Parametric Excitation with an Asymmetric Characteristic in a Self-exciting System*

(1st Report, Behaviors of Region of Resonance of Order 1/2)

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Behaviors of a self-exciting system of van der Pol type subjected to a parametric excitation are investigated. In the present paper the parametric excitation is expressed by the product of a nonlinear function of deflection with an asymmetric characteristic and a periodic function of time. A resonance of order 1/2 and the solution in the neighborhood of the resonance are obtained by the averaging method and the effects of nonlinearity are investigated. Since a squared nonlinearity merely makes the resonance have a constant component, it is found that a cubic nonlinearity plays a more important part in the occurrence of the resonance.

Key Words : Theory of Vibration, Parametric Excitation, Self-exciting System, Resonance of Order 1/2, Asymmetry

1. Introduction

Phenomena of interaction between self-excited and parametric vibrations appear in the vibrational problems where the motions of structural systems and fluid ones are related to each other. Tondl⁽¹⁾ and the authors⁽²⁾⁻⁽⁵⁾ have been studying their qualitative behaviors through fundamental analyses of mathematical models. Then self-excitation and parametric excitation are idealized as a nonlinear resistance of van der Pol type and as a restoring force expressed by both the sum and the product of a nonlinear function of deflection y and a periodic one of time τ . Resonance phenomena and behaviors occurring in these vibration systems are investigated in details. It has been revealed that the resonance phenomena are induced by the product $y^n \cos 2\eta\tau$ consisting of the n th power of y and a periodic function, where n is a positive integer and η is a fundamental frequency of parametric excitation. The mechanisms of the occurrence are evidently different from those of nonlinear forced vibration systems. In a self-exciting system of van der Pol type with only an exciting term $y \cos 2\eta\tau$ (i.e. $n=1$), resonances of orders 1 and 2 occur in the regions of frequencies corresponding to the unstable regions of orders 1 and 2 in a linear equations of Mathieu type. On the other hand, subharmonic resonances of orders 1/2

and 1/3 with frequencies $\eta/2$ and $\eta/3$ respectively and furthermore subsuperharmonic ones of order 2/3 besides orders 1 and 2 occur in a self-exciting system with a cubic exciting term $y^3 \cos 2\eta\tau$ (i.e. $n=3$). These results are derived from analyses of the case that the restoring force of structural system is symmetric with respect to the origin. In the case of an asymmetric one no results are derived.

In this paper the authors shall analyze a self-exciting system with an asymmetric property of restoring force including the nonlinear term y^2 and consider the effects of the squared nonlinearity and the squared exciting term $y^2 \cos 2\eta\tau$ on the regions where the resonances occur and on behaviors in their neighborhood. Firstly it is reported about the subharmonic resonances of order 1/2 and behaviors in their neighborhood.

2. Equation of Motion

The authors described the following nondimensional equation of motion in Refs. (2) to (4), expressing self-excitation by a nonlinear resistance of van der Pol type and parametric excitation by the product of a nonlinear function of deflection and a periodic one of time.

$$\ddot{y} - (\beta - \delta y^2)\dot{y} + (1 + \gamma y^2)(1 + \mu \cos 2\eta\tau)y = 0;$$

$$\dot{y} = dy/d\tau \quad \dots\dots\dots (1)$$

System (1) is the case that a structural system has a symmetric property of the restoring force. In the present paper the authors shall analyze a system with an asymmetric one described as follows:

$$\ddot{y} - (\beta - \delta y^2)\dot{y} + (1 + \alpha y + \gamma y^2) \times (1 + \mu \cos 2\eta\tau)y = 0 \quad \dots\dots\dots (2)$$

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Taking notice of exciting mechanisms, system (1) has two exciting terms $y^n \cos 2n\tau$ ($n=1$ and 3) and system (2) has also a term $n=2$ besides them. The authors investigate the influence of squared nonlinearity (i.e. asymmetry) included in the restoring force on the resonance of order $1/2$ by comparing it with that of cubic nonlinearity (3). In order to investigate whether a resonance of order $1/2$ occurs in a system with a squared nonlinearity and without a cubic one, the authors also consider a system as follows:

$$\ddot{y} - (\beta - \delta y^2)\dot{y} + (1 + \alpha y)(1 + \mu \cos 2\eta\tau)y = 0 \quad \dots\dots\dots(3)$$

In system (2) the authors assume that the parameters β , δ , α , γ and μ are sufficiently smaller than unity, which is a natural frequency of linear system. The parameters β , δ and μ are positive and the parameter of self-excitation β is assumed to be smaller than that of parametric excitation μ .

3. Behaviors in the Regions of Resonances of Order $1/2$

The authors shall call frequencies of both a resonance and its neighborhood the regions of the resonance and firstly investigate behaviors in the regions of resonance of order $1/2$. Their analyses are performed on the basis of such averaging method as used in Ref.(2). Namely using a transformation of variables such as

$$y = y_1, \quad 2\dot{y}/\eta = y_2 \quad \dots\dots\dots(4)$$

Equation (2) is expressed by a set of first order differential equations for y_1 and y_2 . The authors put the solution as follows:

$$\begin{cases} y_1 = \xi \cos(\eta\tau/2) + \zeta \sin(\eta\tau/2) + z_1 \\ y_2 = -\xi \sin(\eta\tau/2) + \zeta \cos(\eta\tau/2) + z_2 \end{cases} \quad \dots\dots(5)$$

where the components $z_1(\tau)$ and $z_2(\tau)$ are included because the existence of a constant component is expected. Applying the averaging method, we obtain the following on the assumption that the components z_1^2 and z_2^2 are fairly smaller than the component $A^2 (= \xi^2 + \zeta^2)$.

$$\begin{aligned} \dot{\xi} = & \frac{1}{2} \left(\beta - \frac{\delta}{4} A^2 \right) \xi + \frac{1}{\eta} \left\{ 1 - \left(\frac{\eta}{2} \right)^2 + 2\alpha z_1 \right. \\ & \left. + \frac{3\gamma}{4} A^2 - \frac{\mu\gamma}{8} (3\xi^2 - \zeta^2) \right\} \zeta \quad \dots\dots\dots(6.a) \end{aligned}$$

$$\begin{aligned} \dot{\zeta} = & -\frac{1}{\eta} \left\{ 1 - \left(\frac{\eta}{2} \right)^2 + 2\alpha z_1 + \frac{3\gamma}{4} A^2 \right. \\ & \left. + \frac{\mu\gamma}{8} (\xi^2 - 3\zeta^2) \right\} \xi + \frac{1}{2} \left(\beta - \frac{\delta}{4} A^2 \right) \zeta \quad \dots\dots\dots(6.b) \end{aligned}$$

$$\dot{z}_1 = (\eta/2) z_2 \quad \dots\dots\dots(6.c)$$

$$\dot{z}_2 = \left(\beta - \frac{\delta}{2} A^2 \right) z_2 - \frac{2}{\eta} \left\{ \left(1 + \frac{3\gamma}{2} A^2 \right) z_1 + \frac{\alpha A^2}{2} \right\} \quad \dots\dots\dots(6.d)$$

Using the relation of $\xi = A \cos \varphi$ and $\zeta = A \sin \varphi$, Eqs.(6.a) and (6.b) are also expressed by differential equations in the polar coordinates (A, φ) as follows:

$$\dot{A} = \frac{1}{2} \left(\beta - \frac{\delta}{4} A^2 \right) A - \frac{\mu\gamma}{8\eta} A^3 \sin 4\varphi \quad \dots\dots\dots(7)$$

$$\begin{aligned} \dot{\varphi} = & -\frac{1}{\eta} \left\{ 1 - \left(\frac{\eta}{2} \right)^2 + \frac{3\gamma}{4} A^2 \right\} - \frac{\mu\gamma}{8\eta} A^2 \cos 4\varphi \\ & - (2/\eta) \alpha z_1 \quad \dots\dots\dots(8) \end{aligned}$$

Paying attention to each term of Eq.(6), it is seen that a squared exciting term does not affect the occurrence of a resonance of order $1/2$ because the parameter of its term $\alpha\mu$ does not appear anywhere. In contrast to this, a cubic exciting term (the parameter $\mu\gamma$) is necessary for the occurrence of a resonance of order $1/2$ as mentioned in Ref.(3). Accordingly in system (3) a resonance of order $1/2$ does not become a subject of discussion for lack of a cubic one. It is interesting to discuss behaviors of the regions of this resonance in comparison with those of a resonance of order 2 in symmetric system (1) because of a similar solution including the components z_1 and z_2 to solution (5).

Now differential equations (6) include two kinds of solutions as follows:

- (1) Steady state solution expressing a resonance ($\dot{\xi} = \dot{\zeta} = \dot{z}_1 = \dot{z}_2 = 0$)
- (2) Solution expressing a limit cycle in the plane $\xi - \zeta$

The authors shall obtain approximate solutions of them in Chapters 4 and 5.

4. Resonance of Order $1/2$

From the steady solution of Eq.(6) we obtain

$$\begin{aligned} \eta^4 + 4\eta^2 \left\{ \left(\beta - \frac{\delta}{4} A^2 \right)^2 - 2 \left(1 + \frac{3\gamma}{4} A^2 \right) \right. \\ \left. - \frac{\alpha^2 A^2}{1 + \frac{3\gamma}{4} A^2 / 2} \right\} + 16 \left(1 + \frac{3\gamma}{4} A^2 - \frac{\alpha^2 A^2}{1 + \frac{3\gamma}{4} A^2 / 2} \right)^2 \\ - (\mu\gamma/2)^2 A^4 = 0 \quad \dots\dots\dots(9) \end{aligned}$$

and

$$z_1 = -\frac{\alpha A^2}{2} / \left(1 + \frac{3\gamma}{2} A^2 \right) \quad \dots\dots\dots(10)$$

In this case z_1 component becomes a constant. Resonance curves are expressed for an exciting frequency η by

$$A + z_1, \quad A - z_1 \quad \dots\dots\dots(11)$$

It is recognized that constant components (z_1) are always negative when the parameter α is positive (and so is γ). Equation (11) expresses amplitudes observed in the positive direction and the negative one. Then the phase φ is determined by

$$\begin{aligned} \tan 4\varphi = & -\frac{\eta \left(\beta - \frac{\delta}{4} A^2 \right)}{2} / \left\{ 1 - \left(\frac{\eta}{2} \right)^2 + \frac{3\gamma}{4} A^2 \right. \\ & \left. - \frac{\alpha^2 A^2}{1 + \frac{3\gamma}{4} A^2 / 2} \right\} \quad \dots\dots\dots(12) \end{aligned}$$

Components ξ_s and ζ_s of steady solutions obtained above are expressed by $\xi_s = A_s \cos \varphi_s$ and $\zeta_s = A_s \sin \varphi_s$, where the subscript s stands for steady solutions.

Next the stability criterion of steady solutions is established. Introducing small vibrations from steady solutions and putting $\xi = \xi_s + \Delta\xi$ and $\zeta = \zeta_s + \Delta\zeta$, we obtain characteristic equations for $\Delta\xi$ and $\Delta\zeta$ from Eqs. (6.a) and (6.b) and an algebraic equation derived from Eq. (6.a). According to the theorem of Routh-Hurwitz, stability conditions are as follows:

$$\left. \begin{aligned} \text{(i)} \quad & \beta - \delta A_s^2/2 - (cg + fh)/i < 0 \\ \text{(ii)} \quad & ae - bd + (cd - af)h/i + g(fb - ce)/i > 0 \end{aligned} \right\} \quad \text{.....(13)}$$

where

$$\begin{aligned} a &= \frac{1}{2} \left(\beta - \frac{\delta}{4} A_s^2 - \frac{\delta}{2} \xi_s^2 + \frac{3\gamma}{\eta} \left(1 - \frac{\mu}{2} \right) \xi_s \zeta_s \right), \\ b &= -\frac{\delta}{4} \xi_s \zeta_s + \frac{1}{\eta} \left(1 - \left(\frac{\eta}{2} \right)^2 + \frac{3\gamma}{4} A_s^2 - \frac{3\mu\gamma}{8} (\xi_s^2 - \zeta_s^2) + \frac{3\gamma}{2} \xi_s^2 + 2az_{1s} \right), \\ c &= 2a\xi_s/\eta, \\ d &= -\frac{1}{\eta} \left(1 - \left(\frac{\eta}{2} \right)^2 + \frac{3\gamma}{4} A_s^2 + \frac{3\mu\gamma}{8} (\xi_s^2 - \zeta_s^2) + \frac{3\gamma}{2} \xi_s^2 + 2az_{1s} \right) - \frac{\delta}{4} \xi_s \zeta_s, \\ e &= \frac{1}{2} \left(\beta - \frac{\delta}{4} A_s^2 - \frac{\delta}{2} \zeta_s^2 - \frac{3\gamma}{\eta} \left(1 - \frac{\mu}{2} \right) \xi_s \zeta_s \right), \\ f &= -2a\xi_s/\eta, \quad g = (12\gamma z_{1s} + \alpha)\xi_s, \\ h &= (12\gamma z_{1s} + \alpha)\zeta_s, \quad i = 4(1 + 3\gamma A_s^2/2). \end{aligned}$$

It is found that the trivial solution ($\xi_s = \zeta_s = z_{1s} = 0$) is always unstable.

5. Solutions in the Neighborhood of Resonance of Order 1/2

The authors shall obtain another kind of solution expressing a limit cycle in the plane ξ - ζ (A - φ). Using the relation of $2A\dot{A} = (A^2)'$, firstly Eq. (7) is transformed into an equation for A^2 . Now the authors express the variable amplitude A^2 by the sum as follows:

$$A^2 = A_0^2 + \Delta A^2; \quad A_0^2 = 4\beta/\delta \quad \text{.....(14)}$$

where ΔA^2 is a variable component and A_0 is a constant amplitude of van der Pol equation included in system (2). Assuming that ΔA^2 component is quite smaller than A_0^2 and ignoring small terms of higher orders, we obtain

$$\frac{d(\Delta A^2)}{d\tau} = -\beta \Delta A^2 - \frac{\mu\gamma}{4\eta} A_0^2 \sin 4\varphi. \quad \text{.....(15)}$$

Furthermore assuming that z_1 component is also quite small, the equation for φ becomes

$$\frac{d\varphi}{d\tau} = -\frac{1}{\eta} \left\{ 1 - \left(\frac{\eta}{2} \right)^2 + \frac{3\gamma}{4} A_0^2 \right\} - \frac{\mu\gamma}{8\eta} A_0^2 \cos 4\varphi. \quad \text{.....(16)}$$

Within the frequencies where the second term of Eq. (16) is quite smaller than the first one, the phase φ is approximated by

$$\varphi = Q_0 \tau + \varphi_0; \quad Q_0 = -\frac{1}{\eta} \left\{ 1 - \left(\frac{\eta}{2} \right)^2 + \frac{3\gamma\beta}{\delta} \right\}. \quad \text{.....(17)}$$

Then from Eq. (15) the component ΔA^2 is expressed by

$$\begin{aligned} \Delta A^2 &= Q \cos \{ 4(Q_0 \tau + \varphi_0) - \Psi \}; \\ Q &= \frac{\mu\gamma}{\eta} \left(\frac{2\beta}{\delta} \right)^2 / \sqrt{16Q_0^2 + \beta^2}, \\ \tan \Psi &= -\beta/4Q_0. \end{aligned} \quad \text{.....(18)}$$

As to the solution in the plane z_1 - z_2 , the following second order differential equation for z_1 is derived from Eqs. (6.c) and (6.d).

$$\ddot{z}_1 - \left(\beta - \frac{\delta}{2} A^2 \right) \dot{z}_1 + \left(1 + \frac{3\gamma}{2} A^2 \right) z_1 = -\frac{\alpha}{2} A^2 \quad \text{.....(19)}$$

Now assuming that $A^2 \approx A_0^2$, we obtain an equation with only a constant external force.

$$\ddot{z}_1 + \beta \dot{z}_1 + (1 + 6\gamma\beta/\delta) z_1 = -2\alpha\beta/\delta \quad \text{.....(20)}$$

Steady solutions of Eq. (20) have a constant value independent of frequencies as follows:

$$z_1 = -2\alpha\beta/(\delta + 6\gamma\beta) \quad \text{.....(21)}$$

As a result, approximate solutions near the resonance are expressed by

$$\begin{aligned} y &= A(\tau) \cos \{ (\eta/2 - Q_0)\tau - \varphi_0 \} + z_1; \\ A(\tau) &= \sqrt{4\beta/\delta + Q \cos \{ 4(Q_0 \tau + \varphi_0) - \Psi \}}. \end{aligned} \quad \text{.....(22)}$$

Then the maximum and minimum amplitudes observed in both positive and negative directions are given by

$$\sqrt{4\beta/\delta + Q} \pm z_1, \quad \sqrt{4\beta/\delta - Q} \pm z_1. \quad \text{.....(23)}$$

Since z_1 becomes negative when $\alpha > 0$ (and also $\gamma > 0$), both amplitudes of negative direction become larger than those of positive one.

The authors investigate the stability of solution (22). Since β (the coefficient of ΔA^2) is of course positive, it is seen that solution (22) is always as stable as ΔA^2 component. It is also clear from Eq. (7) that the trivial solution is unstable because $\dot{A} > 0$ when A approaches zero.

The above analysis predicts the results as follows: z_1 component has constant values inside and outside the resonance. The squared nonlinearity does not

affect the variable amplitude $A(\tau)$, that is, the beat character of vibrations. This result is contrary to the result that z_1 component of system (1) is a vibrational one with a long period and affects the character of vibration wave in the neighborhood of resonance of order 2. It is obvious from Eq.(15) that the cubic exciting term creates the beat character.

6. Numerical Results and Consideration

The authors compare approximate solutions with numerical ones of Eq.(2) by means of Runge-Kutta-Gill method, which are called R.K.G. solutions. Parameters are as follows:

$$\text{Case I: } \beta = 0.05, \delta = 0.02, \alpha = 0.04 \\ \mu = \gamma = 0.2$$

In this case the coefficient of cubic non-linearity γ is positive and that of squared non-linearity α is less than γ . The other parameters except for α are the same as those used in system (1). Symbols $[y]$ express amplitudes of resonance for both positive and negative directions [Eq. (11)] and the maximum and minimum amplitudes of beat [Eq.(23)].

6.1 Consideration of resonance of order 1/2

Figure 1 shows $[y]-\eta$ diagram for both directions. Solid lines stand for stable solutions and broken ones for unstable solutions. Amplitudes of resonance by R.K.G. solutions are marked by thick circles and ones of beat are by pairs of thin

Fig.1 Diagram $[y]-\eta$ (case I)

circles. Two vertical lines connecting thin circles show the frequencies of boundaries where beat vibrations occur and therefore within the frequencies only resonances occur. In the case of approximate solutions their frequencies are estimated by the points of intersection of unstable parts of resonance and amplitude curves of beat as mentioned in Refs.(2) and (5). Figure 2(a) shows absolute values of constant components of resonance and Fig.2(b) shows components ξ_s and ζ_s plotted against the frequency η as parameter. In Case I z_1 is always negative and its absolute values have a tendency to increase together with the amplitudes of resonance. The region of frequencies where the resonance of system (2) is only a little smaller than that of symmetric system (1) but no significant difference between them is found.

Figure 3 shows trajectories in the stroboscope phase plane⁽⁵⁾ at $\eta = 3.1$. Four stable foci (F_s) represent stable solutions of resonance and four saddle points (SP) represent unstable ones. Waveforms of resonance starting from initial conditions near the points A to D (as shown in Fig.4 for the point A) prove that a negative constant component is in existence. Namely

(a) Constant components $|z_1|$

(b) Frequency characteristic of resonance vector

Fig.2 Resonance characteristics

constant components of resonance of order $1/2$ have a sign independent of initial conditions and their absolute values are quite smaller than those of order 2 of symmetric system (1) dependent on initial conditions.

6.2 Consideration of solutions in the neighborhood of resonance

Figure 5 shows trajectories of Eq.(6) at $\eta=3.4$. A limit cycle (L_S) appears in the plane $\xi-\zeta$ and in the plane z_1-z_2 the trajectory converges to a point about $z_1 = -0.05$ on the z_1 axis, whose value coincides with that calculated by Eq.(21). It is confirmed from Eq.(23) and R.K.G. solutions shown in Fig.1 that the maximum and minimum amplitudes of beat are different between positive and negative directions. Figure 6 shows waveforms by both Eq.(22) and R.K.G. solutions at $\eta=3.4$, their aspects of beat being similar. No significant difference between waveforms of resonance of order $1/2$ in systems (2) and (1) [see Fig.13 in Ref.(2)] is recognized for the same η . Therefore it is found that the squared non-linearity does not affect the character of beat.

6.3 Numerical results for some cases

The authors consider the effect of squared non-linearity (the parameter α) for the cases as follows:

Fig. 3 Trajectories of resonance of order $1/2$ in the stroboscope phase plane ($\eta=3.1$)

Fig. 4 Waveform of resonance by R.K.G. [$\eta=3.1$; point A; $y(0)=2$, $\dot{y}(0)=3.72$]

(a) In the plane $\xi-\zeta$

(b) In the plane z_1-z_2

Fig. 5 Trajectories in the neighborhood of resonance [$\eta=3.4$; $\xi(0)=3.3$, $\zeta(0)=0$, $z_1(0)=-0.05$, $z_2(0)=0$]

(a) Eq. (22) ($\varphi_0=0$)

(b) R.K.G. [$y(0)=3.3$, $\dot{y}(0)=0$]

Fig. 6 Waveforms of vibrations in the neighborhood of resonance ($\eta=3.4$)

Case Ia : $\alpha = -0.04$

Case Ib : $\alpha = 0.1$ (Fig.7)

In Case Ia that α has the same absolute value and an opposite sign as Case I, calculated results show $[y]-\eta$ diagram such as Fig.1 which is turned upside down with respect to η axis. Absolute values of z_1 component are the same as shown in Fig.2(a) but their signs are always positive. The character like this is also recognized from R.K.G. solutions. Taking notice of the parameter α in Eq.(19), it is seen that the sign of α does not affect essen-

(a) Diagram $[y]-\eta$

(b) Constant component of resonance $|z_1|$

Fig.7 Relations between $[y]$ and η , $|z_1|$ and η (case Ib)

tially the character of resonance curve because α is included only in the form of α^2 .

Next Fig.7 shows $[y]-\eta$ diagram and constant components of resonance $|z_1|$ for Case Ib. Values of z_1 are always negative and those in the neighborhood are -0.125 independently of η . Absolute values of constant components become large with larger α and then amplitudes $[y]$ in the negative direction become slightly large. The region of frequencies where resonances occur is about the same as in Case I but has a tendency to reduce only a little.

Thus it is obvious that the cubic non-linearity plays a more important part in the occurrence of resonance of order $1/2$ and in the character of vibrations near the resonance. Contrary to this, the squared non-linearity plays an auxiliary part in the occurrence of constant component.

7. Conclusions

The following conclusions are obtained from comparison of two systems with and without a squared non-linearity of restoring force:

(1) The cubic exciting term $y^3 \cos 2\eta t$ is necessary to the occurrence of a resonance of order $1/2$ but the squared one is not necessary in particular.

(2) As the squared non-linearity increases, a constant component of the resonance increases and then the region of frequencies where the resonance occurs has a tendency to become narrow a little. The importance of that non-linearity is slight compared with the influence of the cubic one⁽³⁾ on the resonance characteristic.

(3) Owing to the squared non-linearity, a constant component independent of frequencies appears in the vibrations near the resonance. The beat character of vibration itself is produced by the cubic exciting term.

(4) The sign of coefficient α (of the squared non-linearity determines whether the values of constant components are negative or positive, but does not have a resonance curve and the aspect of vibrations near the resonance is different in essence.

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