Approximate Solution of a Gear System Subjected to Random Excitation*

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Forced vibration of a gear system excited by transmission error having a period equal to the meshing period and by a random external force, is analyzed approximately by means of an averaging method. Some numerical examples are given. To facilitate analytical treatment, the time varying parameter system is transformed into a time invariant parameter system.

Key Words: Gear, Vibration, Transmission Error, Random Force, Averaging Method

1. Introduction

Many studies on the forced vibration of a gear system excited by the fluctuation of an external torque have been presented in the past. It is difficult to obtain a closed form solution because the equation of motion contains a nonlinear term representing backlash or a periodic time varying parameter representing tooth stiffness. Various methods using simulation techniques were suggested in order to obtain a solution⁽¹⁾⁻⁽³⁾.

Recently, however, approximate solutions⁽⁴⁾⁽¹⁴⁾ were obtained for a linear system subjected to a harmonic excitation where backlash term was neglected. These results agree well with the numerical solution obtained by simulation techniques. In this report, an approximate solution of a linear gear system subjected to random excitation such as the case of a transmission error containing random components⁽⁵⁾⁽⁶⁾ or an external torque fluctuating randomly⁽⁷⁾⁽⁸⁾ is given using an averaging method⁽⁹⁾⁽¹⁰⁾⁽¹³⁾.

2. Equation of motion and its approximate solution

Provided that the shaft stiffness is small compared to the torsional stiffness of

From Eqs. (1) and (2), the following equations are obtained:

$$\left. \begin{aligned} dA/d\tau &= -\{B \sin(\nu\tau + \phi) \sin \phi + \sin \phi + \xi(\tau) \sin \phi - A \cdot k(\tau) \sin \phi \cos \phi + 2\delta \\ &\quad \times A\nu \sin^2 \phi + A\nu^2 \sin \phi \cos \phi - Ck(\tau) \sin \phi\} / \nu \\ d\phi/d\tau &= -\{B \sin(\nu\tau + \phi) \cos \phi + \cos \phi + \xi(\tau) \cos \phi - A \cdot k(\tau) \cos^2 \phi + 2\delta A\nu \\ &\quad \times \sin \phi \cos \phi + A\nu^2 \cos^2 \phi - Ck(\tau) \cos \phi\} / (A \cdot \nu) \end{aligned} \right\} \dots\dots\dots (3)$$

the gear teeth, the equation of motion neglecting backlash is represented by Eq. (1)⁽¹⁾⁽⁵⁾. It is assumed that the transmission error is of sinusoidal form with one period per mesh⁽⁸⁾⁽¹²⁾ and the external torque changes randomly.

A nondimensional variable, y , which is composed of the summation of the relative angular displacement and the forced displacement and is normalized by the static torque, is introduced. Then we have

$$\frac{d^2 y}{d\tau^2} + 2\delta \frac{dy}{d\tau} + k(\tau)y = B \sin(\nu\tau + \phi) + 1 + \xi(\tau) \dots\dots\dots (1)$$

where δ is damping coefficient, $k(\tau)$ is variable spring constant as shown in Fig. 1, ν is nondimensional frequency, $\xi(\tau)$ is random exciting term, and so on. As both $k(\tau)$ and the periodic term in the right hand side of Eq. (1) have the same frequency ν , the solution of Eq. (1) and the derivative may be assumed as follows:

$$\begin{aligned} y(\tau) &= A(\tau) \cos(\nu\tau + \phi(\tau)) + C \\ dy/d\tau &= -\nu A(\tau) \sin(\nu\tau + \phi(\tau)) \end{aligned} \dots\dots\dots (2)$$

Fig. 1 Time variation of spring constant

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where $\phi(\tau) = \nu\tau + \phi(\tau)$.

Considering $A(\tau)$ and $\phi(\tau)$ to be slowly changing functions of time in comparison with $\sin \nu\tau$, we can simplify Eq.(3) using the averaging method(9)(10)(13) as follows.

$$\left. \begin{aligned} \frac{dA}{d\tau} &= -\frac{B}{2\nu} \cos(\psi - \varphi) - \frac{1}{\nu} \xi(\tau) \sin \varphi + \frac{Ab}{8\pi\nu} (\cos 2\varphi - \cos(2\nu l_1 + 2\varphi)) \\ &\quad - A\delta + \frac{Cb}{2\pi\nu} (\cos \varphi - \cos(\nu l_1 + \varphi)) \\ \frac{d\varphi}{d\tau} &= -\frac{B}{2A\nu} \sin(\psi - \varphi) - \frac{1}{A\nu} \xi(\tau) \cos \varphi + \frac{1}{4\pi\nu} (b\nu l_1 + 2\pi k_1) \\ &\quad + \frac{b}{8\pi\nu} (\sin(2\nu l_1 + 2\varphi) - \sin 2\varphi) - \frac{1}{2}\nu + \frac{Cb}{2\pi A\nu} (\sin(\nu l_1 + \varphi) - \sin \varphi) \end{aligned} \right\} \dots\dots\dots (4)$$

where

$$C = \frac{2\pi - Ab(\sin(\nu l_1 + \varphi) - \sin \varphi)}{b\nu l_1 + 2\pi k_1} \dots\dots\dots (5)$$

Furthermore, using the transformation

$$\begin{aligned} z_1 &= A \cos \varphi, \quad z_2 = A \sin \varphi \\ \frac{dz_1}{d\tau} &= \frac{dA}{d\tau} \cos \varphi - \frac{d\varphi}{d\tau} A \sin \varphi, \quad \frac{dz_2}{d\tau} = \frac{dA}{d\tau} \sin \varphi + \frac{d\varphi}{d\tau} A \cos \varphi \end{aligned} \dots\dots\dots (6)$$

we can obtain the following equations:

$$\left. \begin{aligned} \frac{dz_1}{d\tau} &= d_0 + d_1 z_1 + d_2 z_2 - \frac{1}{\nu} \xi(\tau) \sin \nu\tau \\ \frac{dz_2}{d\tau} &= c_0 + c_1 z_1 + c_2 z_2 - \frac{1}{\nu} \xi(\tau) \cos \nu\tau \end{aligned} \right\} \dots\dots\dots (7)$$

where

$$\left. \begin{aligned} d_0 &= -\frac{B}{2\nu} \cos \psi + \frac{b(1 - \cos \nu l_1)}{\nu(b\nu l_1 + 2\pi k_1)} \\ d_1 &= \frac{b}{8\pi\nu} (1 - \cos 2\nu l_1) - \delta - \frac{b^2(1 - \cos \nu l_1) \sin \nu l_1}{2\pi\nu(b\nu l_1 + 2\pi k_1)} \\ d_2 &= \frac{\nu}{2} - \frac{1}{4\pi\nu} (b\nu l_1 + 2\pi k_1) + \frac{b}{8\pi\nu} \sin 2\nu l_1 + \frac{b^2}{2\pi\nu} \frac{(1 - \cos \nu l_1)^2}{(b\nu l_1 + 2\pi k_1)} \\ c_0 &= -\frac{B}{2\nu} \sin \psi + \frac{b \sin \nu l_1}{\nu(b\nu l_1 + 2\pi k_1)} \\ c_1 &= \frac{1}{4\pi\nu} (b\nu l_1 + 2\pi k_1) + \frac{b}{8\pi\nu} \sin 2\nu l_1 - \frac{1}{2}\nu - \frac{b^2 \sin^2 \nu l_1}{2\pi\nu(b\nu l_1 + 2\pi k_1)} \\ c_2 &= \frac{b}{8\pi\nu} (\cos 2\nu l_1 - 1) - \delta - \frac{b^2 \sin \nu l_1 (\cos \nu l_1 - 1)}{2\pi\nu(b\nu l_1 + 2\pi k_1)} \end{aligned} \right\} \dots\dots\dots (8)$$

and putting

$$\zeta_1(\tau) = -\xi(\tau) \sin \nu\tau/\nu, \quad \zeta_2(\tau) = -\xi(\tau) \cos \nu\tau/\nu \dots\dots\dots (9)$$

we may approximate the auto- and cross-correlation functions as the following equation(9):

$$\left. \begin{aligned} \langle \zeta_1(\tau) \zeta_1(\tau + \rho) \rangle &\equiv \langle \zeta_1 \zeta_{1\rho} \rangle = K\delta(\rho) = \langle \zeta_2(\tau) \zeta_2(\tau + \rho) \rangle \\ \langle \zeta_1(\tau) \rangle &= \langle \zeta_2(\tau) \rangle = 0, \quad \langle \zeta_1 \zeta_{2\rho} \rangle = \langle \zeta_{1\rho} \zeta_2 \rangle = 0 \end{aligned} \right\} \dots\dots\dots (10)$$

where

$$K = \frac{1}{2\nu^2} \int_{-\infty}^{\infty} \langle \xi \xi_{\rho} \rangle \cos \nu\rho d\rho = \frac{1}{2\nu^2} \kappa(\nu)$$

$\kappa(\tau)$ represents the spectral density function of the $\xi(\tau)$ process at frequency ν , and $\delta(\rho)$ is Dirac's delta function. Here, the condition $K = \kappa(\nu)/2k(\tau)$ is assumed for convenience' sake, which is accurate enough in the limit case of $k(\tau) = \text{constant}$, where

$$\overline{k(\tau)} = \frac{1}{2\pi} (\nu l_1 + k_1(2\pi - \nu l_1)) \dots\dots\dots (11)$$

If the initial time is taken as $\tau=0$, the initial vector as $X_0^1 = (z_{10}, z_{20})$, and the transition matrix as $\bar{\Psi}(\tau)$, then we have

$$\left. \begin{aligned} X(\tau) &= \bar{\Psi}(\tau) X_0 + \int_0^\tau \bar{\Psi}(\tau - \sigma) W(\sigma) d\sigma \\ W'(\tau) &= (\zeta_1(\tau) + d_0, \quad \zeta_2(\tau) + c_0) \end{aligned} \right\} \dots\dots\dots (12)$$

From Eq.(7), the stability conditions are

$$d_1 + c_2 < 0, \quad d_1 c_2 - c_1 d_2 > 0$$

From Eqs.(10)-(12), the mean and the covariance are represented as follows:

$$\langle X(\tau) \rangle = \bar{\Psi}(\tau) X_0 + \int_0^\tau \bar{\Psi}(\tau - \sigma) C_0 d\sigma, \quad C_0 = (d_0, c_0) \dots \dots \dots (13)$$

$$\langle \{X(\tau) - \langle X(\tau) \rangle\} \{X(\tau) - \langle X(\tau) \rangle\}' \rangle$$

$$= \int_0^\tau \int_0^\tau \bar{\Psi}(\tau - \sigma) \langle \zeta(\sigma) \zeta'(\eta) \rangle \bar{\Psi}'(\tau - \eta) d\sigma d\eta = K \int_0^\tau \bar{\Psi}(\tau - \eta) \bar{\Psi}'(\tau - \eta) d\eta \dots \dots \dots (14)$$

where

$$\zeta'(\eta) = (\zeta_1(\eta), \zeta_2(\eta))$$

And from Eqs.(2), (5), (6), the following equations are obtained:

$$y(\tau) = z_1(b_1 + \cos \nu \tau) - z_2(\sin \nu \tau - b_2) + b_0$$

$$dy/d\tau = -\nu z_1 \sin \nu \tau - \nu z_2 \cos \nu \tau \dots \dots \dots (15)$$

where

$$\left. \begin{aligned} b_0 &= 2\pi / (b\nu l_1 + 2\pi k_1), \quad b_1 = -b \sin \nu l_1 / (b\nu l_1 + 2\pi k_1) \\ b_2 &= -b(\cos \nu l_1 - 1) / (b\nu l_1 + 2\pi k_1) \end{aligned} \right\} \dots \dots \dots (16)$$

Therefore, the mean and the variance of $y(\tau)$ are represented as follows, respectively.

$$\langle y(\tau) \rangle = \langle z_1 \rangle (b_1 + \cos \nu \tau) - \langle z_2 \rangle (\sin \nu \tau - b_2) + b_0$$

$$\begin{aligned} \langle \{y(\tau) - \langle y(\tau) \rangle\}^2 \rangle &= \langle \{z_1 - \langle z_1 \rangle\}^2 \rangle (b_1 + \cos \nu \tau)^2 + \langle \{z_2 - \langle z_2 \rangle\}^2 \rangle (\sin \nu \tau - b_2)^2 \\ &\quad - 2 \langle \{z_1 - \langle z_1 \rangle\} \{z_2 - \langle z_2 \rangle\} \rangle (\sin \nu \tau - b_2)(\cos \nu \tau + b_1) \dots \dots \dots (17) \end{aligned}$$

3. Numerical examples

The variance and the mean obtained by solving Eq.(17) analytically are shown by chain-dash lines in Figs.2,3 and 4. The solid lines show the numerical solution by Runge-Kutta-Gill method of the simultaneous differential equations (20) derived from the following Fokker-Planck equation.

$$\frac{\partial P}{\partial \tau} = -\frac{\partial}{\partial y_1} \{y_2 P\} - \frac{\partial}{\partial y_2} \{[-k(\tau)y_1 - 2\delta y_2 + B \sin(\nu \tau + \phi) + 1]P\} + \frac{D}{2} \frac{\partial^2 P}{\partial y_1^2} \dots \dots \dots (18)$$

where P is a transition probability density function, $y_1 = y$ and $y_2 = dy_1/d\tau$. From Eq.(18), the following equation is derived:

$$\begin{aligned} d\langle y_1^n y_2^{n-1} \rangle / d\tau &= n_1 \langle y_1^{n-1} y_2^{n-1} \rangle + n_2 \langle y_1^n y_2^{n-2} \rangle (-k(\tau)y_1 - 2\delta y_2 \\ &\quad + B \sin(\nu \tau + \phi) + 1) + n_2(n_2 - 1) D \langle y_1^n y_2^{n-2} \rangle / 2 \dots \dots \dots (19) \end{aligned}$$

Finally, the following equations are obtained from Eq.(19).

$$\begin{aligned} dm_1/d\tau &= m_2, \quad dm_2/d\tau = -k(\tau)m_1 - 2\delta m_2 + B \sin(\nu \tau + \phi) + 1 \\ d\sigma_{11}/d\tau &= 2\sigma_{12}, \quad d\sigma_{12}/d\tau = \sigma_{22} - k(\tau)\sigma_{11} - 2\delta\sigma_{12}, \quad d\sigma_{22}/d\tau = -2k(\tau)\sigma_{12} - 4\delta\sigma_{22} + D \dots \dots \dots (20) \end{aligned}$$

where $k_1 = 0.6$, $\delta = 0.1$, $B = K = 1$, $\psi = 0$, $D = \kappa(\nu)$.

These numerical values except for the random value K have been commonly used in other investigations⁽⁵⁾⁽⁶⁾.

As statistical average of the response is proportional to the magnitude of K , the value K set equal to unity for simplicity. As shown in Figures 2 through 4, fluctuations with high frequency do not appear, particularly in the case of the variances, because the approximate solution depends only on the fundamental frequency. As the calculated result coincides, as a whole, with the solution directly integrated, however, the approximate solution is of use to know a rough approximation of variation. The first passage time probability density function is obtained by reference (11) and shown in Fig.5, since Eq.(2) is of the form of narrow band random process. The calculation condition

is as in Fig.2, and a single barrier $b' = (1/k(\tau) + 2\sqrt{1/(2\delta)})$. The ordinate and the abscissa represent the first passage time probability density function $f(n)$ and the time $n = \tau/T$, ($T = 2\pi/\nu$), respectively. Fig.5 (a) is the result for stationary start and (b) is for zero start.

4. Conclusions

An approximate solution of the forced vibration of a gear system with the transmission error having a period equal to the meshing period subjected to random excitation is given. Though the accuracy of the solution is not necessarily enough, it is of use for obtaining the averaged variation, especially in the application to the first passage time problem or the random fatigue problem.

Fig.2 Mean and variance of y , and spring constant

Fig.3 Mean and variance of y , and spring constant

Fig.4 Mean and variance of y , and spring constant

Fig.5 Example of the first passage time probability density function; (a) stationary start, (b) zero start

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