Vibration Analysis of Rotating Cylindrical Shells

Based on the Timoshenko Beam Theory*

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The basic equations for the beam-type vibration of a rotating cylindrical shell are derived by the Timoshenko beam theory. The frequency analysis is presented for three kinds of boundary conditions and the validity of this model is examined by comparing the results with those based on the cylindrical shell theory. It is found that as for the natural frequencies of non-rotating, and rotating cylindrical shells and for the critical speeds reasonable results can be obtained by solving the problem on the basis of the Timoshenko beam theory.

Key words: Vibration of Continuous System, Timoshenko Beam Theory, Rotating Cylindrical Shell, Vibration Analysis, Critical Speed.

1. Introduction

Thin circular cylindrical shells are employed in many engineering applications, such as high-speed centrifugal separators. In their beam-type vibration for the number of circumferential waves 1, a critical speed exists when the rotating speed is nearly equal to the natural frequency of the non-rotating cylindrical shell and an unstable phenomenon may happen. So it is necessary for practical use to make clear the behavior of this mode in the rotating state. The vibration of a rotating cylindrical shell can be analyzed strictly by use of the basic equations which the authors derived in the previous paper.(1),(2) But in the case of the beam-type vibration, it is considered that it is possible to treat a circular cylindrical shell as a beam, and useful to analyze a rotating cylindrical shell model based on beam theory for simplification of the problem. Up to date, several papers dealing with the beam-type vibration of a circular cylindrical shell have been published and some compared shell theory and beam theory.(3),(4) However, there are few papers in which vibration analysis for various boundary conditions was performed systematically.

The purpose of the present paper is to derive the basic equations for the beam-type vibration of a rotating cylindrical shell model and to examine the

propriety of this model concerning the vibration phenomenon by comparing with the analytical results based on shells theory.

In the formulation for the beam-type vibration of a rotating cylindrical shell, we derive the basic equations based on the Timoshenko beam theory which considers rotatory inertia and shear deflection. And we analyze the frequencies under three kinds of boundary conditions: (a) both ends clamped; (b) simply supported without axial constraint; and (c) both ends covered with side disks; and obtain the values of the frequencies of a non-rotating and rotating cylindrical shell and the critical speeds and moreover compare them with those based on the rotating cylindrical shell theory. Numerical value of the shear coefficient is calculated as $\nu=0.3$ by the formula derived by Cowper.(5)

2. A Rotating Cylindrical Shell Model

2.1 A rotating cylindrical shell model

As shown in Fig.1, assuming that a thin hollow beam rotating at the angular velocity Ω has length l , mean radius a (outer diameter d_1 , inner diameter d_2) and thickness h , the dynamic coordinate system xyz is fixed on the beam. Moreover, for the boundary condition (c) both ends

Fig.1 Coordinate system

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covered with side disks it is assumed that both ends of the beam are covered with rigid circular disks and they are supported by a negligibly soft shaft.

2.2 Basic equations

On the basis of the Timoshenko beam theory, applying Hamilton's principle, we have derived basic equations. First, we consider the flexural displacement v and the angular rotation ψ_v in xy -plane, and those w , ψ_z in zx -plane and respectively calculate the strain energy E_v including shear deflections and the kinetic energy E_r including rotatory inertia. And from the following Hamilton's principle

$$\int_{t_0}^{t_1} (\delta E_r - \delta E_v) dt = 0 \quad (1)$$

we can obtain Euler's equations and the boundary conditions. Moreover, considering the orthogonality between v and w , ψ_v and ψ_z , we can put as follows.

$$\left. \begin{aligned} u &= v + iw \\ \psi &= \psi_v + i\psi_z \end{aligned} \right\} \quad (2)$$

where i is an imaginary unit. Rewriting u as w , Euler's equations are expressed as follows.

$$\left. \begin{aligned} \rho I \frac{\partial^2 \psi}{\partial t^2} - EI \frac{\partial^2 \psi}{\partial x^2} - k' GA \left(\frac{\partial w}{\partial x} - \psi \right) &= 0 \\ \rho \left(\frac{\partial^2 w}{\partial t^2} - 2i\Omega \frac{\partial w}{\partial t} - \Omega^2 w \right) - k' GA \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x} \right) &= 0 \end{aligned} \right\} \quad (3)$$

where $E, G, \rho, I = \pi(d_1^4 - d_2^4)/64$, $A = \pi(d_1^2 - d_2^2)/4$, and k' are modulus of longitudinal elasticity, transverse one, density, second moment of area, sectional area and shear coefficient of a beam respectively. For $\Omega = 0$ equation (3) is equal to that of Timoshenko beam. (6)

2.3 Boundary conditions

From the boundary conditions which we can obtain by applying Hamilton's principle in the preceding chapter, we can determine the boundary conditions, (a) both ends clamped and (b) simply supported without axial constraint. But in another boundary condition, (c) both ends covered with side disks, the rigid circular disks covering both sides have an additional kinetic energy E_r . So, fixing the dynamic coordinate system $\bar{x}\bar{y}\bar{z}$ on the rigid circular disk and considering the rigid displacement $\bar{v}$, the angular rotation $\bar{\psi}_v$ in $\bar{x}\bar{y}$ -plane and those $\bar{w}$, $\bar{\psi}_z$ in $\bar{x}\bar{z}$ -plane, the kinetic energy E_r is obtained. Using the following instead of eq. (1),

$$\int_{t_0}^{t_1} (\delta E_r - \delta E_v + \delta E_r) dt = 0 \quad (4)$$

And putting

$$\left. \begin{aligned} \bar{u} &= \bar{v} + i\bar{w} \\ \bar{\psi} &= \bar{\psi}_v + i\bar{\psi}_z \end{aligned} \right\} \quad (5)$$

we consider the conditions $\bar{u} = w$ and $\bar{\psi} = \psi$ at $x = \pm l/2$, where the rigid circular disks and the beam move as one body. Finally, the boundary conditions are denoted as follows. At $x = \pm l/2$,

$$\left. \begin{aligned} (a) \quad w &= 0, \quad \psi = 0 \\ (b) \quad w &= 0, \quad \frac{\partial \psi}{\partial x} = 0 \\ (c) \quad EI \frac{\partial \psi}{\partial x} + \bar{\rho} \bar{h} \left\{ (\bar{I}_1 + \bar{I}_2) \frac{\partial^2 \psi}{\partial t^2} - 2i\Omega \bar{I}_2 \frac{\partial \psi}{\partial t} - \bar{I}_2 \Omega^2 \psi \right\} &= 0, \\ k' AG \left(\frac{\partial w}{\partial x} - \psi \right) + \bar{\rho} \bar{h} \bar{A} \left(\frac{\partial^2 w}{\partial t^2} - 2i\Omega \frac{\partial w}{\partial t} - \Omega^2 w \right) &= 0 \end{aligned} \right\} \quad \dots (6)$$

where $\bar{\rho}, \bar{h}, \bar{I}_1 = \pi d_1^4/64$ and $\bar{A} = \pi d_1^2/4$ are density, thickness, second moment of area, sectional area of a rigid circular disk, and $\bar{I}_2 = \pi d_1^2 \bar{h}^2/48$.

3. Frequency Analysis

Substituting the following equations

$$\left. \begin{aligned} w &= W_0 e^{\lambda x} e^{i p t} \\ \psi &= \Psi_0 e^{\lambda x} e^{i p t} \end{aligned} \right\} \quad (7)$$

into eq. (3), we have the following general solutions.

$$\left. \begin{aligned} w &= \left[\sum_{j=1}^2 (W_j^1 \cosh \lambda_j x + W_j^2 \sinh \lambda_j x) \right] e^{i p t} \\ \psi &= \left[\sum_{j=1}^2 (W_j^1 x_j \sinh \lambda_j x + W_j^2 x_j \cosh \lambda_j x) \right] e^{i p t} \end{aligned} \right\} \quad (8)$$

where p is the natural frequency and

$$\left. \begin{aligned} \lambda_j^2 &= -\frac{1}{2} \left\{ \frac{\rho p^2}{E} + \frac{\rho}{k' G} (p - \Omega)^2 \right\} \\ &\pm \frac{1}{2} \left\{ \left[\frac{\rho p^2}{E} - \frac{\rho}{k' G} (p - \Omega)^2 \right]^2 + 4(p - \Omega)^2 \frac{\rho A}{EI} \right\}^{1/2} \\ x_j &= \frac{-k' GA \lambda_j}{EI \lambda_j^3 + \rho I p^2 - k' GA} \end{aligned} \right\} \quad (9)$$

Moreover, undetermined W_j^1 and W_j^2 correspond to such a mode that the flexural displacement amplitudes are respectively symmetric and anti-symmetric with respect to the center of a beam.

Substituting eq. (8) into the boundary conditions eq. (6) leads to the homogeneous equations concerning W_j^1 and W_j^2 . If the determinant of the coefficient matrix is set equal to zero, it leads to a frequency equation, which can be obtained for each boundary condition. The details are omitted here.

The trial method is used to obtain the value of the natural frequency. In the actual numerical computation as in previous papers, in order to compare with the analytical results based on circular cylindrical shells theory, the dimensionless forms of angular frequency p and rotating angular velocity Ω are treated as follows.

$$\bar{p} = \sqrt{\gamma} p, \quad \bar{\Omega} = \sqrt{\gamma} \Omega \quad (10)$$

where $\gamma = \rho a^2 (1 - \nu^2)/E$. The same analysis is performed on the basis of rotating cylindrical shells theory (for the detail of the process, see literature.) (1), (2)

4. Examination of Numerical Computational Results

4.1 The natural angular frequency in the non-rotating state

The natural angular frequencies of a non-rotating cylindrical shell are calculated in this section. The results for dimensionless thickness $k=h/2a=0.02$ are shown. The dependence of the natural angular frequencies β_1 upon the

dimensionless length $\alpha=l/2a$ are shown in Fig.2 (a), (b), (c)-1 and (c)-2. For the symmetric mode ($m=1$) and the anti-symmetric one ($m=2$) with respect to the flexural displacement of a circular cylindrical shell, the results are shown for each boundary condition; (a) both ends clamped, (b) simply supported without axial constraint and (c) both ends covered with side disks. Concerning the boundary condition (c), two kinds of the results

(a)

(c)-1

(b)

(c)-2

Fig.2 Variation of natural frequency in the non-rotating state ($k=0.02$)

Fig.3 Mode shape in the axial direction
($\alpha=5$, $\kappa=0.02$)

for the ratio of side disk thickness to the cylindrical shell one, $\bar{h}/h=4$ ((c)-1) and $\bar{h}/h=1$ ((c)-2) are shown. In all figures, the solid line and the broken one respectively represent the results based on the Timoshenko beam theory and those on circular cylindrical shells one. Both curves of the figures are sufficiently identical and except for the case of an extremely short cylindrical shell, it is considered that the reasonable results can be obtained by means of the Timoshenko beam theory. Then, in Fig.2 (c)-2 ($\bar{h}/h=1$), the place where both lines widely separate (e.g. near by $\alpha=1.3$ in $m=1$) are found. The reason is that the side disks are treated as elastic body in circular cylindrical shells and the mode of a circular plate changes here. For example at the discontinuous point near $\alpha=3$ in $m=1$, the number of nodal circles changes from 0 to 1. On the other hand, the axial mode shape for the same conditions are shown in Fig.3 (a), (b), (c)-1 and (c)-2, where $\xi=x/a$. The flexural displacement w and the rotatory angle of the section Ψ in the Timoshenko beam theory, and the

axial, circumferential and radial displacements u, v and w and moreover for the boundary condition (c), the flexural displacement of side disk $\bar{w}$ in circular cylindrical shells theory are shown. Comparing the flexural displacement w in the present theory with the radial displacement w in circular cylindrical shells one, both curves of the shapes are sufficiently identical. Moreover, it is found that the curves of the rotatory angle of the section Ψ is quite similar to that of the axial displacement u of a circular cylindrical shell.

4.2 The natural frequency in the rotating state

The dependence of the natural angular frequencies upon the rotating speed is shown in Fig.4 (a), (b), (c)-1 and (c)-2. The results for the same conditions as in the preceding section with $\alpha=l/2a=5$, $k=h/2a=0.02$ are shown. In all figures, the rotating angular velocities $\bar{\Omega}$ and the natural frequencies $\bar{p}$ are normalized by the natural angular frequencies $\bar{p}_i$ in the non-rotating state. The frequencies of the forward waves are generally $\bar{p}<0$, and

Fig.4 Frequency of traveling waves
(part 1)

so the absolute values are shown in these figures. As in section 4.1, the solid line and the broken one respectively represent the results based on the Timoshenko beam theory and those on circular cylindrical shells theory. It is found that as the rotating speed increased, both curves gradually separate from each other but they are sufficiently identical in the domain where the rotating speed is not high.

4.3 The critical speed

Moreover, under the same conditions the critical speeds are calculated. Here, the rotating speed where the natural frequency is equal to zero, is called the critical speed. Putting $\bar{\omega}=0$ in the frequency equation, the values can be obtained in the same way as in sections 4.1 and 4.2. The computational results for dimensionless thickness $\kappa=h/2a=0.02$ are shown. The dependence of the dimensionless critical speed upon the dimensionless length $\alpha=l/2a$ for the same conditions

Fig.4 Frequency of traveling waves
(part 2)

as in the preceding section is shown in Fig.5 (a), (b), (c)-1 and (c)-2. As in the preceding sections, the solid line and the broken one respectively represent the results based on the present theory and those on cylindrical shells theory. It is found that as the length shortens, both results for both ends clamped in which boundary condition is constrained in the axial direction, gradually separate, but the results for the other boundary conditions are quite identical.

5. Conclusions

Concerning the beam-type vibration of a rotating cylindrical shell model, the basic equations were derived by the Timoshenko beam theory. Comparing with the rotating cylindrical shell theory, we examined the natural frequencies of a non-rotating and rotating cylindrical shell and the critical speeds were examined:

(1) As for the natural frequencies of a non-rotating cylindrical shell, the reasonable results can be obtained by the Timoshenko beam theory for three boundary conditions except for the case of an

Fig.5 Variation of critical speed

extremely short cylindrical shell.

(2) As for the natural frequencies of a rotating cylindrical shell, the reasonable results can be obtained by the present theory for three boundary conditions in the domain where the rotating speed is not so high.

(3) As for the critical speed, the reasonable results can be obtained for both ends clamped except for the case of an extremely short cylindrical shell and the other boundary conditions.

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