Theoretical Study of Influence of Plastic
Deformation upon Acoustoelastic Effect*
(An Acoustic Theory for Measuring Stress Dis-
tributions in Slightly Orthotropic Materials)

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In the present paper, the acoustical effects caused by elasto-plastic finite deformations in slightly orthotropic materials are theoretically investigated using the theories of finite deformations and plasticity in which the plastic strain increments are expressed with stress deviations and their increments.

From analytical results, it is found that the expressions for the elastic-plastic strain-velocity relations of acoustic waves propagating in slightly orthotropic materials can be derived skillfully with aid of Jaumann co-rotational rate and can be reduced to the theoretical results derived by Iwashimizu and Kubomura in the elastic state. Further in order to remove the previous discrepancies between the theoretical results and the experimental ones about the influence of plastic deformations upon the acoustoelastic effects, the author discusses the influence of the deviation of stress increments on the plastic strain increments.

Key Words : Acoustoelastic Effect, Predeformed Elastic-plastic Material, Slightly Orthotropic Material, Ultrasonic Velocity, Theory of Finite Deformation, Plasticity, Elasticity, Sound

1. Introduction

The velocities of ultrasonic waves in a medium are known to be dependent on stress and strain states in the deformed medium. Because the acoustic effects called acoustoelasticity can be used to measure the stress distributions in a body subject to externally applied loading and to determine residual stresses induced by inhomogeneous plastic deformation, the basic research of these effects has been made by many investigators since the early work by Rollins⁽¹⁾ and Li and Torikai⁽²⁾. In particular, Fukuoka and Toda⁽³⁾⁻⁽⁶⁾ are making a dynamical study of the practical measuring procedure to evaluate residual stress states in practical engineering materials and Fukuoka⁽⁷⁾ reviewed recent developments of acoustoelasticity.

Most practical engineering materials exhibit the inhomogeneous anisotropy caused by material texture or micro-structure which is introduced by the manufacturing process such as rolling process and heat treatment. Because such anisotropy and inhomogeneity influence the ultrasonic wave propagation, a few theories for wave propagation through slightly anisotropic stressed materials

were developed by Tokuoka and Saito⁽⁸⁾, Iwashimizu and Kubomura⁽⁹⁾ and Okada⁽¹⁰⁾. In order to separate the influences of stress and the original anisotropy upon the acoustic measurements, King and Fortunko⁽¹¹⁾ proposed a new measurement technique with horizontally polarized shear waves (SH-waves), and Fukuoka and Toda⁽¹²⁾ proposed R-value method. However, acoustic measurements to evaluate residual stresses which are caused by inhomogeneous plastic deformation in a material are not always reliable. Therefore, further theoretical development of acoustoelasticity is required for a plastic state. The influence of plastic deformation on the propagation of ultrasonic waves is considered in the newly proposed theories of Johnson⁽¹³⁾ and Kobayashi⁽¹⁴⁾, and Gamer and Pao⁽¹⁵⁾. Fukuoka⁽⁷⁾ pointed out that experimental results can not be explained quantitatively well by these theories yet.

The author⁽¹⁴⁾ has investigated the acoustical effects caused by elasto-plastic deformations of solids using both theories of finite deformation and plasticity; he has indicated that general expressions for the strain-velocity relations of waves propagating in elasto-plastically deformed solids can be derived skillfully with aid of Jaumann co-rotational rate. In the previous work⁽¹⁶⁾ on the numerical calculations for both cases of aluminum and nickel-steel to be characterized by

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the Ramberg-Osgood type stress-strain relation, the author has shown that the influence of stress increment terms constituting plastic strain increments on the wave velocities is considerably larger than that of stress terms constituting ones.

In this paper, we present the theoretical results of the acoustoelastic effects in the elasto-plastically deformed solids due to slightly orthotropic elastic moduli and anisotropy of initial strains and examine the coefficient of the stress increment terms to explain the experimental results theoretically.

2. Theory of Acoustoelasticity: Media of Orthotropic Symmetry and Elasto-plastic Deformation

2.1 Constitutive relations

Dependence of the velocity of elastic waves (ultrasonic waves) on the stresses and strains in a medium can be considered from a point of view of the nonlinear interactions between small disturbances (ultrasonic waves) and the predeformation of a medium. Therefore, we examine the velocity changes of the elastic waves using both theories of

finite deformations and plasticity in order to consider the effects of geometric and physical nonlinearity. If the medium is homogeneous and slightly orthotropic, the Green-St.Venant elastic strain tensor E_{ij}^e as a function of the second- and third-order elastic constants is given by

$$E_{ij}^e = -\frac{\lambda}{2G(3\lambda+2G)}\theta_1\delta_{ij} + \frac{1}{2G\eta}S_{ij} + (\kappa_1\theta_1^2 + \kappa_2\theta_2)\delta_{ij} + \kappa_2\theta_1 S_{ij} + \kappa_3 S_{ij}S_{jk}S_{ki} \quad (i, j=1, 2, 3) \quad (1)$$

where summation is understood to be carried out over repeated small letter subscripts and capital letter subscripts are free indexes. The variables $\kappa_1 \sim \kappa_3$ are determined as a function of the second- and third-order elastic constants as described later and symbol δ_{ij} denotes the Kronecker delta. In Eq.(1) the second Piola-Kirchhoff stress tensor is denoted by S_{ij} and its invariants $\theta_1 \sim \theta_3$ are expressed as follows:

$$\theta_1 = S_{kk}, \theta_2 = \frac{1}{2}S_{ij}S_{ij}, \theta_3 = \frac{1}{3}S_{ij}S_{jk}S_{ki} \quad (2)$$

For a slightly orthotropic case the tensor G_{IJK} , the symmetric second-order tensor, is related to Lamé's constants λ , G and elastic stiffness C_{ij} as follows:

$$G_{ij} = G + g_{ij}, |g_{ij}/G| \ll 1 \quad (3)$$

$$C_{ij} = \frac{1}{\Delta} \left\{ \frac{1}{2}\delta_{ij}\delta_{kk}\delta_{ll}\delta_{mm} \left(\frac{1}{2G_{kk}} - \frac{\lambda}{2G(3\lambda+2G)} \right) \left(\frac{1}{2G_{ll}} - \frac{\lambda}{2G(3\lambda+2G)} \right) - \delta_{ij} \left(\frac{\lambda}{2G(3\lambda+2G)} \right)^2 + \frac{\delta_{ij}\delta_{kk}\delta_{ll}\delta_{mm}}{4G_{kk} \cdot G(3\lambda+2G)} \lambda \right\} \quad (i, j=1, 2, 3) \quad (4)$$

$$C_{11} = G_{22}, C_{33} = G_{11}, C_{66} = G_{12}$$

where δ_{ij} is

$$\delta_{ij} = 1 \quad i=j, \quad \delta_{ij} = 0 \quad i \neq j$$

and Δ is the value of the following determinant:

$$\Delta = \begin{vmatrix} \frac{1}{2G_{11}} - \frac{\lambda}{2G(3\lambda+2G)} & -\frac{\lambda}{2G(3\lambda+2G)} & -\frac{\lambda}{2G(3\lambda+2G)} \\ -\frac{\lambda}{2G(3\lambda+2G)} & \frac{1}{2G_{22}} - \frac{\lambda}{2G(3\lambda+2G)} & -\frac{\lambda}{2G(3\lambda+2G)} \\ -\frac{\lambda}{2G(3\lambda+2G)} & -\frac{\lambda}{2G(3\lambda+2G)} & \frac{1}{2G_{33}} - \frac{\lambda}{2G(3\lambda+2G)} \end{vmatrix}$$

We now assume that the finite elastic-plastic deformation, characterized by the displacement u_i from the stress free state X_i , is in static equilibrium. Due to propagation of the ultrasonic wave superimposed on the predeformation, the current position $\tilde{x}_i$ of a material point x_i can be expressed by the small disturbance $d\tilde{u}_i$ as follows:

$$\tilde{x}_i = x_i + d\tilde{u}_i, \quad x_i = X_i + u_i \quad (i=1, 2, 3) \quad (5)$$

The perturbation of the elastic strain produced by propagation of the ultrasonic wave can be obtained from Eq.(1) as follows:

$$dE_{ij}^e = \left\{ \frac{1}{2G_{ij}} + \left(\kappa_2 + \frac{2}{3}\kappa_3 \right) S_{kk} \right\} dS_{ij} + \left(\kappa_2 + \frac{2}{3}\kappa_3 \right) S_{ij} dS_{kk} + \kappa_3 d\theta_1 \delta_{ij} + \frac{1}{3}\delta_{ij} \left[\left\{ \frac{1}{2G_{ij}} - \frac{3\lambda}{2G(3\lambda+2G)} \right\} (6\kappa_1 + 3\kappa_2 + \frac{2}{3}\kappa_3) S_{kk} \right] dS_{rr} + (3\kappa_2 + 2\kappa_3) S_{rr} dS_{ij} \quad (6)$$

where S_{ij}' , dS_{ij}' and $d\theta_1'$ are a stress deviation tensor, the first-order and the second-order increment of the stress deviation tensor which are defined by the following expressions respectively,

$$S_{ij}' = S_{ij} - 1/3\delta_{ij}S_{kk}, \quad dS_{ij}' = dS_{ij} - 1/3\delta_{ij}dS_{kk}, \quad d\theta_1' = S_{kk}'dS_{kk} + S_{ij}'dS_{ij} - 2/3\delta_{ij}S_{ij}'dS_{kk} \quad (7)$$

In the next place the definition of the increments of the plastic strain dE_{ij}^p caused to change the velocities of the ultrasonic waves in the plastically deformed medium is given as follows:

$$dE_{ij}^p = \tilde{\lambda}_{ij}dS_{ij} - \frac{1}{3}\delta_{ij}\tilde{\lambda}_{kk}dS_{kk} + S_{ij}'d\tilde{\lambda} \quad (8)$$

In Eq.(8) $\tilde{\lambda}_{IJK}$ is a symmetric second-order tensor and dE_{ij}^p satisfies the condition of incompressibility. $\tilde{\lambda}_{IJK}$ in Eq.(8) equals zero for Prandtl-Reuss materials but this case can not explain the experimental results quantitatively. Therefore, we consider that a variable $\tilde{\lambda}_{IJK}$ can be defined as a function of the plastic strains and we will describe the details later. In Eq.(8) $d\tilde{\lambda}$ can be expressed as follows:

$$d\tilde{\lambda} = \tilde{\lambda}_{ij}dS_{ij}, \quad \tilde{\lambda} = \tilde{\lambda}(J_2, J_3), \quad \tilde{\lambda}_{ij} = \partial\tilde{\lambda}/\partial J_2' S_{ij}' + \partial\tilde{\lambda}/\partial J_3' t_{ij}' \quad (9)$$

where J_2' and J_3' are the second and the third invariants of deviatoric stresses and t_{ij}' is a second-order deviatoric stress given by

$$J_2' = 1/2S_{ij}'S_{ij}', \quad J_3' = 1/3S_{ij}'S_{jk}'S_{ki}', \quad t_{ij}' = S_{ij}'S_{kk}' - 2/3\delta_{ij}J_2' \quad (10)$$

respectively. On account of propagation of the ultrasonic waves the total strain increment $d\tilde{E}_{ij}$ and the spin $d\tilde{\Omega}_{ij}$ arise and they are expressed by the disturbance of the waves $d\tilde{u}_i$ as follows:

$$d\tilde{E}_{ij} = \frac{1}{2} \left(\frac{\partial d\tilde{u}_i}{\partial x_j} + \frac{\partial d\tilde{u}_j}{\partial x_i} \right), \quad d\tilde{\Omega}_{ij} = \frac{1}{2} \left(\frac{\partial d\tilde{u}_i}{\partial x_j} - \frac{\partial d\tilde{u}_j}{\partial x_i} \right). \quad (11)$$

When the elastic deformation is small, by using Jaumann co-rotational rate⁽¹⁷⁾ the total strain increment $d\tilde{E}_{ij}$ can be defined as follows⁽¹⁸⁾:

$$\begin{aligned} \text{where } E_{ij}^e \text{ is elastic strain. Hence, substitution of Eqs. (6) and (8) into Eq. (12) leads to} \\ d\tilde{E}_{ij} + E_{ik}^e d\tilde{\Omega}_{ki} + E_{ik}^e d\tilde{\Omega}_{kj} = \left(\frac{1}{2G_{ij}} + \left(\kappa_2 + \frac{2}{3}\kappa_3 \right) S_{kk} + \lambda_{ij} \right) dS_{ij} + \left(\kappa_2 + \frac{2}{3}\kappa_3 \right) S_{ij} dS_{kk} \\ + \kappa_3 d\lambda_{ij} + S_{ij} \lambda_{rs} dS_{rs} + \frac{1}{3} \delta_{ij} \left[\left(\frac{1}{2G_{ij}} - \frac{3\lambda}{2G(3\lambda+2G)} + \left(6\kappa_1 + \frac{2}{3}\kappa_3 \right) S_{kk} \right) dS_{rr} \right. \\ \left. - \lambda_{rk} dS_{kk} + (3\kappa_2 + 2\kappa_3) S_{rs} dS_{rs} \right]. \end{aligned} \quad (13)$$

From Eq. (13) the isotropic and deviatoric components of $d\tilde{E}_{ij}$ can be written respectively as

$$\begin{aligned} d\tilde{E}_{kk} + 2E_{kk}^e d\tilde{\Omega}_{kk} = \frac{1}{2G_{kk}} dS_{kk} + \left(\frac{1}{6G_{kk}} - \frac{3\lambda}{2G(3\lambda+2G)} + \left(6\kappa_1 + 3\kappa_2 + \frac{2}{3}\kappa_3 \right) S_{kk} \right) dS_{rr} \\ + (3\kappa_2 + 2\kappa_3) S'_{rs} dS'_{rs} \end{aligned} \quad (14)$$

$$\begin{aligned} d\tilde{E}_{ij} + E_{ik}^e d\tilde{\Omega}_{ki} + E_{ik}^e d\tilde{\Omega}_{kj} - \frac{2}{3} \delta_{ij} E_{kk}^e d\tilde{\Omega}_{kk} = \frac{1}{2G} dS_{ij} + \left(\frac{1}{2G_{ij}} - \frac{1}{2G} + \lambda_{ij} + \left(\kappa_2 + \frac{2}{3}\kappa_3 \right) S_{kk} \right) dS_{ij} \\ - \frac{1}{3} \delta_{ij} \left(\lambda_{kk} + \frac{1}{2G_{kk}} \right) dS_{kk} + \left(\frac{1}{6} \delta_{ij} \left(\frac{1}{G_{ij}} - \frac{1}{3G_{kk}} \right) + \left(\kappa_2 + \frac{2}{3}\kappa_3 \right) S_{ij} \right) dS_{kk} + \kappa_3 d\lambda_{ij} + S_{ij} \lambda_{rs} dS'_{rs} \end{aligned} \quad (15)$$

where $d\tilde{E}_{ij}$ is a deviatoric strain increment defined as follows:

$$d\tilde{E}_{ij} = d\tilde{E}_{ij} - \frac{1}{3} \delta_{ij} d\tilde{E}_{kk}$$

To evaluate the deviatoric stress increment, we put the expression of dS'_{ij} as

$$dS'_{ij} = 2G(d\tilde{E}_{ij} + E_{ik}^e d\tilde{\Omega}_{ki} + E_{ik}^e d\tilde{\Omega}_{kj} - \frac{2}{3} \delta_{ij} E_{kk}^e d\tilde{\Omega}_{kk}) + \Delta_0 S_{ij} + 2G\lambda_{ij} \quad (16)$$

where the magnitudes of Δ_0 , Δ_{ij} and $1/G_{ij} - 1/G$ are $O(dE)$, $O(EdE)$ and $O(E)$ in this order. Substituting Eq. (16) into Eq. (15) and neglecting the terms higher than $E^2 dE$ order, expressions of Δ_0 and Δ_{ij} are obtained as follows:

$$\begin{aligned} \Delta_0 = -2G \cdot \tilde{G} \left(\kappa_2 + \frac{2}{3}\kappa_3 \right) \left\{ d\tilde{E}_{kk} - \frac{G}{G_{kk}} \left(2 - \frac{G}{G_{kk}} \right) d\tilde{E}_{kk} \right\} - 2G\lambda_{rs} d\tilde{E}_{rs} \\ \Delta_{ij} = \left\{ 1 - \frac{G}{G_{ij}} - 2G\lambda_{ij} - 2G \left(\kappa_2 + \frac{2}{3}\kappa_3 \right) S_{kk} \right\} d\tilde{E}_{ij} + \frac{1}{3} \delta_{ij} \left(\frac{G}{G_{kk}} + 2G\lambda_{kk} \right) d\tilde{E}_{kk} \\ + \frac{G}{6} \delta_{ij} \left(\frac{1}{3G_{kk}} - \frac{1}{G_{ij}} \right) \left\{ d\tilde{E}_{kk} - \frac{G}{G_{kk}} \left(2 - \frac{G}{G_{kk}} \right) d\tilde{E}_{kk} \right\} - 2G\kappa_3 \left(S'_{ik} d\tilde{E}_{kj} + S_{kj} d\tilde{E}_{ik} - \frac{2}{3} \delta_{ij} S'_{rs} d\tilde{E}_{rs} \right) \\ \frac{1}{G} = \frac{1}{6G_{kk}} - \frac{3\lambda}{2G(3\lambda+2G)}. \end{aligned} \quad (17)$$

Hence, substitution of Eqs. (17) into Eq. (16) leads to

$$\begin{aligned} dS'_{ij} = 2G \left\{ 2 - \frac{G}{G_{ij}} - 2G\lambda_{ij} - 2G \left(\kappa_2 + \frac{2}{3}\kappa_3 \right) S_{kk} \right\} d\tilde{E}_{ij} + 2G(E_{ik}^e d\tilde{\Omega}_{ki} + E_{ik}^e d\tilde{\Omega}_{kj}) \\ - 2G \cdot \tilde{G} \left(\kappa_2 + \frac{2}{3}\kappa_3 \right) S'_{ij} \left\{ d\tilde{E}_{kk} - \frac{G}{G_{kk}} \left(2 - \frac{G}{G_{kk}} \right) d\tilde{E}_{kk} \right\} - 4G^2 S'_{ij} \lambda_{rs} d\tilde{E}_{rs} \\ - 4G^2 \kappa_3 (S'_{ik} d\tilde{E}_{kj} + S_{kj} d\tilde{E}_{ik}) + \frac{1}{3} \delta_{ij} G \left[\tilde{G} \left(\frac{1}{3G_{kk}} - \frac{1}{G_{ij}} \right) d\tilde{E}_{kk} + G \left\{ \frac{2}{G_{kk}} - \frac{G}{G_{kk}} \left(\frac{1}{3G_{kk}} - \frac{1}{G_{ij}} \right) \right\} \left(2 - \frac{G}{G_{kk}} \right) \right. \\ \left. + 4\lambda_{kk} \right\} d\tilde{E}_{kk} - 4E_{rs}^e d\tilde{\Omega}_{rs} + 8G\kappa_3 S'_{rs} d\tilde{E}_{rs} \end{aligned} \quad (18)$$

Similarly, from Eqs. (14) and (18) dS_{kk} can be obtained as

$$\begin{aligned} dS_{kk} = \tilde{G} \left\{ 1 - \tilde{G} \left(6\kappa_1 + 3\kappa_2 + \frac{2}{3}\kappa_3 \right) S_{kk} \right\} d\tilde{E}_{rr} + 2E_{rs}^e d\tilde{\Omega}_{rs} - \left\{ \frac{G}{G_{kk}} - \tilde{G} \left(6\kappa_1 + 3\kappa_2 + \frac{2}{3}\kappa_3 \right) \right. \\ \left. \times \frac{G}{G_{kk}} \left(2 - \frac{G}{G_{kk}} \right) S_{rr} \right\} d\tilde{E}_{kk} - 2G(3\kappa_2 + 2\kappa_3) S'_{rs} d\tilde{E}_{rs} \end{aligned} \quad (19)$$

From Eqs. (18) and (19) the increment of the second Piola-Kirchhoff stress tensor dS_{ij} is given as

$$\begin{aligned} dS_{ij} = 2G \left\{ \left(2 - \frac{G}{G_{ij}} - 2G\lambda_{ij} - 2G \left(\kappa_2 + \frac{2}{3}\kappa_3 \right) S_{kk} \right) d\tilde{E}_{ij} + E_{ik}^e d\tilde{\Omega}_{ki} + E_{ik}^e d\tilde{\Omega}_{kj} - 2GS'_{ij} \lambda_{rs} d\tilde{E}_{rs} \right. \\ \left. - \tilde{G} \left(\kappa_2 + \frac{2}{3}\kappa_3 \right) S'_{ij} \left\{ d\tilde{E}_{kk} - \frac{G}{G_{kk}} \left(2 - \frac{G}{G_{kk}} \right) d\tilde{E}_{kk} \right\} - 2G\kappa_3 (S'_{ik} d\tilde{E}_{kj} + S_{kj} d\tilde{E}_{ik}) \right\} \\ + \frac{1}{3} \delta_{ij} G \left\{ \left(1 + G \left(\frac{1}{3G_{kk}} - \frac{1}{G_{ij}} \right) - \tilde{G} \left(6\kappa_1 + 3\kappa_2 + \frac{2}{3}\kappa_3 \right) S_{kk} \right) d\tilde{E}_{rr} + \left(2 - 4\frac{G}{G} \right) E_{rs}^e d\tilde{\Omega}_{rs} \right. \\ \left. + G \left\{ \frac{2G}{G \cdot G_{kk}} - \frac{1}{G_{kk}} - \frac{G}{G_{kk}} \left(\frac{1}{3G_{kk}} - \frac{1}{G_{ij}} \right) \right\} \left(2 - \frac{G}{G_{kk}} \right) + \left(6\kappa_1 + 3\kappa_2 + \frac{2}{3}\kappa_3 \right) \right. \\ \left. \times \frac{G}{G_{kk}} \left(2 - \frac{G}{G_{kk}} \right) S_{rr} + 4\frac{G}{G} \lambda_{kk} \right\} d\tilde{E}_{kk} - 2G \left(3\kappa_2 + 2\kappa_3 - 4\frac{G}{G} \kappa_3 \right) S'_{rs} d\tilde{E}_{rs} \end{aligned} \quad (20)$$

2.2 Equation of motion

In the next place a stress disturbance $d\tilde{\sigma}_{ij}$, which arises due to the wave propagation in the current state $\tilde{x}_i$, will be derived from the increment of the second Piola-Kirchhoff stress tensor dS_{ij} expressed by Eq. (20). Here, the Euler-Cauchy stress tensor is denoted by σ_{ij} and is related to the second Piola-Kirchhoff stress tensor S_{ij} as follows:

$$\begin{aligned} \sigma_{ij} = \frac{\rho}{\rho_0} \left\{ S_{ij} + \left(\delta_{ir} \frac{\partial u_j}{\partial x_r} + \delta_{js} \frac{\partial u_i}{\partial x_s} - \frac{\partial u_i}{\partial x_s} \frac{\partial u_j}{\partial x_r} \right) S_{rs} \right\} \\ \frac{\rho_0}{\rho} = \det \left| \frac{\partial x_i}{\partial X_j} \right| \end{aligned} \quad (21)$$

where ρ_0 and ρ are densities of the stress free and current states respectively. From the above equation and by neglecting small terms of higher order, the increment of the

Euler-Cauchy stress tensor $d\sigma_{ij}$ becomes

$$d\sigma_{ij} = dS_{ij} - E_{ijk} dS_{ij} - S_{ij} d\tilde{E}_{ijk} + \left(\delta_{ir} \frac{\partial d\tilde{u}_j}{\partial x_r} + \delta_{js} \frac{\partial d\tilde{u}_i}{\partial x_s} \right) S_{rs} + \left(\delta_{ir} \frac{\partial u_j}{\partial x_s} + \delta_{js} \frac{\partial u_i}{\partial x_r} \right) dS_{rs} \quad (22)$$

Hence Jaumann co-rotational stress rate σ'_{ij} is

$$\sigma'_{ij} = D\sigma_{ij}/Dt - \sigma_{ik}\Omega_{kj} - \sigma_{jk}\Omega_{ki} \quad (23)$$

where D/Dt and Ω_{ij} are the material derivative and the spin tensor respectively defined as follows:

$$\frac{D}{Dt} \sigma_{ij} = \frac{d\sigma_{ij}}{dt} + \frac{\partial \sigma_{ij}}{\partial x_k} \frac{\partial x_k}{\partial t} \quad (24)$$

$$\Omega_{ij} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} \frac{d\tilde{u}_j}{dt} - \frac{\partial}{\partial x_j} \frac{d\tilde{u}_i}{dt} \right) \quad (25)$$

Therefore, the stress disturbance $d\tilde{\sigma}_{ij}$ arising by the wave propagation is obtained as

$$d\tilde{\sigma}_{ij} = \sigma'_{ij} dt = dS_{ij} - E_{ijk} dS_{ij} - S_{ij} d\tilde{E}_{ijk} + \left(\delta_{ir} \frac{\partial d\tilde{u}_j}{\partial x_r} + \delta_{js} \frac{\partial d\tilde{u}_i}{\partial x_s} \right) S_{rs} + \left(\delta_{ir} \frac{\partial u_j}{\partial x_s} + \delta_{js} \frac{\partial u_i}{\partial x_r} \right) dS_{rs} + \delta_{ij} \frac{\partial u_k}{\partial x_k} dS_{kk} + \frac{1}{2} \sigma_{ik} \left(\frac{\partial d\tilde{u}_k}{\partial x_j} - \frac{\partial d\tilde{u}_j}{\partial x_k} \right) + \frac{1}{2} \sigma_{jk} \left(\frac{\partial d\tilde{u}_k}{\partial x_i} - \frac{\partial d\tilde{u}_i}{\partial x_k} \right) \quad (26)$$

and equations of motion for the wave propagation can be expressed by the form of a increment as follows:

$$\frac{\partial d\tilde{\sigma}_{ij}}{\partial x_j} = \rho \frac{\partial^2 d\tilde{u}_i}{\partial t^2} \quad (i=1,2,3) \quad (27)$$

When the distributions of the displacements and the stresses vary negligibly in the region of a distance of one wave-length, by substituting Eq.(26) into Eq.(27) and linearizing them with respect to $d\tilde{u}_i$, we can obtain the following equations:

$$\begin{aligned} \frac{\partial dS_{ij}}{\partial x_j} - E_{ijk} \frac{\partial dS_{ij}}{\partial x_j} - S_{ij} \frac{\partial d\tilde{E}_{ijk}}{\partial x_j} + S_{rs} \left(\delta_{ir} \frac{\partial^2 d\tilde{u}_j}{\partial x_s \partial x_j} + \delta_{js} \frac{\partial^2 d\tilde{u}_i}{\partial x_r \partial x_j} \right) + \left(\delta_{ir} \frac{\partial u_j}{\partial x_s} + \delta_{js} \frac{\partial u_i}{\partial x_r} \right) \frac{\partial dS_{rs}}{\partial x_j} \\ + \delta_{ij} \frac{\partial u_k}{\partial x_k} \frac{\partial dS_{kk}}{\partial x_j} + \frac{1}{2} S_{ik} \left(\frac{\partial^2 d\tilde{u}_k}{\partial x_j^2} - \frac{\partial^2 d\tilde{u}_j}{\partial x_k \partial x_j} \right) + \frac{1}{2} S_{jk} \left(\frac{\partial^2 d\tilde{u}_k}{\partial x_i \partial x_j} - \frac{\partial^2 d\tilde{u}_i}{\partial x_k \partial x_j} \right) = \rho \frac{\partial^2 d\tilde{u}_i}{\partial t^2} \end{aligned} \quad (i=1,2,3) \quad (28)$$

where dS_{ij} is defined by Eq.(20) and $d\tilde{u}_i$ can be defined by both the propagation direction and the amplitude of the wave. Here σ_{ij} is assumed to be close to S_{ij} and the terms higher than second order of u_i are omitted. Eq.(28) is a basic equation governing the propagation of ultrasonic waves in a plastically deformed state characterized by the displacement u_i and the stress S_{ij} .

2.3 Discussion of $\tilde{\lambda}_{ij}$

In the previous paper⁽¹⁶⁾, it was proved that the influence of stress increment terms constituting plastic strain increments on the wave velocities is considerably larger than that of stress terms constituting ones owing to the numerical calculations for both cases of flow theory and the rate type form of deformation theory using Ramberg-Osgood type stress-strain relation. It is suggested that the stress increment terms constituting plastic strain increments represent the strain hardening behavior of materials and have some influence on the acousto-elastic effect even if those terms make a negligible amount of contribution to the plastic strain. The above characteristic is analogous to the hardening mechanism⁽¹⁹⁾ of an irradiation damaged material: the point defects such as vacancies and interstitial atoms increased by the irradiation damage are capable of suppressing the moving dislocations. It is well known that plastic deformation increases vacancies and the density of the vacancies C_v can be related with the plastic strain ϵ^p in the following expressions⁽²⁰⁾:

$$C_v = a(\epsilon^p)^n, \quad \epsilon^p < 0.1$$

$$a \approx 10^{-4}, \quad n \approx 1;$$

$$\epsilon^p > 0.1$$

$$a \approx 10^{-2}, \quad n \approx 2.$$

.....(29)

Considering the hardening introduced by an increase of the vacancies, we now assume that the equivalent stress increment $d\bar{\sigma}$ can be put in the form

$$d\bar{\sigma} = H' d\bar{\epsilon}^p + K dC_v, \quad H' = d\bar{\sigma}/d\bar{\epsilon}^p \quad (30)$$

where $d\bar{\epsilon}^p$ is an equivalent plastic strain increment and K is a coefficient conceived as a function of the equivalent plastic strain $\bar{\epsilon}^p$. In the above form, H' and $d\bar{\epsilon}^p$ are a hardening coefficient and an equivalent plastic strain increment respectively in the case of no effects of the vacancies. Because the increment of the vacancies makes a negligible amount of contribution to the plastic strain increment, i.e. $d\bar{\epsilon}^p \approx d\bar{\epsilon}_0^p$, we get the following restriction:

$$\left| \frac{K}{H'} \cdot \frac{dC_v}{d\bar{\epsilon}^p} \right| \ll 1 \quad (31)$$

From Eq.(30), $d\bar{\epsilon}^p$ can be written as

$$\begin{aligned} d\bar{\epsilon}^p &= \frac{1}{H'} d\bar{\sigma} - \frac{K}{(H')^2} \frac{dC_v}{d\bar{\epsilon}^p} d\bar{\sigma} \\ &= d\bar{\epsilon}_0^p - \frac{K}{(H')^2} \frac{dC_v}{d\bar{\epsilon}^p} d\bar{\sigma} \end{aligned} \quad (32)$$

Here, $d\bar{\sigma}$ and $d\bar{\epsilon}^p$ are expressed by an expansion in the deviatoric stress tensor S'_{ij} and the plastic strain tensor E^p_{ij} respectively as follows:

$$d\bar{\sigma} = \frac{\partial \bar{\sigma}}{\partial S'_{ij}} dS'_{ij}, \quad d\bar{\epsilon}^p = \frac{\partial \bar{\epsilon}^p}{\partial E^p_{ij}} dE^p_{ij} \quad (33)$$

where dS'_{ij} is an increment of S'_{ij} and dE^p_{ij} is that of E^p_{ij} . If dE^p_{ij} is $(dE^p_{ij})_0$ and dS'_{ij} are dS'_{ij} one to one correspondent, substitution of Eqs.(33) into Eq.(32) leads to

$$dE^p_{ij} = (dE^p_{ij})_0 - \frac{K}{(H')^2} \frac{dC_v}{d\bar{\epsilon}^p} \frac{\partial \bar{\sigma}}{\partial E^p_{ij}} \frac{\partial S'_{ij}}{\partial E^p_{ij}} dS'_{ij} \quad (34)$$

where $(dE^p_{ij})_0$ is dE^p_{ij} with no effects of vacancies. Modifying Eq.(34) to satisfy incompressibility of dE^p_{ij} yields the expression of dE^p_{ij} as follows

$$dE_{ij} = (dE_{ij})_e - \frac{K}{(H')^2} \frac{dC_p}{d\epsilon^p} \left(\frac{\partial \bar{\epsilon}}{\partial \epsilon^p} \frac{\partial S_{ij}}{\partial E_{ij}} dS_{ij} - \frac{1}{3} \frac{\partial \bar{\epsilon}}{\partial \epsilon^p} \frac{\partial S_{kk}}{\partial E_{kk}} dS_{kk} \right) \dots (35)$$

Therefore, by comparison of Eq.(35) with Eq.(8), the following relations can be derived

$$\left. \begin{aligned} S_{ij} d\lambda &= (dE_{ij})_e \\ \lambda_{ij} &= -\frac{K}{(H')^2} \frac{dC_p}{d\epsilon^p} \frac{\partial \bar{\epsilon}}{\partial \epsilon^p} \frac{\partial S_{ij}}{\partial E_{ij}} \end{aligned} \right\} \dots (36)$$

The possible forms of a concrete example of λ_{IJ} are restricted by the thermodynamics of irreversible deformation processes as pointed by Gotoh⁽²⁾. Hence we leave a detailed discussion about those forms for another opportunity.

3. Plane Waves Propagating in a Principal Direction

3.1 Velocities in an elasto-plastic state

Consider the case in which the coordinate axes x_i are aligned with the principal directions of stress and a plane wave propagates along a principal direction x_3 . In this case the corresponding deformation gradients $\partial u_i / \partial X_i$, elastic strains E_{ij} and stresses S_{ij} have the following forms:

$$\left. \begin{aligned} \frac{\partial u_1}{\partial X_1} = \frac{\partial u_2}{\partial X_2} = \frac{\partial u_3}{\partial X_3} = 0 \\ E_{11} = E_{22} = 0, S_{11} = S_{22} = 0 \end{aligned} \right\} \dots (37)$$

and the components of the small disturbance $\tilde{u}_i$ caused by the wave propagation would then be

$$\tilde{u}_i = \tilde{U}_i \exp(i(\gamma x_3 - \omega t)) \quad i=1, \dots, (38)$$

where γ is the wave number, ω the angular frequency, and $\tilde{U}_i$ denotes an unspecified amplitude j vector component. The wave velocity V is given by $V = \omega / \gamma$. Substituting Eqs.(20), (37) and (38) into Eqs.(28) and rearranging them, we find the equations to be solved for the wave velocities as follows

$$\left\{ \rho_0 V^2 - G \left(2 - \frac{G}{G_{11}} \right) + 2G^2 \lambda_{11} - 2G \left(2 - \frac{G}{G_{11}} \right) (E_{11} + E_{33}) + G(E_{11} - E_{33}) - \frac{1}{2} (S_{11} + S_{33}) + 2G^2 (\lambda_{22} + \lambda_{33}) S_{33} - 2G^2 \lambda_{33} S_{11} \right\} \tilde{U}_1 - \left\{ 2G \left(2 - \frac{G}{G_{11}} \right) E_{12} - GE_{12} + \frac{1}{2} S_{12} - 2G^2 \lambda_{33} S_{12} \right\} \tilde{U}_2 = 0, \dots (39)$$

$$\left\{ 2G \left(2 - \frac{G}{G_{11}} \right) E_{12} - GE_{12} + \frac{1}{2} S_{12} - 2G^2 \lambda_{33} S_{12} \right\} \tilde{U}_1 - \left\{ \rho_0 V^2 - G \left(2 - \frac{G}{G_{11}} \right) + 2G^2 \lambda_{11} - 2G \left(2 - \frac{G}{G_{11}} \right) (E_{11} + E_{33}) + G(E_{11} - E_{33}) - \frac{1}{2} (S_{11} + S_{33}) + 2G^2 (\lambda_{22} + \lambda_{33}) S_{33} - 2G^2 \lambda_{33} S_{11} \right\} \tilde{U}_2 = 0, \dots (40)$$

$$\left\{ \rho_0 V^2 - \frac{1}{3} G - \frac{4}{3} G \left(2 - \frac{G}{G_{11}} \right) - \frac{1}{9} G \cdot G \left(\frac{1}{3G_{11}} - \frac{1}{G_{11}} \right) (3 + \Delta g_1) + \frac{1}{9} G (2G - \bar{G}) \Delta g_2 + \frac{8}{3} G^2 \lambda_{11} + \frac{4}{9} G^2 (\lambda_{11} + \lambda_{22} - 2\lambda_{33}) - \left\{ 1 - \frac{2}{3} G \cdot G \left(\lambda_{22} + \frac{2}{3} \lambda_{33} \right) (6 + \Delta g_1) - \frac{8}{3} G^2 \lambda_{33} - 4G^2 \lambda_{13} \right\} S_{33} + \frac{1}{3} \left\{ G^2 (2\lambda_{11} + \lambda_{22} + \frac{2}{3} \lambda_{33}) (3 + \Delta g_1) + \frac{2}{3} G \cdot G \left(\lambda_{22} + \frac{2}{3} \lambda_{33} \right) \left(12 \frac{G}{G} - 6 - \Delta g_1 \right) - \frac{8}{3} G^2 \lambda_{33} - 4G^2 \lambda_{13} \right\} S_{33} - 4 \left\{ \frac{1}{3} G + \frac{4}{3} G \left(2 - \frac{G}{G_{11}} \right) + \frac{1}{9} G \cdot G \left(\frac{1}{3G_{11}} - \frac{1}{G_{11}} \right) (3 + \Delta g_1) - \frac{1}{9} G (2G - \bar{G}) \Delta g_2 \right\} E_{33} \right\} \tilde{U}_3 = 0, \dots (41)$$

where Δg_1 and Δg_2 are expressed as

$$\Delta g_1 = \frac{G}{G_{11}} \left(2 - \frac{G}{G_{11}} \right) + \frac{G}{G_{22}} \left(2 - \frac{G}{G_{22}} \right) - 2 \frac{G}{G_{33}} \left(2 - \frac{G}{G_{33}} \right),$$

$$\Delta g_2 = 1/G_{11} + 1/G_{22} - 2/G_{33}.$$

A set of Eqs.(39)-(41) is a third order characteristic equation for V^2 and $\tilde{U}_i$ is the i -th component of the eigenvector associated with the eigenvalue V^2 . Here we express the components of the stresses S_{ij} , the elastic strains E_{ij}^e and the strains E_{ij} by the principal stresses S_i , the principal plastic strains E_i^p and the principal initial strains E_i^i as follows:

$$\left. \begin{aligned} S_{11} &= \frac{1}{2} (S_1 + S_2) + \frac{1}{2} (S_1 - S_2) \cos 2\theta, S_{22} = \frac{1}{2} (S_1 + S_2) - \frac{1}{2} (S_1 - S_2) \cos 2\theta \\ S_{12} &= \frac{1}{2} (S_1 - S_2) \sin 2\theta, S_{33} = S_3 \end{aligned} \right\} \dots (42)$$

$$\left. \begin{aligned} E_{11}^e &= \left\{ \frac{1}{4G_{11}} - \frac{\lambda}{2G(3\lambda+2G)} \right\} (S_1 + S_2) - \frac{\lambda}{2G(3\lambda+2G)} S_3 + \frac{1}{4G_{11}} (S_1 - S_2) \cos 2\theta \\ E_{12}^e &= \left\{ \frac{1}{4G_{22}} - \frac{\lambda}{2G(3\lambda+2G)} \right\} (S_1 + S_2) - \frac{\lambda}{2G(3\lambda+2G)} S_3 - \frac{1}{4G_{22}} (S_1 - S_2) \cos 2\theta \\ E_{12}^p &= \frac{1}{4G_{12}} (S_1 - S_2) \sin 2\theta \\ E_{33}^e &= \frac{1}{2G_{33}} S_3 - \frac{\lambda}{2G(3\lambda+2G)} (S_1 + S_2 + S_3) \end{aligned} \right\} \dots (43)$$

$$\left. \begin{aligned} E_{11} &= \left\{ \frac{1}{4G_{11}} - \frac{\lambda}{2G(3\lambda+2G)} \right\} (S_1 + S_2) - \frac{\lambda}{2G(3\lambda+2G)} S_3 + \frac{1}{4G_{11}} (S_1 - S_2) \cos 2\theta \\ &\quad + \frac{1}{2} (E_1^p + E_2^p + E_3^p + E_4^p) + \frac{1}{2} (E_1^p - E_2^p) \cos 2\theta + \frac{1}{2} (E_1^p - E_4^p) \cos 2\varphi \\ E_{12} &= \left\{ \frac{1}{4G_{22}} - \frac{\lambda}{2G(3\lambda+2G)} \right\} (S_1 + S_2) - \frac{\lambda}{2G(3\lambda+2G)} S_3 - \frac{1}{4G_{22}} (S_1 - S_2) \cos 2\theta \\ &\quad + \frac{1}{2} (E_1^p + E_2^p + E_3^p + E_4^p) - \frac{1}{2} (E_1^p - E_2^p) \cos 2\theta - \frac{1}{2} (E_1^p - E_4^p) \cos 2\varphi \\ E_{12}^p &= \frac{1}{4G_{12}} (S_1 - S_2) \sin 2\theta + \frac{1}{2} (E_1^p - E_2^p) \sin 2\theta + \frac{1}{2} (E_1^p - E_4^p) \sin 2\varphi \end{aligned} \right\} \dots (44)$$

$$E_{33} = \frac{1}{2G_{33}} S_3 - \frac{\lambda}{2G(3\lambda+2G)} (S_1 + S_2 + S_3) + E_1^p + E_2^p$$

where θ , θ_e and ψ are the angles between the x_1 -axis and the directions of the principal stress, the principal plastic strain and the principal initial strain respectively. By substituting Eqs.(42)~(44) into Eqs.(39)~(41), we can immediately obtain the following expressions for the longitudinal wave velocity V_L and the transverse wave velocity V_T :

$$\begin{aligned} \rho_0 V_L^2 = & \frac{1}{3} \bar{G} + \frac{4}{3} \bar{G} \left(2 - \frac{G}{G_{33}} \right) + \frac{1}{9} G \cdot \bar{G} \left(\frac{1}{3G_{33}} - \frac{1}{G_{33}} \right) (3 + \Delta g_1) - \frac{1}{9} G(2G - \bar{G}) \Delta g_2 - \frac{8}{3} G^2 \bar{\lambda}_{33} - \frac{4}{9} G^2 (\bar{\lambda}_{11} \\ & + \bar{\lambda}_{22} - 2\bar{\lambda}_{33}) + \left[1 + \frac{2}{9G_{33}} \left\{ 3\bar{G} + 12G \left(2 - \frac{G}{G_{33}} \right) + G \cdot \bar{G} \left(\frac{1}{3G_{33}} - \frac{1}{G_{33}} \right) (3 + \Delta g_1) - G(2G - \bar{G}) \Delta g_2 \right\} \right. \\ & \left. - \frac{2}{3} G \cdot \bar{G} \left(\kappa_2 + \frac{2}{3} \kappa_3 \right) (6 + \Delta g_1) - \frac{8}{3} G^2 \kappa_3 - 4G^2 \bar{\lambda}_{33} \right] S_3 - \frac{1}{3} \left[\frac{2\lambda}{3G(3\lambda+2G)} \right. \\ & \times \left\{ 3\bar{G} + 12G \left(2 - \frac{G}{G_{33}} \right) + G \cdot \bar{G} \left(\frac{1}{3G_{33}} - \frac{1}{G_{33}} \right) (3 + \Delta g_1) - G(2G - \bar{G}) \Delta g_2 \right\} + G^2 \left(2\kappa_1 + \kappa_2 + \frac{2}{9} \kappa_3 \right) \\ & \times (3 + \Delta g_1) + \frac{2}{3} G \cdot \bar{G} \left(\kappa_2 + \frac{2}{3} \kappa_3 \right) \left(12 \frac{G}{G} - 6 - \Delta g_1 \right) - \frac{8}{3} G^2 \kappa_3 - 4G^2 \bar{\lambda}_{33} \right] (S_1 + S_2 + S_3) \\ & \left. + \frac{4}{9} \left\{ 3\bar{G} + 12G \left(2 - \frac{G}{G_{33}} \right) + G \cdot \bar{G} \left(\frac{1}{3G_{33}} - \frac{1}{G_{33}} \right) (3 + \Delta g_1) - G(2G - \bar{G}) \Delta g_2 \right\} (E_1^p + E_2^p) \right\}, \end{aligned}$$

$$\begin{aligned} (\rho_0 V_T^2)^2 = & (\bar{\Gamma}_1 + \bar{\Gamma}_2 + (\Gamma_{11} - \Gamma_{22}) \cos 2\theta + (\Gamma_{12}^2 - \Gamma_{22}^2) \cos 2\theta_e + (\Gamma_{13}^2 - \Gamma_{23}^2) \cos 2\varphi) (\rho_0 V_T^2) \\ & + (\bar{\Gamma}_1 + \Gamma_{11} \cos 2\theta + \Gamma_{12}^2 \cos 2\theta_e + \Gamma_{13}^2 \cos 2\varphi) (\bar{\Gamma}_2 - \Gamma_{22} \cos 2\theta - \Gamma_{23}^2 \cos 2\theta_e \\ & - \Gamma_{23}^2 \cos 2\varphi) - (\Gamma_{12} \sin 2\theta + \Gamma_{22}^2 \sin 2\theta_e + \Gamma_{23}^2 \sin 2\varphi) (\Gamma_{13} \sin 2\theta + \Gamma_{23}^2 \sin 2\theta_e \\ & + \Gamma_{23}^2 \sin 2\varphi) = 0 \end{aligned} \quad (45)$$

where Γ_1 , Γ_2 , ..., and Γ_{23}^1 are given by

$$\begin{aligned} \bar{\Gamma}_1 = & G \left(2 - \frac{G}{G_{33}} \right) - 2G^2 \bar{\lambda}_{33} + 2G \left(2 - \frac{G}{G_{33}} \right) \left\{ \frac{1}{4G_{11}} (S_1 + S_2) + \frac{1}{2G_{33}} S_3 - \frac{\lambda}{G(3\lambda+2G)} (S_1 + S_2 + S_3) \right. \\ & \left. + \frac{1}{2} (E_1^p + E_2^p + 2E_3^p + E_1^i + E_2^i + 2E_3^i) \right\} - \frac{G}{4G_{11}} (S_1 + S_2) + \frac{G}{2G_{33}} S_3 \\ & + \frac{1}{4} (S_1 + S_2 + 2S_3) - 2G^2 (\kappa_2 + \kappa_3) (S_1 + S_2 + S_3) + G^2 \kappa_3 (S_1 + S_2), \\ \bar{\Gamma}_2 = & G \left(2 - \frac{G}{G_{33}} \right) - 2G^2 \bar{\lambda}_{33} + 2G \left(2 - \frac{G}{G_{33}} \right) \left\{ \frac{1}{4G_{22}} (S_1 + S_2) + \frac{1}{2G_{33}} S_3 - \frac{\lambda}{G(3\lambda+2G)} (S_1 + S_2 + S_3) \right. \\ & \left. + \frac{1}{2} (E_1^p + E_2^p + 2E_3^p + E_1^i + E_2^i + 2E_3^i) \right\} - \frac{G}{4G_{22}} (S_1 + S_2) + \frac{G}{2G_{33}} S_3 \\ & + \frac{1}{4} (S_1 + S_2 + 2S_3) - 2G^2 (\kappa_2 + \kappa_3) (S_1 + S_2 + S_3) + G^2 \kappa_3 (S_1 + S_2), \\ \Gamma_{11} = & \left\{ \frac{G}{4G_{11}} \left(3 - 2 \frac{G}{G_{33}} \right) + \frac{1}{4} - G^2 \kappa_3 \right\} (S_1 - S_2), \\ \Gamma_{22} = & \left\{ \frac{G}{4G_{22}} \left(3 - 2 \frac{G}{G_{33}} \right) + \frac{1}{4} - G^2 \kappa_3 \right\} (S_1 - S_2), \\ \Gamma_{12} = & \left\{ \frac{G}{4G_{12}} \left(3 - 2 \frac{G}{G_{33}} \right) + \frac{1}{4} - G^2 \kappa_3 \right\} (S_1 - S_2), \\ \Gamma_{21} = & \left\{ \frac{G}{4G_{12}} \left(3 - 2 \frac{G}{G_{33}} \right) + \frac{1}{4} - G^2 \kappa_3 \right\} (S_1 - S_2), \\ \Gamma_{13}^1 = & G \left(2 - \frac{G}{G_{33}} \right) (E_1^p - E_2^p), \quad \Gamma_{23}^1 = G \left(2 - \frac{G}{G_{33}} \right) (E_2^p - E_1^p), \\ \Gamma_{13}^2 = & G \left(2 - \frac{G}{G_{33}} \right) (E_1^i - E_2^i), \quad \Gamma_{23}^2 = G \left(2 - \frac{G}{G_{33}} \right) (E_2^i - E_1^i). \end{aligned}$$

Moreover rearranging the above expressions by using Eq.(3) and neglecting small terms of higher order, we obtain for slightly orthotropic materials as follows:

$$\begin{aligned} \bar{\Gamma}_1 = & \bar{\Gamma} + g_{11} - 2G^2 \bar{\lambda}_{33}, \quad \bar{\Gamma}_2 = \bar{\Gamma} + g_{22} - 2G^2 \bar{\lambda}_{33}, \\ \bar{\Gamma} = & G - \left\{ \frac{\lambda - 2G}{2(3\lambda + 2G)} + G^2 (2\kappa_2 + \kappa_3) \right\} (S_1 + S_2 + S_3) + \left(\frac{3}{2} - G^2 \kappa_3 \right) S_3 + G(E_1^p + E_2^p + 2E_3^p \\ & + E_1^i + E_2^i + 2E_3^i), \\ \Gamma = & \Gamma_{11} = \Gamma_{22} = \Gamma_{12} = \Gamma_{21} = \left(\frac{1}{2} - G^2 \kappa_3 \right) (S_1 - S_2), \\ \Gamma^p = & \Gamma_{13}^1 = \Gamma_{23}^1 = G(E_1^p - E_2^p), \quad \Gamma^i = \Gamma_{13}^2 = \Gamma_{23}^2 = G(E_1^i - E_2^i). \end{aligned} \quad (47)$$

Substituting Eqs.(47) into Eq.(46) we obtain the following expressions for the transverse wave velocities V_{T1} and V_{T2}

$$\begin{aligned} \rho_0 V_{T1}^2 = & \bar{\Gamma} + (g_{11} + g_{22} - 2G^2 (\bar{\lambda}_{11} + \bar{\lambda}_{22})) / 2 + [\Gamma^2 + (\Gamma^p)^2 + (\Gamma^i)^2 + 2\Gamma\Gamma^p \cos 2(\theta - \theta_e) + 2\Gamma\Gamma^i \\ & \times \cos 2(\theta - \varphi) + 2\Gamma^p\Gamma^i \cos 2(\theta_e - \varphi) + (g_{11} - g_{22} - 2G^2 (\bar{\lambda}_{11} - \bar{\lambda}_{22})) (\Gamma \cos 2\theta + \Gamma^p \cos 2\theta_e + \Gamma^i \cos 2\varphi) \\ & + (g_{11} - g_{22} - 2G^2 (\bar{\lambda}_{11} - \bar{\lambda}_{22}))^2 / 4]^{1/2}, \quad (48) \\ \rho_0 V_{T2}^2 = & \bar{\Gamma} + (g_{11} + g_{22} - 2G^2 (\bar{\lambda}_{11} + \bar{\lambda}_{22})) / 2 - [\Gamma^2 + (\Gamma^p)^2 + (\Gamma^i)^2 + 2\Gamma\Gamma^p \cos 2(\theta - \theta_e) + 2\Gamma\Gamma^i \\ & \times \cos 2(\theta - \varphi) + 2\Gamma^p\Gamma^i \cos 2(\theta_e - \varphi) + (g_{11} - g_{22} - 2G^2 (\bar{\lambda}_{11} - \bar{\lambda}_{22})) (\Gamma \cos 2\theta + \Gamma^p \cos 2\theta_e + \Gamma^i \cos 2\varphi) \\ & + (g_{11} - g_{22} - 2G^2 (\bar{\lambda}_{11} - \bar{\lambda}_{22}))^2 / 4]^{1/2}. \quad (49) \end{aligned}$$

Similarly, from Eq.(45) the longitudinal wave velocity V_L reduces to

$$\begin{aligned} \rho_0 V_L^2 = & \lambda + 2G + \frac{(\lambda + 2G)^2}{2G^2} (g_{11} + g_{22} + g_{33}) - 2(1 + \frac{\lambda}{G}) (g_{11} + g_{22}) - \frac{4}{9} G^2 (\bar{\lambda}_{11} + \bar{\lambda}_{22}) - \frac{16}{9} G^2 \bar{\lambda}_{33} \\ & + \left\{ 5 + 2 \frac{\lambda}{G} - 4G(3\lambda + 2G) \left(\kappa_2 + \frac{2}{3} \kappa_3 \right) - \frac{8}{3} G^2 \kappa_3 - 4G^2 \bar{\lambda}_{33} \right\} S_3 - \frac{1}{3} \left\{ \frac{6\lambda}{3\lambda + 2G} \left(\frac{\lambda}{G} + 2 \right) \right. \\ & \left. + (3\lambda + 2G)^2 (6\kappa_1 + 3\kappa_2 + \frac{2}{3} \kappa_3) - 4G(3\kappa_2 + 2\kappa_3) - \frac{8}{3} G^2 \kappa_3 - 4G^2 \bar{\lambda}_{33} \right\} (S_1 + S_2 + S_3) \\ & + 4(\lambda + 2G)(E_1^p + E_2^p). \quad (50) \end{aligned}$$

Here, according to the assumption given in Eq.(3) the constants κ_1, κ_2 , are expressed by the third-order Murnaghan constants l, m and n as follows:

$$\left. \begin{aligned} \kappa_1 &= \frac{1}{(3\lambda+2G)^2} \left\{ -l + \frac{(3\lambda+2G)^2}{4G^2} m - \frac{1}{8G^2} (4G^3 + 12\lambda G^2 + 15\lambda^2 G + 6\lambda^3) n \right\}, \\ \kappa_2 &= \frac{1}{2G^2(3\lambda+2G)} \left\{ -Gm + \frac{1}{2}(\lambda+G)n \right\}, \\ \kappa_3 &= -\frac{1}{8G^2} n. \end{aligned} \right\} \quad (51)$$

For further details of the derivation of above expressions the reader should refer to the author's previous report⁽¹⁴⁾.

3.2 Polarization directions of transverse waves

Taking β_1 as the angle between the x_1 -axis and the polarization direction of the transverse wave V_{T1} and $\bar{U}$ as the amplitude of its wave, we write the eigenvector components of a vector x_1 and a vector x_2 as

$$\bar{U}_1 = \bar{U} \cos \beta_1, \quad \bar{U}_2 = \bar{U} \sin \beta_1, \quad (52)$$

Substitution of Eqs.(48) and (52) into Eqs.(39) and (40) gives

$$\begin{aligned} \tan \beta_1 &= (\Gamma' \sin 2\theta + \Gamma'' \sin 2\theta_e + \Gamma' \sin 2\varphi) / (\Gamma' \cos 2\theta + \Gamma'' \cos 2\theta_e + \Gamma' \cos 2\varphi) \\ &\quad + (g_{13} - g_{23} - 2G^2(\bar{\lambda}_{13} - \bar{\lambda}_{23})) / 2 + [\Gamma^2 + (\Gamma'')^2 + (\Gamma')^2 + 2\Gamma\Gamma'' \cos 2(\theta - \theta_e) + 2\Gamma\Gamma' \cos 2(\theta - \varphi) \\ &\quad + 2\Gamma''\Gamma' \cos 2(\theta_e - \varphi) + (g_{13} - g_{23} - 2G^2(\bar{\lambda}_{13} - \bar{\lambda}_{23}))(\Gamma' \cos 2\theta + \Gamma'' \cos 2\theta_e + \Gamma' \cos 2\varphi) \\ &\quad + (g_{13} - g_{23} - 2G^2(\bar{\lambda}_{13} - \bar{\lambda}_{23}))^2 / 4]^{1/2}. \end{aligned} \quad (53)$$

Similarly, taking β_2 as the angle between the x_1 -axis and the polarization direction of the transverse wave V_{T2} , we obtain

$$\begin{aligned} \tan \beta_2 &= (\Gamma' \sin 2\theta + \Gamma'' \sin 2\theta_e + \Gamma' \sin 2\varphi) / (\Gamma' \cos 2\theta + \Gamma'' \cos 2\theta_e + \Gamma' \cos 2\varphi) \\ &\quad + (g_{13} - g_{23} - 2G^2(\bar{\lambda}_{13} - \bar{\lambda}_{23})) / 2 - [\Gamma^2 + (\Gamma'')^2 + (\Gamma')^2 + 2\Gamma\Gamma'' \cos 2(\theta - \theta_e) \\ &\quad + 2\Gamma\Gamma' \cos 2(\theta - \varphi) + 2\Gamma''\Gamma' \cos 2(\theta_e - \varphi) + (g_{13} - g_{23} - 2G^2(\bar{\lambda}_{13} - \bar{\lambda}_{23}))(\Gamma' \cos 2\theta \\ &\quad + \Gamma'' \cos 2\theta_e + \Gamma' \cos 2\varphi) + (g_{13} - g_{23} - 2G^2(\bar{\lambda}_{13} - \bar{\lambda}_{23}))^2 / 4]^{1/2}. \end{aligned} \quad (54)$$

3.3 Relationship between theory proposed in this paper and theoretical results derived by Iwashimizu and Kubomura

For a material that is slightly orthotropic in its natural state, the theoretical results about the acousto-elastic effects have been reported by Iwashimizu and Kubomura⁽⁹⁾. It will be shown that in a special case such as an elastic state the results of the present theory agree with those of Iwashimizu and Kubomura's theory.

Supposing a medium is slightly orthotropic, from Eqs.(3) and (4) we can obtain the following relations between C'_{ij} in Ref.(9) and g_{ij} in this paper as

$$\begin{aligned} C'_{13} &= \frac{(\lambda+2G)^2}{2G^2} (g_{11} + g_{22} + g_{33}) \\ &\quad - 2\left(1 + \frac{\lambda}{G}\right)(g_{11} + g_{22}), \\ C'_{11} &= g_{11}, \quad C'_{22} = g_{22}. \end{aligned} \quad (55)$$

Hence, by substituting Eqs.(51) and (55) into Eqs.(48)~(59), putting $\bar{\lambda}_{ij} = \Gamma^p = \Gamma' = 0$ and replacing both the stress components and the third-order Murnaghan constants by the strain components and the third-order constants defined by Toupin and Bernstein, we can confirm that in an elastic state the results of the present paper agree with those of Iwashimizu and Kubomura's report⁽⁹⁾. Here, the third-order Murnaghan constants l, m and n are related to the third-order Toupin-Bernstein constants v_1, v_2 and v_3 in Ref.(22) in the following expressions:

$$l = v_1/2 + v_2, \quad m = v_2 + 2v_3, \quad n = 4v_3. \quad (56)$$

4. Conclusions

In the present paper, acousto-elastic effects in elasto-plastically deformed solids due to slightly orthotropic elastic moduli and anisotropy of initial strains are analyzed

theoretically; it is shown that in a special case such as a elastic state the results agree with those derived by Iwashimizu and Kubomura.

It has been pointed out that there are considerable discrepancies between the theoretical results and the experimental ones about the influence of plastic deformations upon the acousto-elastic effects. Here it is suggested that a new parameter $\bar{\lambda}_{ij}$ introduced in the present paper relates an anisotropic behavior induced by plastic deformations and it can be applied to improve those discrepancies.

The experimental studies on acousto-elastic effects due to plastic deformations are being continued dynamically: for example Refs.(23)~(26), but experimental data to date are not yet adequate enough to examine accuracy of the theory. Therefore, a detailed definition of $\bar{\lambda}_{ij}$ and an examination of accuracy of the proposed theory will be made at a later date.

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