A Hybrid Type of Virtual Work Principle for Impact Contact Problems of Two Bodies *

By Naoki ASANO **

A variational principle for impact contact of two bodies generally requires subsidiary contact conditions for the bodies on a contact area, because of the continuity of displacement between the bodies. Various Lagrange multipliers are used to include the subsidiary conditions in the principle. This paper presents an extended multiplier applicable to the subsidiary contact conditions for both contact displacement and contact acceleration of the two bodies in the principle. This multiplier can also formulate a hybrid type of virtual work principle for various impact contact states of two bodies. A finite element method (FEM) based on this principle is applied to a two-dimensional response for longitudinal impact of two uniform rods in order to check the validity of impact force based on this multiplier in the principle. The mean value of impact force by FEM agrees well with the value by the theory of propagation of one-dimensional elastic stress wave. Moreover, this paper illustrates the time variation of impact contact pressure distribution.

Key Words: Shock, Impact Problems, Virtual Work Principle, FEM, Extended Lagrange Multiplier, Variational Principle, Problem of Two Bodies, Elasticity, Structural Analysis, Contact Structures.

1. Introduction

Impact contact occurs in many machine parts such as meshing gears, rolling bearings, clutches, brakes, and so on. Impact contact of these structures sometimes causes vibration, noise, and wear. It is also difficult to measure and/or calculate the impact contact behavior of the structures, because they have many complicated geometries and the non-linearity of contact due to the change in both the contact area and the contact states.

With the increase of memory capacity and the speed of execution in a computer, a finite element method (FEM) is applied to many areas. In the area of structural analysis, especially, the FEM is applied to two-dimensional, three-dimensional, and non-linear structures with a complicated geometry, in comparison with some analytic solutions under simple geometries and special boundary conditions [1-3]. A FEM for contact behavior with frictional effects has been developed recently and applied to various contact structures [4-18].

The formulation of FEMs is very difficult in the analysis of structures including contact and non-linear effects, but the use of a variational principle makes it easy to formulate various FEMs [15]. A virtual work principle for impact contact of two bodies generally requires

subsidiary contact conditions for the two bodies on a contact area, because of the continuity of displacement between the bodies [9,13,20]. Various Lagrange multipliers are used to include the subsidiary contact conditions in the principle [17].

This paper presents an extended Lagrange multiplier applicable to the subsidiary conditions for both contact displacement and contact acceleration of the two bodies in the principle. This multiplier can formulate a hybrid type of virtual work principle for various impact contact states of two bodies. A two-dimensional response for longitudinal impact of two uniform rods is also calculated by using the FEM based on this principle, in order to check the validity of impact force based on multipliers in this principle.

2. An Extended Lagrange Multiplier in a Virtual Work Principle for Two Bodies

For simplicity of treatment, we first consider the motion of two bodies, $\alpha = 1, 2$, which interconnect on a contact area S_c in Fig.1. It is assumed in this chapter that no slip occurs on S_c between the two bodies. Rectangular Cartesian coordinates are taken in the n -direction normal to S_c , the t_a - and t_b -directions tangential to S_c . From the continuity of displacement, subsidiary contact conditions between the two bodies are given by

$$\begin{aligned} u_c^{(1)} &= u_c^{(2)}, & \dot{u}_c^{(1)} &= \dot{u}_c^{(2)}, & \ddot{u}_c^{(1)} &= \ddot{u}_c^{(2)}, \\ c &= n, t_a, t_b \text{ on } S_c, \dots\dots\dots (1) - (3) \end{aligned}$$

* Received 5th April, 1984.

** Professor, Faculty of Engineering, Tamagawa University, Machida, Tokyo 194, JAPAN.

where $u_c(\alpha)$, $\dot{u}_c(\alpha)$, and $\ddot{u}_c(\alpha)$ are the displacement, velocity, and acceleration vectors on the contact surface of the body (α) in a fixed coordinate system ($O-X_1, X_2, X_3$), respectively. The subscript c is the direction of the contact coordinates on S_c .

Lagrange multipliers are generally used to include the subsidiary contact conditions such as Eqs.(1)-(3) in a variational principle for impact contact of two bodies. However, some of the multipliers are physically undefined and the relationship among the multipliers has not yet been clarified, as pointed out in Ref.[17,19-21].

If Eq.(2) is required as a subsidiary contact condition and the damping effect of two bodies is neglected, Eqs.(1) and (3) can be included in a virtual work principle for two bodies by using multiplier λ_c [21]:

$$\int_{\tau_1}^{\tau_2} \left\{ \sum_{a=1}^2 (\delta E^{(a)} + \delta T^{(a)} - \delta W^{(a)}) + \delta \int_{S_c} \lambda_c (u_c^{(1)} - u_c^{(2)}) dS + \delta \int_{S_c} \lambda_c (\dot{u}_c^{(1)} - \dot{u}_c^{(2)}) dS \right\} d\tau = 0,$$

sum for $c = n, t_a, t_b$ on S_c ,(4)

where τ_1 and τ_2 are time instants, δ is a variational operator. $\delta T^{(a)}$, $\delta E^{(a)}$, and $\delta W^{(a)}$ are the virtual kinetic, internal and external works of body (α), respectively. These virtual works are expressed by

$$\left. \begin{aligned} \delta E^{(a)} &= \int_{V^{(a)}} \sigma_{ij}^{(a)} \cdot \delta \varepsilon_{ij}^{(a)} dV, \\ \delta T^{(a)} &= \int_{V^{(a)}} \rho^{(a)} \cdot \dot{u}_i^{(a)} \cdot \delta u_i^{(a)} dV, \\ \delta W^{(a)} &= \int_{V^{(a)}} F_i^{(a)} \cdot \delta u_i^{(a)} dV + \int_{S_{\sigma}^{(a)}} P_i^{(a)} \delta u_i^{(a)} dS, \end{aligned} \right\} \text{.....(5)}$$

where $V^{(a)}$ and $\rho^{(a)}$ denote the volume and the density of the body (α). $\sigma_{ij}^{(a)}$ is a stress tensor, and $\varepsilon_{ij}^{(a)}$ a virtual strain tensor. $u_i^{(a)}$, $\dot{u}_i^{(a)}$, and $\ddot{u}_i^{(a)}$ are the displacement, the acceleration, and the virtual displacement tensors in the fixed coordinates, respectively. $F_i^{(a)}$ is a force tensor, $P_i^{(a)}$ a traction tensor, and $S_{\sigma}^{(a)}$ the area on which $P_i^{(a)}$ acts.

In comparison with a non-hybrid type virtual work principle for impact contact of two bodies [20], Eq.(4) represents the impact contact force R_c as follows [21]:

Fig.1 Rectangular cartesian coordinates on a contact area.

$$\int_{\tau_1}^{\tau_2} \left\{ \sum_{a=1}^2 (\delta E^{(a)} + \delta T^{(a)} - \delta W^{(a)}) + \int_{S_c} R_c (\delta u_c^{(1)} - \delta u_c^{(2)}) dS \right\} d\tau = 0,$$

$$R_c = \lambda_c + \dot{\lambda}_c, \text{ sum for } c = n, t_a, t_b \text{ on } S_c, \text{.....(6)}$$

where λ_c means a stationary contact force due to contact displacement, and $\dot{\lambda}_c$ denotes a time variation of contact force corresponding to contact acceleration.

The formulation of Eqs.(4) and (6) will be described in appendix. Equation (4) expresses a hybrid type of virtual work principle in a stick state with no sliding permitted behavior of the two bodies. On the contact boundary of the two bodies, a Lagrange multiplier generally represents a contact displacement in the stationary solution by a force method, and a contact force by a displacement method [23]. A virtual work principle belongs to the displacement method, but the term with the multipliers in this principle contains the components of the force method. Hence, the principle with the multipliers is hereinafter defined as a hybrid type.

3. A Hybrid Type of Virtual Work Principle for Impact Contact and Separate States

This chapter describes a hybrid type of virtual work principle for various impact contact and separate states of two bodies, based on Eq.(4). Various virtual works used here are calculated by using Eq.(5). For simplicity of treatment, the damping and thermal effects of two bodies are neglected in this paper.

The contact area S_c of the bodies, as shown in Fig.1, is represented as

$$S_c = C_{st} + C_s + C_m + C_o, \text{.....(7)}$$

where C_{st} is locally the stick area, C_s the slip area, C_m the mixed area, and C_o the separate area. The details of various impact contact states will be explained later. Virtual contact works on S_c are represented by δQ_{st} , δQ_s , δQ_m , and δQ_o , respectively, corresponding to C_{st} , C_s , C_m , and C_o . The virtual work principle using Lagrange multipliers is formulated as

$$\int_{\tau_1}^{\tau_2} \left\{ \sum_{a=1}^2 (\delta E^{(a)} + \delta T^{(a)} - \delta W^{(a)}) + \delta Q_{st} + \delta Q_s + \delta Q_m + \delta Q_o \right\} d\tau = 0, \text{.....(8)}$$

where the details of various virtual contact works are expressed in sections 3.1 to 3.4.

3.1 Stick state of two bodies

We consider the motion of two bodies interconnecting on a local contact area C_{st} . Assume that the stick state is discriminated by Coulomb's law. Using Eq.(6), we obtain

$$\left. \begin{aligned} |\lambda_{ta} + \ddot{\lambda}_{ta}| &< |\mu_{ta}(\lambda_n + \ddot{\lambda}_n)|, \\ |\lambda_{tb} + \ddot{\lambda}_{tb}| &< |\mu_{tb}(\lambda_n + \ddot{\lambda}_n)|, \end{aligned} \right\} \dots\dots\dots (9)$$

where μ_{ta} and μ_{tb} denote frictional coefficients in the ta - and tb -directions, respectively.

δQ_{st} in Eq.(8) is expressed by

$$\delta Q_{st} = \delta Q_n + \delta Q_{ta} + \delta Q_{tb}, \dots\dots\dots (10)$$

$$\begin{aligned} \delta Q_c &= \delta \int_{C_{st}} \lambda_c (\dot{u}_c^{(1)} - \dot{u}_c^{(2)}) dS \\ &\quad + \delta \int_{C_{st}} \lambda_c (\ddot{u}_c^{(1)} - \ddot{u}_c^{(2)}) dS, \end{aligned}$$

$$c = n, ta, tb \text{ on } C_{st},$$

where a subsidiary condition for contact velocity is

$$\dot{u}_c^{(1)} = \dot{u}_c^{(2)}, \quad c = n, ta, tb \text{ on } C_{st}. \dots\dots (11)$$

3.2 Slip state of two bodies

Let us consider the motion of two bodies slipping on a local contact area C_s . Slipping conditions in both the ta - and tb -directions are given by Coulomb's law. Using Eq.(6), we obtain

$$\left. \begin{aligned} |\lambda_{ta} + \ddot{\lambda}_{ta}| &\geq |\mu_{ta}(\lambda_n + \ddot{\lambda}_n)|, \\ |\lambda_{tb} + \ddot{\lambda}_{tb}| &\geq |\mu_{tb}(\lambda_n + \ddot{\lambda}_n)|, \end{aligned} \right\} \dots\dots\dots (12)$$

We have a discrepancy on each paired contact node in FEM analysis. If the discrepancy increases, a pair of contact nodes must be corrected by re-discretizing the two bodies or interpolating the results obtained. Then, the use of relative movement components makes it easy to check the discrepancy. Denoting the relative displacement, velocity, and acceleration by r_c , $\dot{r}_c$, and $\ddot{r}_c$, respectively, δQ_s in Eq.(8) is represented as

$$\delta Q_s = \delta Q_n + \delta Q_{ta} + \delta Q_{tb}, \dots\dots\dots (13)$$

$$\delta Q_n = \delta \int_{C_s} \lambda_n (\dot{u}_n^{(1)} - \dot{u}_n^{(2)}) dS$$

$$+ \delta \int_{C_s} \lambda_n (\ddot{u}_n^{(1)} - \ddot{u}_n^{(2)}) dS,$$

$$\delta Q_c = \delta \int_{C_s} \mu_c (\lambda_n + \ddot{\lambda}_n) (\delta \dot{u}_c^{(1)} - \delta \dot{u}_c^{(2)}) dS$$

$$+ \delta \int_{C_s} \delta \lambda_c (\dot{u}_c^{(1)} - \dot{u}_c^{(2)} - r_c) dS$$

$$+ \delta \int_{C_s} \delta \lambda_c (\ddot{u}_c^{(1)} - \ddot{u}_c^{(2)} - \ddot{r}_c) dS,$$

$$c = ta, tb,$$

where subsidiary contact conditions for velocity are

$$\dot{u}_n^{(1)} = \dot{u}_n^{(2)}, \quad \dot{u}_c^{(1)} = \dot{u}_c^{(2)} + \dot{r}_c,$$

$$c = ta, tb \text{ on } C_s. \dots\dots\dots (14)$$

The matrix size of FEM equation remains constant even if contact states change [17]. In addition, the use of the relative movement components effectively decreases the computing time [13].

3.3 Mixed state of stick and slip

Let us consider the mixed motion of two bodies in both stick and slip states on a contact area C_m , such as the lateral slipping behavior of a tire on a road. For example, the mixed state is treated as a stick state in the ta -direction and a slip state in the tb -direction. From Coulomb's law, mixed conditions in both the ta - and tb -directions are discriminated as

$$\left. \begin{aligned} |\lambda_{ta} + \ddot{\lambda}_{ta}| &< |\mu_{ta}(\lambda_n + \ddot{\lambda}_n)|, \\ |\lambda_{tb} + \ddot{\lambda}_{tb}| &\geq |\mu_{tb}(\lambda_n + \ddot{\lambda}_n)|. \end{aligned} \right\} \dots\dots\dots (15)$$

δQ_m in Eq.(8) is represented as

$$\delta Q_m = \delta Q_n + \delta Q_{ta} + \delta Q_{tb}, \dots\dots\dots (16)$$

$$\delta Q_c = \delta \int_{C_m} \lambda_c (\dot{u}_c^{(1)} - \dot{u}_c^{(2)}) dS$$

$$+ \delta \int_{C_m} \lambda_c (\ddot{u}_c^{(1)} - \ddot{u}_c^{(2)}) dS, \quad c = n, ta,$$

$$\delta Q_{tb} = \delta \int_{C_m} \mu_{tb} (\lambda_n + \ddot{\lambda}_n) (\delta \dot{u}_{tb}^{(1)} - \delta \dot{u}_{tb}^{(2)}) dS$$

$$+ \delta \int_{C_m} \delta \lambda_{tb} (\dot{u}_{tb}^{(1)} - \dot{u}_{tb}^{(2)} - r_{tb}) dS$$

$$+ \delta \int_{C_m} \delta \lambda_{tb} (\ddot{u}_{tb}^{(1)} - \ddot{u}_{tb}^{(2)} - \ddot{r}_{tb}) dS,$$

where subsidiary contact conditions for velocity are

$$\dot{u}_c^{(1)} = \dot{u}_c^{(2)}, \quad \dot{u}_{tb}^{(1)} = \dot{u}_{tb}^{(2)} + \dot{r}_{tb},$$

$$c = n, ta \text{ on } C_m. \dots\dots\dots (17)$$

3.4 Separate state of two bodies

When the whole or local area C_o between two bodies is in separation, impact contact force leads to zero. Consequently, δQ_o in Eq.(8) becomes equal to zero, so that the relation between the bodies is independent, because there exist no subsidiary contact conditions.

Let us consider an alternating behavior between contact and separation, such as observed on a beam struck with a rod, on meshing gears, and so on. Then, the use of the clearance between two bodies makes it easy to discriminate between contact and separation on a contact area, and makes it possible to decrease the computing time in FEM analysis [13].

Introducing relative movement components equivalent to the clearance, δQ_o in Eq.(8) is expressed by

$$\delta Q_o = \delta Q_n + \delta Q_{ta} + \delta Q_{tb}, \dots\dots\dots (18)$$

$$\delta Q_c = \delta \int_{C_o} \delta \lambda_c (\dot{u}_c^{(1)} - \dot{u}_c^{(2)} - r_c) dS$$

$$+ \delta \int_{C_o} \delta \lambda_c (\ddot{u}_c^{(1)} - \ddot{u}_c^{(2)} - \ddot{r}_c) dS,$$

$$c = n, ta, tb,$$

where a subsidiary contact condition for velocity is

$$\dot{u}_c^{(1)} = \dot{u}_c^{(2)} + \dot{r}_c, \quad c = n, ta, tb \text{ on } C_o. \dots\dots (19)$$

4. Investigation of Contact Behavior in Longitudinal Impact of Two Rods

The impact contact force in Eq.(6) can be checked using a FEM based on the principle in chapter 3. The formulation of the FEM is omitted, owing to limited space.

4.1 Conditions for analysis

A two-dimensional finite element model in longitudinal impact of two uniform rods is shown in Fig.2. The end of the rod (1) is fixed on a rigid wall, and the rod (2) collides with the rod (1). The two elastic rods different in length are subjected to impact at their flat end faces. Since the contact area between the rods remains constant, calculations are carried out for an impact velocity $V = 1$ m/s of the rod (2). The frictional coefficient on the contact area is assumed to be 0.2.

The upper half of the rods is calculated, because of the symmetry of the rods in shape. The rods are divided into a constant strain type triangular elements.

The two rods are assumed to be made of mild steel. Runge-Kutta-Gill method is used to obtain the numerical solutions of differential equation. The time interval for integration is 0.1 μ sec. The lumped mass matrix (LM) is used to decrease both memory capacity and computing time [8,9,12-17].

4.2 Behavior of impact contact force

Figure 3 illustrates both the total elastic impact contact force on all contact nodes by the FEM and the value from the theory of propagation of one-dimensional elastic stress wave (1D theory) [8,9,12-17]. FEM results show a periodic change which

Fig.2 A finite element model for two-dimensional analysis of longitudinal impact.

Fig.3 Elastic impact contact force

depends on the deformability of elements due to the lateral effect of inertia as a two-dimensional stress wave propagation [1-3], and a high frequency fluctuation depends on a discretized error in element division [18]. However, the mean value of FEM results agrees well with the value from 1D theory idealized.

The results in Fig.3 agree precisely with those of other FEMs [9,13-16] based on the principles [19,20], and the validity of the results of the other FEMs is recognized in comparison with experimental results [22]. Therefore, the validity of impact contact force in Eq.(6) is clear from both the results in Fig.3 and those of the other FEMs.

4.3 Behavior of impact contact pressure

A time variation of contact pressure distribution is illustrated in Fig.4. The contact pressure near the outside of the contact area fluctuates greater than that near the central part. This is because the lateral stress near the outside becomes less than that near the central part. A lateral stress distribution on the contact area is omitted, owing to limited space.

After 80 μ sec. the distribution of contact pressures changes greatly, because of the interference between the stress wave reflected from the fixed end of the rod (1) and that from the free end of the rod (2).

5. Concluding Remarks

If Eqs.(2) and (3) are treated as a subsidiary contact condition, an approximate hybrid type virtual work principle can be formulated [17], so that Eq.(6) expresses just λ_c only. The approximate principle can formulate the FEM by Hughes et al [7]. However, λ_c of the principle is valid with a maximum $\pm 10.4\%$ error, in two-dimensional FEM analysis analogous to the model in Fig. 2 [17].

As compared with the approximate principle, Eq.(4) expresses an improved principle, because Eqs.(1) and (3) are included in this principle. However, it is not clear whether or not Eq.(2) may be included in Eq.(4) by using a same multiplier for Eqs.(1) and (3).

Fig.4 A time variation of contact pressure distribution.

APPENDIX (1) Derivation of Eqs.(4) and (6)
Based on Hamilton's Principle

Hamilton's principle for the body (α) is written as

$$\delta \int_{\tau_1}^{\tau_2} \{T^{(\alpha)} - \Pi^{(\alpha)}\} d\tau = 0, \dots\dots\dots (20)$$

$$\Pi^{(\alpha)} = E^{(\alpha)} - W^{(\alpha)},$$

$$T^{(\alpha)} = \int_{V^{(\alpha)}} \frac{1}{2} \rho^{(\alpha)} \cdot u_i^{(\alpha)} \cdot u_i^{(\alpha)} dV,$$

$$E^{(\alpha)} = \int_{V^{(\alpha)}} \frac{1}{2} \sigma_{ij}^{(\alpha)} \cdot \epsilon_{ij}^{(\alpha)} dV,$$

$$W^{(\alpha)} = \int_{V^{(\alpha)}} F_i^{(\alpha)} \cdot u_i^{(\alpha)} dV + \int_{S_c^{(\alpha)}} P_i^{(\alpha)} \cdot u_i^{(\alpha)} dS,$$

where $T^{(\alpha)}$ and $\Pi^{(\alpha)}$ denote, respectively, a kinetic energy and a potential one [23]. All the other symbols are similar to those in Eq.(5).

Hamilton's principle for impact contact of two bodies [21,23] can be expressed by

$$\delta \int_{\tau_1}^{\tau_2} \{T_c - \Pi_c\} d\tau = 0, \dots\dots\dots (21)$$

$$T_c = \sum_{\alpha=1}^2 T^{(\alpha)} + \int_{S_c} \lambda_c (\dot{u}_c^{(1)} - \dot{u}_c^{(2)}) dS,$$

$$\Pi_c = \sum_{\alpha=1}^2 \Pi^{(\alpha)} + \int_{S_c} \lambda_c (u_c^{(1)} - u_c^{(2)}) dS,$$

sum for $c = n, t_a, t_b$ on S_c ,

where

$$\lambda_c = \frac{d}{d\tau} \lambda_c. \dots\dots\dots (22)$$

λ_c in Eq.(21) represents physically a stationary contact force [21,23], and $\int_{S_c} \lambda_c u_c^{(\alpha)} dS$ expresses the analogue of the potential energy of body (α), so that a part of Eq.(21)

$$\int_{S_c} \lambda_c (u_c^{(1)} - u_c^{(2)}) dS$$

means a potential energy difference between the two bodies, and corresponds to the energy transmitted from one body to the other one.

From Eq.(22), $\dot{\lambda}_c$ denotes the time variation of contact force [21], and a part of Eq.(21)

$$\int_{S_c} \dot{\lambda}_c (\dot{u}_c^{(1)} - \dot{u}_c^{(2)}) dS$$

also means a kinetic energy difference between the two bodies and corresponds to the energy transmitted from one body to the other one.

Therefore, Eq.(4) can be easily derived from Eq.(21) under following conditions [21]:

$$\delta \lambda_c(\tau_1) = \delta \lambda_c(\tau_2) = 0,$$

$$\delta \dot{\lambda}_c(\tau_1) = \delta \dot{\lambda}_c(\tau_2) = 0,$$

$$\delta u_c^{(\alpha)}(\tau_1) = \delta u_c^{(\alpha)}(\tau_2) = 0,$$

$$\delta \dot{u}_c^{(\alpha)}(\tau_1) = \delta \dot{u}_c^{(\alpha)}(\tau_2) = 0,$$

$$\alpha = 1, 2, c = n, t_a, t_b \text{ on } S_c. \dots\dots\dots (23)$$

On the other hand, the following is derived from Eqs.(4) and (23):

$$\begin{aligned} & \int_{\tau_1}^{\tau_2} \left\{ \sum_{\alpha=1}^2 (\delta E^{(\alpha)} + \delta T^{(\alpha)} - \delta W^{(\alpha)}) \right. \\ & \quad \left. + \delta \int_{S_c} (\lambda_c + \dot{\lambda}_c) (u_c^{(1)} - u_c^{(2)}) dS \right\} d\tau = 0, \\ & \text{sum for } c = n, t_a, t_b \text{ on } S_c, \dots\dots (24) \end{aligned}$$

From Eq.(24), Eq.(6) can be obtained by using Eqs.(1) and (23).

APPENDIX (2) Derivation of Eqs.(4) and (6)
Based on FEM equation

If the damping effect of each body is neglected, a matrix equation based on a variational principle is formulate as follows:

$$[M][\ddot{X}] + [K][X] = [F], \dots\dots\dots (25)$$

where M is the mass matrix, K the stiffness matrix, and F the external matrix. $\ddot{X}$ and X are, respectively, the acceleration matrix and the displacement one.

Using Eq.(25), a matrix equation of two bodies (1) and (2) in a stick state is

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \begin{bmatrix} -G \\ G \end{bmatrix}, \dots\dots\dots (26)$$

where G denotes an unknown impact contact force matrix [9,17,21]. Subscripts 1 and 2 refer to bodies (1) and (2), respectively.

In Eq.(26) G can be decomposed into G_m , G_k , and G_f [17], so that

$$G = G_m + G_k + G_f, \dots\dots\dots (27)$$

where

$$\left. \begin{aligned} G_m &= -M_1 \ddot{X}_1, \quad G_k = -K_1 X_1, \quad G_f = F_1, \\ \text{or} \quad G_m &= M_2 \ddot{X}_2, \quad G_k = K_2 X_2, \quad G_f = -F_2. \end{aligned} \right\} \dots\dots (28)$$

In Eqs.(27) and (28), G_m is a variable component of the impact force matrix dependent upon $\ddot{X}_1$ or $\ddot{X}_2$, G_k a stationary component dependent upon X_1 or X_2 , and G_f an external force matrix acting on a contact area. We can put $G_f = 0$ on a contact area of various contact impact structures. Matrices G_m and G_k change during any impact contact time [17,21]. Then, Eq.(27) is rewritten as

$$G = G_m + G_k, \quad G_f = 0. \dots\dots (29)$$

From Eqs.(26) and (29) the following expression is obtained as

$$\begin{bmatrix} M_1 & 0 & I_c \\ 0 & M_2 & -I_c \\ I_c^T & -I_c^T & 0 \end{bmatrix} \begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \\ G_m \end{bmatrix} + \begin{bmatrix} K_1 & 0 & I_c \\ 0 & K_2 & -I_c \\ I_c^T & -I_c^T & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ G_k \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix}, \quad I_c^T = [0, I], \quad \dots\dots\dots (30)$$

where I is a unit matrix, and I_c a unit matrix related to contact nodes in FEM analysis. But the relationship between G_m and G_k has not yet been clarified.

Equation (28) indicates that

$$\text{or } \left. \begin{array}{l} G_m \propto \ddot{X}_1, \text{ and } \ddot{X}_2, \\ G_k \propto X_1, \text{ and } X_2. \end{array} \right\} \dots\dots\dots (31)$$

Using Eq.(31), the relationship between G_m and G_k can be rewritten as

$$G_m \propto \ddot{G}_k. \quad \dots\dots\dots (32)$$

In Eq.(32) the following expression is required for Eq.(30) in order to make a form of differential equation:

$$G_m = \ddot{G}_k. \quad \dots\dots\dots (33)$$

Using Eq.(33), Eq.(30) is represented as

$$\begin{bmatrix} M_1 & 0 & I_c \\ 0 & M_2 & -I_c \\ I_c^T & -I_c^T & 0 \end{bmatrix} \begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \\ \ddot{G}_k \end{bmatrix} + \begin{bmatrix} K_1 & 0 & I_c \\ 0 & K_2 & -I_c \\ I_c^T & -I_c^T & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ G_k \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix}, \quad I_c^T = [0, I]. \quad \dots\dots\dots (34)$$

Equation (34) can be transformed into the following variational equation:

$$\int_{\tau_1}^{\tau_2} \left[\begin{array}{l} \delta X_1 \\ \delta X_2 \\ \delta G_k \end{array} \right]^T \left\{ \begin{array}{l} \left[\begin{array}{ccc} M_1 & 0 & I_c \\ 0 & M_2 & -I_c \\ I_c^T & -I_c^T & 0 \end{array} \right] \begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \\ \ddot{G}_k \end{bmatrix} \\ + \left[\begin{array}{ccc} K_1 & 0 & I_c \\ 0 & K_2 & -I_c \\ I_c^T & -I_c^T & 0 \end{array} \right] \begin{bmatrix} X_1 \\ X_2 \\ G_k \end{bmatrix} - \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix} \end{array} \right\} d\tau = 0. \quad \dots\dots\dots (35)$$

Using Eq.(5), Eqs.(4) and (6) can be derived from Eq.(35).

References

- 1 Skalak, R., ASME Journal of Applied Mechanics, Vol.24, 1957, pp.59-64.
- 2 Jones, O.E. and Ellis, A.T., ASME Journal of Applied Mechanics, Vol.30, 1963, pp.51-60; *ibid*, pp.61-69.
- 3 Tanaka, K. and Motoyama, C., Transactions of the JSME (in Japanese), Vol.40, No.330, 1974, pp.382-386.
- 4 Ohte, S., Bulletin of the JSME, Vol.16, No.95, 1973, pp.797-804.
- 5 Tsuta, T. and Yamaji, S., Journal of the JSME (in Japanese), Vol.76, No.651, 1973, pp.348-358.
- 6 Okamoto, N. and Nakazawa, M., Int.J.Num. Meth.Engng., Vol.14, 1979, pp.337-357.
- 7 Hughes, T.J.R., Taylor, R.L. and Kanoknukulchai, W., U.S.-Germany Symp. on FEM at MIT, Aug. 1976, (Formulation and Computational Algorithms in Finite Element Analysis, by Bathe, K.J., Oden, T.J., and Wunderlich, W.), pp.468-495.
- 8 Doi, O. and Asano, N., Proceedings of the JSME (in Japanese), No.760-17, 1976, pp.141-143; *ibid*, pp.144-146.
- 9 Doi, O. and Asano, N., Bulletin of the JSME, Vol.24, No.189, 1981, pp.528-533.
- 10 Yagawa, G., Hirayama, H. and Ando, Y., Transactions of the JSME (in Japanese), Vol.46, No.411, Series A., 1980, pp.1120-1129.
- 11 Kikuchi, N. and Oden, J.T., Ticom Report 79-8, 1979.
- 12 Asano, N., Bulletin of the JSME, Vol.23, No.184, 1980, pp.1575-1580.
- 13 Asano, N., Bulletin of the JSME, Vol.24, No.193, 1981, pp.1123-1129.
- 14 Asano, N., Bulletin of the JSME, Vol.25, No.201, 1982, pp.328-333.
- 15 Asano, N., Bulletin of the JSME, Vol.25, No.202, 1982, pp.501-507.
- 16 Asano, N., Proceedings of the JSME (in Japanese), No.788-3, 1978, pp.97-104.
- 17 Asano, N., Bulletin of the JSME, Vol.26, No.221, 1983, pp.1849-1856.
- 18 Yagawa, G., Kanto, Y. and Ando, Y., Transactions of the JSME (in Japanese), Vol.49, No.448, Series A, 1983, pp.1581-1589.
- 19 Asano, N., Proceedings of the JSME (in Japanese), No.790-3, 1979, pp.1-3.
- 20 Asano, N., Bulletin of the JSME, Vol.26, No.220, 1983, pp.1687-1695.
- 21 Asano, N., Proceedings of the JSME (in Japanese), No.840-1, 1984, pp.89-91.
- 22 Asano, N., 18th Symp. on Impact and Dynamic Response, Proceedings of the JSME (in Japanese), No.800-9, 1980, pp.95-100.
- 23 Washizu, K., Variational Methods in Elasticity & Plasticity, third edition, 1982, Pergamon Press.