inevitably assumed in place of infinite flow region. In connection with boundary conditions at far downstream, Takaisi ⁽⁸⁾ examined the following two cases: (i) ψ_∞ and ζ_∞ are given by the values of the first term of Imai's asymptotic solution ⁽⁹⁾ where ζ_∞ and ψ_∞ are the vorticity

and the perturbed stream function at specified outer boundary, respectively, and (ii) both ψ_∞ and ζ_∞ are zero. He calculated drag coefficient C_D of a circular cylinder for a flow in the range of $0.5 \leq Re \leq 100$, by using the SLOR method in the finite difference calculations. From this calculation, Takaisi concluded that the value of C_D hardly depends on the outer boundary conditions when (i) and (ii) are assumed. However, his numerical solutions do not converge to the analytical solutions of Oseen's approximation at sufficiently small Reynolds numbers Re . Therefore, his conclusion seems to require further investigation. Meanwhile, Fornberg ⁽³⁾ examined this problem in more detail; as for the vorticity, $\partial\zeta/\partial r|_{r \rightarrow \infty} = 0$ for every case and for the perturbed stream function ψ , (i) $\psi_\infty = 0$, (ii) ψ_∞ is equal to the first term of Imai's asymptotic solution, (iii) $\partial\psi/\partial r|_{r \rightarrow \infty} = 0$, (iv) a mixed condition in which above mentioned (i) and (iii) are combined. Then, he concluded that the results depend little on the choice of the outer boundary conditions provided the outer boundary is far enough from the body and that for the outer boundary kept at some specified distance from the body, the suitable boundary conditions are (iii) for $Re \leq 40$ and (iv) for $Re > 40$. The generality of this conclusion, however, is an open question.

In this report, we studied the flow around a circular cylinder to find the smallest distance of the outer boundary from the cylinder and the suitable outer boundary conditions necessary to keep high accuracy of the numerical calculation. We adopted smaller meshes in finite difference lattices than any others of the available literature, and examined the convergence of a solution, by checking up the convergence of the vorticity on the surface of a circular cylinder and ψ and ζ in the wake that converge most slowly. The results we obtained in this

* Received 1st October, 1984.

** Lecturer, Faculty of Engineering, Setsunan Univ., 17-8 Ikeda-Nakamachi, Neyagawa.

*** Professor, Faculty of Engineering, Osaka Univ., 2-1 Yamada-Oka, Suita.

† Professor, General of Education, Osaka Institute of Technology, 16-1 Ohmiya, Asahiku, Osaka.

paper are more accurate than any one of previous attempts⁽¹⁾⁻⁽⁸⁾.

2. Basic Equations and Boundary Conditions

To solve the two-dimensional flow around a circular cylinder, let us use polar coordinates (r, θ) whose origin coincides with the axis of the circular cylinder. Radial length r is made dimensionless by the radius a of the circular cylinder. We can transform the (r, θ) system into a (z, θ) system using the following relations

$$z = \log r, \quad \theta = \theta. \quad (1)$$

The velocity of uniform flow at far upstream is put as U . Then the stream function Ψ can be written in the form

$$\Psi = e^z \sin \theta + \psi(z, \theta), \quad (2)$$

where both Ψ and ψ , a perturbed stream function from the uniform flow, are made dimensionless by Ua . Then ψ and vorticity ζ , (that is made dimensionless by a/U), satisfy the following Eqs. (3) and (4):

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial \theta^2} = -e^z \zeta, \quad (3)$$

$$\begin{aligned} \frac{\partial^2 \zeta}{\partial z^2} + \frac{\partial^2 \zeta}{\partial \theta^2} + \frac{R_e e^z}{2} \left(\sin \theta \frac{\partial \zeta}{\partial \theta} - \cos \theta \frac{\partial \zeta}{\partial z} \right) \\ = \frac{R_e}{2} \left(\frac{\partial \psi}{\partial \theta} \frac{\partial \zeta}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \zeta}{\partial \theta} \right), \end{aligned} \quad (4)$$

Where $R_e = 2aU/\nu$ (ν is the kinematic viscosity of the fluid). Since the streamline is symmetrical with respect to the x -axis which is the direction of the uniform stream, we can specify the boundary conditions as follows:

$$\psi = \zeta = 0 \quad \text{at } \theta = 0 \text{ and } \theta = \pi, \quad (5)$$

$$\psi(0, \theta) = \left(\frac{\partial \psi}{\partial z} \right)_{z=0} = -\sin \theta \quad \text{at } z = 0, \quad (6)$$

$$\left. \begin{aligned} \psi(z_\infty, \theta) &= f(C_D; z_\infty, \theta) \\ \zeta(z_\infty, \theta) &= g(C_D; z_\infty, \theta) \end{aligned} \right\} \quad \text{at } z \rightarrow \infty, \quad (7)$$

or

$$\left. \begin{aligned} \psi(z_\infty, \theta) &= \psi^0(z_\infty, \theta) \\ \zeta(z_\infty, \theta) &= \zeta^0(z_\infty, \theta) \end{aligned} \right\} \quad \text{at } z \rightarrow \infty, \quad (8)$$

where

$$\begin{aligned} f(C_D; z, \theta) &= \frac{C_D}{2} \left(\frac{\theta}{\pi} - \operatorname{erf} Q \right) \\ &\quad - \frac{C_D^2 R_e}{16\sqrt{\pi}} \left\{ \frac{1}{P} [\sqrt{2} (\operatorname{erf} \sqrt{2} Q - 1)] \right. \\ &\quad \left. - e^{-Q^2} \operatorname{erf} Q \right\} + \frac{\sqrt{2} P}{P^2 + Q^2} \end{aligned} \quad (9)$$

and

$$\begin{aligned} g(C_D; z, \theta) &= -\frac{C_D R_e}{4\sqrt{\pi}} \frac{Q}{r} e^{-Q^2} + \frac{C_D^2 R_e^2}{64\sqrt{\pi} r} \\ &\quad \times \left\{ -\frac{1}{P} e^{-Q^2} \left[\frac{2}{\sqrt{\pi}} Q e^{-Q^2} + (2Q^2 - 1) \operatorname{erf} Q \right] \right. \\ &\quad \left. + \frac{1}{P^3} [\sqrt{2} (\operatorname{erf} \sqrt{2} Q - 1) - e^{-Q^2} \operatorname{erf} Q] \right\} \end{aligned} \quad (10)$$

with $P = \sqrt{R_e r/2} \cos(\theta/2)$, $Q = \sqrt{R_e r/2} \sin(\theta/2)$

and $\operatorname{erf} Q = \frac{2}{\sqrt{\pi}} \int_0^Q e^{-s^2} ds$.

Takami-Keller⁽⁶⁾ obtained the above Eq. (7) by extending Imai's asymptotic expansions⁽⁹⁾ in the dead water region into the whole flow field. The first terms of both f and g are the first asymptotic solutions to Oseen's approximation, and ψ^0 and ζ^0 represent the values of the stream function and vorticity, both of which are obtained from the analytic solution of Oseen's approximation on the outer boundary specified. In this report, we abbreviate the boundary conditions (7) and (8) as T.B.C. and O.B.C., respectively. Since the wake behind the body is also taken into consideration, these two boundary conditions are considered most reasonable.

3. Finite Difference Equations

Let us use the simplest central differences for finite difference equations. Dennis-Chang⁽⁷⁾ examined the effects of the order of the central difference on the numerical accuracy and concluded that it affects little, i.e., even the simplest central difference yields enough accuracy.

In this report, an alternating directional implicit (ADI) scheme was used for the vorticity equation so that the outer boundary conditions immediately affect the entire flow field. A successive line over-relaxation (SLOR) method was used for the stream function in the θ direction. The finite difference equations employed in this calculation are given as follows:

(1). Stream function

$$\begin{aligned} \alpha_B \psi_{i,j+1}^{n+1} + \alpha_O \psi_{i,j}^{n+1} + \alpha_D \psi_{i,j-1}^{n+1} \\ = -e^{2(i+1)a} \zeta_{i,j}^n - \alpha_A \psi_{i+1,j}^n \\ - \alpha_C \psi_{i-1,j}^n \end{aligned} \quad (11)$$

where $\alpha_A = \alpha_C = 1/a^2$, $\alpha_B = \alpha_D = 1/b^2$ and

$$\alpha_D = -2[(1/a^2) + (1/b^2)] \text{ with } a = \Delta z \text{ and } b = \Delta \theta.$$

Subscripts (i, j) denote lattice point coordinates $\{z = (i-1)a, \theta = (j-1)b\}$.

The stream function $\psi^{n+1/2}$, which is necessary for the finite difference equations of the vorticity, was obtained from

$$\left. \begin{aligned} \psi_{i,j}^{n+1} &= \psi_{i,j}^n + K(\psi_{i,j}^{n+1} - \psi_{i,j}^n) \\ \psi_{i,j}^{n+1/2} &= \frac{1}{2}(\psi_{i,j}^n + \psi_{i,j}^{n+1}) \end{aligned} \right\} \quad (12)$$

where K is the acceleration factor and was taken as 1.6 in this report.

(2). Vorticity

For the first half time-step, $(n+1/2)$, the line relaxation in the θ direction was carried out by using the following finite difference equation:

$$\begin{aligned} \beta_B \zeta_{i,j+1}^{n+1/2} - (\beta_{02} + \rho) \zeta_{i,j}^{n+1/2} + \beta_D \zeta_{i,j-1}^{n+1/2} \\ = -\beta_A \zeta_{i+1,j}^n + (\beta_{01} - \rho) \zeta_{i,j}^n - \beta_C \zeta_{i-1,j}^n \end{aligned} \quad (13)$$

where

$$\begin{aligned} \beta_B &= \frac{1}{b^2} + \frac{R_e e^z}{4b} \sin \theta \\ &\quad + \frac{R_e}{8ab} (\psi_{i+1,j}^{n+1/2} - \psi_{i-1,j}^{n+1/2}) \end{aligned}$$

$$\begin{aligned}
\beta_D &= \frac{1}{b^2} - \frac{Re e^z}{4b} \sin \theta \\
&\quad - \frac{Re}{8ab} (\phi_{i,j+1}^{n+(1/2)} - \phi_{i,j-1}^{n+(1/2)}), \\
\beta_{02} &= \frac{2}{b^2}, \quad \beta_{01} = \frac{2}{a^2}, \\
\beta_A &= \frac{1}{a^2} - \frac{Re e^z}{4a} \cos \theta \\
&\quad - \frac{Re}{8ab} (\phi_{i,j+1}^{n+(1/2)} - \phi_{i,j-1}^{n+(1/2)}), \\
\beta_C &= \frac{1}{a^2} + \frac{Re e^z}{4a} \cos \theta \\
&\quad + \frac{Re}{8ab} (\phi_{i,j+1}^{n+(1/2)} - \phi_{i,j-1}^{n+(1/2)}).
\end{aligned}$$

For second half time-step, $(n+1)$, the line relaxation in the z direction was carried out by using the following finite difference equation:

$$\begin{aligned}
\beta_A \zeta_{i+1,j}^{n+1} - (\beta_{01} + \rho) \zeta_{i,j}^{n+1} \\
+ \beta_C \zeta_{i-1,j}^{n+1} = -\beta_B \zeta_{i,j+1}^{n+(1/2)} \\
+ (\beta_{02} - \rho) \zeta_{i,j}^{n+(1/2)} - \beta_D \zeta_{i,j-1}^{n+(1/2)},
\end{aligned} \quad (14)$$

where ρ is a relaxation parameter and four values, 3.0, 2.5, 2.0 and 1.5, were used consecutively in the cycle of the iteration.

In calculation of Eq.(11), ψ -values specified as the inner boundary conditions on the surface of the body were determined by Eqs.(5) and (6), while ψ -values specified at the outer boundary were determined by Eq.(7) or (8). And in calculations of Eqs.(13) and (14), ζ -values given by Eqs. (7) or (8) were used as the outer boundary conditions, but as the inner boundary conditions, the ζ -value on the surface of the body is required in addition to the ζ -value given by Eq.(5). This ζ -value was obtained in the following form;

$$\begin{aligned}
\zeta_0 &= -\left\{ \frac{2(\psi_2 + \sin \theta)}{a^2} + \frac{2 \sin \theta}{a} + \sin \theta \right\} \\
\text{or} \\
\zeta_0 &= \frac{1}{2a^2} \{ \psi_3 - 8\psi_2 - (2a^2 + 6a + 7) \sin \theta \}
\end{aligned} \quad (15)$$

where subscripts of ψ denote the row-number of a certain finite difference lattice which was numbered from the wall in the positive z direction. It should be noted here that the second equation of Eqs.(15) does not necessarily give better results than the first equation in spite of its more accurate difference quotient. Dennis-Chang⁷ demonstrated an alternative method for the evaluation of the vorticity on the surface of the body, but it is not essentially different from the values of ζ_0 obtained from Eqs.(11) to (15). Computation time for a solution till convergence considerably depends on distributions of initial values of ψ and ζ . In this calculation, we used as the initial value the analytic solutions to Oseen's approximation which had been obtained by Tomotika-Aoi⁽¹⁰⁾. The space between finite difference lattices was taken as $\Delta z = 0.05$ and $\Delta \theta = 3^\circ$ and z_∞ was taken as $z_\infty = 3$ (or $r_\infty = 20.1$) except when the

effect of z_∞ was examined in detail. For the range of $Re < 1$ in which larger z_∞ values are required, however, z_∞ was taken as $z_\infty = 4.8$ ($r_\infty = 121$). The criterion for the convergent degree of calculation is that the rate of variation for the vorticity in the dead water region in the vicinity of the body, which is most difficult to converge, become less than 10^{-4} .

4. Drag Coefficient C_D

Intensity of drag D of a circular cylinder is usually expressed by the drag coefficient C_D , by making D dimensionless with $\rho U^2 a$. C_D is obtained using the following expressions:

$$\begin{aligned}
C_D &= -\frac{4}{Re} \int_0^\pi \zeta_0 \sin \theta d\theta \\
&\quad + \frac{4}{Re} \int_0^\pi \left(\frac{\partial \zeta}{\partial z} \right)_0 \sin \theta d\theta
\end{aligned} \quad (16)$$

or

$$\begin{aligned}
C_D &= -\frac{4}{Re} \int_0^\pi \zeta_0 \sin \theta d\theta \\
&\quad - 2 \int_0^\pi P(1, \theta) \cos \theta d\theta,
\end{aligned} \quad (17)$$

where subscript 0 indicates the value on the surface of the circular cylinder and $P(1, \theta)$ does the usual dimensionless pressure distribution on the surface of the circular cylinder. In the above equations, the first and second terms on the right-hand side represent the friction and pressure drags, respectively.

5. Results of Calculation and Considerations

5.1 Error of finite difference solution

No previous reports on numerical solutions for flows around a circular cylinder present quantitative examination of the accuracy of their calculations. Therefore, we carried out the numerical calculations of the Oseen equations by applying the present finite difference method and compared the solutions with the analytic solutions¹⁰ to the Oseen equations. Figure 1 shows error of the vorticity ζ_0 on the surface of a circular cylinder. In Fig.1,

Fig. 1 Error of numerical solution to the Oseen equations.

r.m.s. is

$$\text{r.m.s.} = \sqrt{\frac{\sum_N [(\zeta_0)_n - (\zeta_0)_a]^2}{N}} \quad (18)$$

where N is the number of lattice points on the surface of the circular cylinder and $N=61$, and $(\zeta_0)_n$ is the numerical solution and $(\zeta_0)_a$ the analytical one of the vorticity, respectively. Furthermore, $|\bar{\zeta}|$ is the mean value of the vorticity $(\zeta_0)_a$ on the surface of the circular cylinder. The error increases approximately linearly with Re implying that the numerical calculations become more difficult with an increase in Re . In this example, distance r_∞ of the outer boundary (a circle) was taken as $r_\infty \approx 20.1$.

Table 1 shows a comparison between the numerical and analytical solutions for the drag coefficient C_D . Here, C_{DF} , C_{DP} and C_{DT} denote the friction-, pressure-, total-drag, respectively. The left-hand column of each Reynolds number is an analytic solution, and the right-hand column is a numerical solution. For the analytical solution, C_{DF} is equal to C_{DP} . The error of C_D which is mainly due to the error of pressure drag also increases with Re . Pressure on the surface of the circular cylinder, $P(1, \theta)$, is obtained by the following relation:

$$P(1, \theta) = \frac{P(1, \theta) - P(r_\infty, \pi)}{\rho U^2 / 2} \\ = 2 + \frac{4}{Re} \int_0^\infty \left(\frac{\partial \zeta}{\partial \theta} \right)_{\theta=\pi} dz$$

Table 1 Comparison between value of drag as predicted by analytical solutions and numerical solutions to the Oseen equations.

	Re = 1		Re = 10		Re = 20		Re = 40	
C_{DF}	5.928	5.927	2.088	2.093	1.733	1.743	1.513	1.532
C_{DP}	5.928	5.922	2.088	2.104	1.733	1.757	1.513	0.944
C_{DT}	11.857	11.849	4.175	4.197	3.466	3.499	3.026	2.475

Fig. 2 Comparison of analytical and numerical pressure distributions obtained from the Oseen equations.

Fig. 3 Comparison of analytical and numerical vorticity distributions, around a circular cylinder obtained from the Oseen equations.

$$-\frac{4}{Re} \int_0^\pi \left(\frac{\partial \zeta}{\partial z} \right)_0 d\theta + 2 \int_0^\pi \zeta_0 \cos \theta d\theta. \quad (19)$$

Figure 2 shows some examples of $P(1, \theta)$. In Fig. 2, the solid lines and dotted lines indicate finite difference solutions and analytic solutions, respectively, and dot-and-dash lines represent the case where only $P(1, \theta)$ is obtained by the finite difference method, while the vorticity is obtained by the analytic method. As seen in Fig. 2, the accuracy of pressure distribution on the surface of the circular cylinder rapidly drops as Re increases. In the case of C_D , however, the error is not so large because the C_D -value is obtained as a difference in two certain integral values between upstream and downstream points with respect to the point of $\theta = \pi/2$, consequently the two errors canceling each other. An error of $P(1, \theta)$ is caused by one in estimating the rate of variation of the vorticity perpendicular to the cylinder wall, i.e., the third term of Eq.(19). Figure 3 shows the vorticity distributions on the circumference at $r=1.00$ (on cylindrical surface), $r=1.05$ and $r=1.105$ for $Re=20$, in which the solid lines show the finite difference solutions and the dotted lines the analytic solutions. As noted in Fig. 3, the error of the vorticity which is directly obtained as an unknown quantity in the finite difference method is small enough. However, since the magnitude of the vorti-

Fig. 4 Comparison of the numerical results of vorticity distribution around a circular cylinder for two (T.B.C. and O.B.C.) outer boundary values.

Table 2 Calculated values for the total drag coefficient;
 $C_D = D / (1/2 \rho U^2 d)$.

Reynolds number ...	0.01	0.05	0.1	1.0	4.0	10	20	40	60	80	100
Oseen	380.366	100.536	58.312	11.857	5.842	4.175	3.466	3.026	2.850	2.750	2.688
T.B.C.	-----	-----	54.248	10.050	4.430	2.769	2.018	1.517	1.298	1.176	1.093
PRESENT (O.B.C.)	379.494	100.221	57.925	10.556	4.444	2.728	1.966	1.462	1.243	1.121	1.038
TAKAMI & KELLER	-----	-----	-----	10.283	4.437	2.754	2.003	1.536	1.325	---	---
DENNIS & CHANG	-----	-----	---	---	---	2.846	2.045	1.522	---	---	---
FORNBERG	-----	-----	---	---	---	---	2.000	1.498	---	---	1.058

Fig. 5 The respective variations of two quantities $1 - (C_D / C_{Doseen})$ and C_D with the Reynolds number.

city decreases rapidly as a point under consideration moves away from the surface of the body, replacement of a differential coefficient by a difference quotient is liable to produce a serious error. Figure 4 shows that in the case of the Navier-Stokes equations (abbreviated as the N-S hereafter) the vorticity variation is much slower than in Oseen's approximation in both tangential and vertical directions to the wall surface. This is because in Oseen's approximation the transport effects of the vorticity in the x-axis are overestimated. Accordingly, it is inferred that in the case of the N-S equations the error of the vorticity by the finite difference approximation is not so large as shown in Fig. 2.

5.2 Outer boundary conditions

Figure 5 shows the results about the variation of the drag coefficient C_D of a circular cylinder as a function of Re . The solid and dot-and-dash lines indicate the results of numerical solution to the N-S equations at $r_\infty = 20.1$ when taking T.B.C. (for solid lines) and O.B.C. (for dot-and-dash lines) as the outer boundary conditions, respectively. The dotted lines indicate analytical solution to Oseen's approximation. Marks o correspond to Tritton's experiments (11). In spite of various probable causes for errors as described in the preceding chapters, our numerical solutions for both outer boundary conditions show that C_D -values are in good agreement with experimental values up to sufficiently high Reynolds numbers. Table 2 lists C_D -values obtained so far by various investigations. Though C_D -values based on Oseen's approximation show considerable discrepancy even for $Re = 1$,

(a) C_D

(b) ζ_{max}

Fig 6 Variations of quantity (a) C_D or (b) ζ_{max} on a circular cylinder with different outer boundary distances, r_∞ (infinity): T, Takami & Keller; D, Dennis & Chang; F, Fornberg.

it is known that the N-S equations can be approximated by Oseen's approximation to good accuracy, when Re is small enough. Thus, Fig. 5 shows the difference between the C_D -values obtained by numerical solutions to the N-S equations and analytical ones C_{Doseen} to Oseen's approximation. From this figure, it becomes evident that C_D -value of the numerical solutions based on O.B.C. approaches C_{Doseen} within 1% difference for $Re < 0.1$. On the contrary, in the case of T.B.C. the difference between the

two solutions remains unchanged even for small Re . The C_D -value based on T.B.C. is larger than that based on O.B.C. for $Re < 4$, while this relation is reversed for $Re > 4$. In general, C_D -value increases in proportion to $1/Re$ as Re decreases. In the case of T.B.C., however, since the boundary conditions are determined by using C_D , correction is required for each cycle of repetition of numerical calculations. As a result, for low Re the calculation becomes difficult to converge because of large C_D -values. In our calculations, the lower limit of Re was 0.04.

Fornberg⁽³⁾ has pointed out that when r_∞ is large enough the outer boundary conditions affect little the results of the numerical calculation. To examine this, we investigated the effect of the distance of r_∞ on the converged values of C_D and $\zeta_{\max}$. And Figs. 6 show the results. Figure 6(b) shows the maximum value of the vorticity, $\zeta_{\max}$, on the surface of the circular cylinder. To make the figure easier to see, in the figures, the values of C_D and ζ are converted into C_{DK} and ζ_K values defined separately for each Re value. In the figures, marks o denote the values shown in Table 2 and T and D represent the values calculated by Takami-Keller⁽⁶⁾ and Dennis-Chang⁽⁷⁾, respectively, and the subscript numbers mean the Reynolds number. As seen in Figs. 6(a) and (b), the values of the other previous reports were not necessarily ones obtained with sufficiently large computation field. From Figs. 6, it follows that (i) when r_∞ -values are large enough, i.e. for $r_\infty \geq 200$, difference in outer boundary conditions does not affect the results of calculations; (ii) in the case of T.B.C., the higher the Re -value is, the larger the required r_∞ -value is, but in the case of O.B.C., the higher the Re -value is, the smaller the required r_∞ -value is; and (iii) in the case of O.B.C., as the r_∞ -value increases, the values of both C_D and $\zeta_{\max}$ decrease and always approach values of saturation. On the other hand, in the case of T.B.C., for $Re=4$ the values of both C_{DK} and ζ_K depend little on the r_∞ -value, but for $Re=1$ they approach increasingly the saturated value with the r_∞ -value, while for $Re=40$, they do so decreasingly. This corresponds to the fact that the two C_D curves for T.B.C. and O.B.C. intersect

Fig. 7 The values of (a) ψ and (b) ζ at $r_\infty \approx 20.1$ under different outer boundary conditions.

each other at $Re=4$ as shown in Fig. 5.

From the above results it can be concluded that the O.B.C. is the most adequate as the outer boundary condition and that the smaller the Re -value is, the larger r_∞ -value is needed. This is because the smaller the Re -value is, the greater the viscosity diffusion effect is, and therefore, the more widely the flow is affected by presence of a body. On the contrary, the higher the Re -value becomes, the narrower the wake region becomes within which the flow field disturbance given by a body is restricted, and thereby the details of the pressure- and vorticity- distribution within the wake region affect little the accuracy of the calculation. Figure 7 shows the distribution of ψ and ζ to be given as the boundary conditions on the outer circumference of $r_\infty \approx 20.1$ for both T.B.C. and O.B.C. Thus, as observed in Fig. 7, T.B.C. assumes a stronger perturbation of the flow than Oseen's approximation for $Re=1$, but weaker ones for $Re=10$ and 40 . Since T.B.C. is an asymptotic expansion with respect to distance r consisting of two terms as exhibited in Eqs. (9) and (10), it does not necessarily give an appropriate flow when the outer boundary is set at finite distance. In other words, though all the three kinds

Fig. 8 Comparison between streamlines and equi-vorticity lines as predicted by numerical solutions under T.B.C. and O.B.C. to the N-S equations.

Fig. 9 Comparison between streamlines and equi-vorticity lines as predicted by analytical solution to the Oseen equations and numerical solutions to the N-S equations.

Fig. 10 Angle of separation.

of dotted curves (●: $Re=1$, ■: $Re=10$, ▲: $Re=40$) in Fig. 7 represent only the first term of Eqs. (9) and (10), they are considerably different from the ones obtained by considering the second term. In the case of a circular cylinder we consider, the exact solutions for Oseen's approximation are available. In the case of a body with an arbitrary sectional form, however, analytical solutions are hard to obtain even for Oseen's approximation. Therefore, in the case of an arbitrary sectional form, it is practical to adopt a sufficiently large distance for the outer boundary and to use Imai's asymptotic solution (the first term).

5.3 Flow around a circular cylinder

The previous section dealt mainly with the problems of the outer boundary for C_D calculation and this section describes the flow patterns and the characteristic quantities for flows around a circular cylinder. Both Figs. 8 and 9 show examples of streamlines (shown in the upper half) and equi-vorticity lines (shown in the lower half) for flows around a circular cylinder. In Fig. 8, the O.B.C. method is compared with the T.B.C. method at $Re=40$. Figure 9 shows the comparison between the numerical solutions (solid line) to the N-S equations with O.B.C. and the analytical solutions (dotted lines) to Oseen's approximation at $Re=10$. As shown in Fig. 8, because of the difference in the intensity of perturbed flows on the outer boundary, the streamlines and equi-vorticity lines for the two cases are somewhat different from each other in the region distant from the body. But they coincide with each other in the neighbourhood of the body. This phenomenon becomes more accentuated as the Reynolds number decreases. On the contrary, in Oseen's approximation, the shorter the distance from the body is, the larger is the error caused by approximating the convection term in Oseen's manner. As a result, a remarkable discrepancy in streamlines and equi-vorticity lines between the two cases, or Oseen's and N-S solutions, is seen in the whole flow field.

Figure 10 shows the departure angle θ of streamlines separation, resulting in twin-vortices. Figure 11 shows the length of the twin-vortices L/a . Here, marks ○ indicate Taneda's experimental values⁽¹¹⁾. The angle of separation θ does not vary at all

Fig. 11 Length of standing vortices

with the form of the boundary conditions. This corresponds to the fact that the flow pattern depends very little on the outer boundary condition in the vicinity of a body as shown in Fig. 8. Meanwhile, in Oseen's approximation, the convection effect is overestimated. This overestimation is equivalent to higher Reynolds numbers being effectively adopted locally in the vicinity of the body. Accordingly, twin-vortices tend to be created at lower Reynolds numbers than in the case of the full N-S equations, in other words the separation region is widened for the same Reynolds number in Oseen's approximation. As for the length of twin-vortices too, the results of calculation based on T.B.C. differ little from those based on O.B.C. and the both results are in good agreement with those of the experiments for the region of Reynolds numbers in which twin-vortices exist. Further, since vortices are forcibly transported downstream in Oseen's approximation, the length of the twin-vortices becomes larger. Oseen's approximation is an appropriate approximation to the N-S equations for $Re < 0.1$, but it does no longer hold for the Reynolds number at which twin-vortices are created.

6. Conclusions

In this study, we examined in detail the distance of the outer boundary conditions in the numerical analysis for the flow around a two-dimensional body and have obtained the conclusions as follows:

(1) When the distance of the outer boundary is large enough, the boundary conditions at the outer boundary do not affect the flow in the vicinity of a body.

(2) When a small flow field is to be taken, it is recommended to use Oseen's approximation flows as the outer boundary conditions. If no Oseen's flow can be obtained, it is useful to adopt Imai's asymptotic solution.

(3) For the region in which the Reynolds number is small, the distance of the outer boundary should be large.

(4) Oseen's approximation to the N-S equation has high accuracy for the Reynolds number less than 0.1, but it produces a certain amount of error for the small Reynolds number at which twin-vortices begin to appear.

References

- (1) Thom, A., Proc. R. Soc., A. 141(1933), P.651.
 - (2) Kawaguti, M., J. Phys. Soc. Jpn., Vol.8, No.6 (1953-11), P.747.
 - (3) Fornberg, B., J. Fluid Mech., Vol.98, No.4 (1980-6), P.819.
 - (4) Apelt, C.J., Aeronaut. Res. Council. Rep. Mem., No.3175 (1961).
 - (5) Keller, H.B. and Takami, H. (Greenspan, Ed.), Numerical Studies of Steady Viscous Flow about Cylinder, (1966), P.115.
 - (6) Takami, H. and Keller, H.B., Phys. Fluid Suppl. II, (1969), P.II-51.
 - (7) Dennis, S.C.R. and Chang, G.Z., J. Fluid Mech., Vol.42, No.3 (1970-2), P.471.
 - (8) Takaisi, Y., Phys. Fluid Suppl. II, (1969), P.II-86.
 - (9) Imai, I., Proc. R. Soc., A.141 (1951), P.471.
 - (10) Tomotika, S. and Aoi, T., Q. J. Mech. Appl. Math., Vol.3, No.2 (1950), P.140.
 - (11) Tritton, D.J., J. Fluid Mech., Vol.6, No.4 (1959), P.547.
 - (12) Taneda, S., J. Phys. Soc. Jpn., Vol.11, No.3 (1956), P.302.
-