An Analysis of Large Deflections in a Symmetrical Three-point Bending of Beam

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The large deflection problem of a thin elastic simply supported beam is analyzed for a symmetrical three-point bending. The derived nonlinear differential equation governing beam deflections is solved by applying the numerical elliptic integrals of the first and the second kinds. Moreover, a reduction technique is proposed to estimate representative flexural quantities such as a maximum deflection, an end slope, and a maximum bending stress in large deflection states from the conventional linear bending theory in place of the exact large deflection theory. An experiment is also performed to confirm the applicability of the proposed large deflection theory. The experimental results agree well with those obtained from the exact large deflection theory.

Key Words: Elasticity, Material Testing, Nonlinear Bending, Large Deflection, Three-point Bending, Bending of Plastics, Elastica

1. Introduction

A point-bending test as a kind of material testing is often used to investigate the properties and behavior of materials because of simplicity and high reliability in testing.

In a simple bending theory assuming a deformation to be infinitesimal, the square of the first derivative is generally ignored supposing it small in comparison with unity in the expression of curvature. So the linear bending theory gives a good approximation when a deflection is small. In particular, this theory is used effectively for analyzing bending problems of the beam manufactured of the metallic materials. On the other hand, a large deflection can be imposed without exceeding the elastic limit in testing specimens made of the materials such as a spring steel and a high polymer (for example, plastics). In this case, the familiar simple linear bending theory is no longer applicable. A general study on the large deflection behavior of the materials with high deformability is of great technological and practical importance.

The large deformation problems have been dealt with by many researchers for a long time in relation to the buckling of members. Examples are the study on a circular plate subjected to a concentrated load by energy method, the successive approximation study on an elliptical plate subjected to a constant load, the studies on a cantilever subjected to a concentrated load and on a cantilever subjected to a distributed load and on a cantilever subjected to a concentrated load and a distributed load, and the study on a cantilever made of nonlinear materials subjected to a concentrated load. Then, the same problems related to a point-bending as taken up in this report are investigated by Conway applying the same method used for an analysis of cantilever and by Frisch-Pay assuming the reaction support to be movable in a horizontal direction. In each analysis, only the vertical balance of force component is taken into account. Therefore, these analyses are effective to the bending problem under a specific bending situation corresponding to the analytic basis. It is not so difficult to manufacture a bending device with movable reaction supports. Most of the usual bending devices, however, do not possess this function. From such a viewpoint, a different theoretical approach is necessary to grasp accurately a large deflection behavior in a widely used compact bending test where the reaction supports are immovable, that is, the supports are hinged. Moreover, it seems that the horizontal balance of force component should be taken into account.

The present report is concerned with a nonlinear large deflection problem on a symmetrical three-point bending of a simply supported beam subjected to a central concentrated load. The derived nonlinear differential equation is solved numerically by using the numerical technique...
(Runge-Kutta-Gill method) and the analytical solutions in terms of elliptic integrals are presented based on the elastica theory.

A large flexural experiment is performed and the experimental results are compared with ones based on the proposed large deflection theory.

A reduction method is proposed newly by which the representative flexural quantities in large deflection states can be obtained easily from the simple linear bending theory and then two reduction factors are defined in application of this method, and each value of these factors is also given graphically.

In a theoretical analysis, it is assumed that 1) this deflection theory holds in an elastic range, 2) Euler-Bernoulli's assumption is applicable, 3) a shear deformation is negligible, 4) an effect of friction between beam and end supports is not taken into account because the supports can rotate freely in the experiment, 5) a deviation of an actual point of force application from the neutral plane of the beam is ignored, and 6) the beam is inextensible.

2. Theoretical Analysis in a Symmetrical Three-point Bending

2.1 Numerical solution

In an elastic range obeying Hook's law, the relation between a bending moment \( M \) and a curvature \( 1/R \) is given by

\[
\frac{1}{R} = \frac{M}{EI}
\]

where, \( E \) is Young's modulus and \( I \) a second moment of area about the neutral axis of the beam.

As shown in Fig.1, the fixed support \( A \) is selected as the origin of the coordinate system, \( x \) is a horizontal distance from the origin, \( y \) a deflection (a vertical displacement in other words), and \( \theta \) a deflection angle.

The exact expression of curvature based on a differential geometry is written as follows:

\[
\frac{1}{R} = \frac{dy}{dx} \left/ \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2} \right. \quad \frac{dy}{dx} = \tan \theta
\]

When the vertical load \( P \) acts at the midspan \( C \), the reaction forces at the end supports \( A \) and \( B \) (a reaction force is a resultant force of a force applied in the vertical direction and a force applied in the horizontal direction) are aligned in the normal direction making a certain angle \( \theta_B \) to the horizontal axis \( x \). So the horizontal balance of these reaction force components, which is neglected in the conventional linear bending theory, must be considered newly. The reaction force \( Q \) at the end supports \( A, B \) and the horizontal component of \( Q \) are \( P/2 \cos \theta_B \), \( P \tan \theta_B/2 \), respectively, where \( \theta_B \) is a deflection angle at the end supports \( A, B \).

Only half portion of the beam (domain \( AC \)) needs to be analyzed because the beam deforms into a symmetrical shape.

The bending moment \( M \) at an arbitrary horizontal distance \( x \) is given as the sum of the moment \( M_j = P \theta / 2 \) due to the vertical load \( P \) and the moment \( M_c = (P \tan \theta_B / 2) \) due to the horizontal component of the reaction force \( Q \).

\[
M = M_1 + M_2 = \frac{P}{2} (x + \tan \theta_B \cdot y)
\]

\[
[0 \leq x \leq l]
\]

From Eqs. (1), (2), and (3), the following nonlinear differential equation is obtained.

\[
\frac{d^3 y}{dx^3} = -\frac{P}{2EI} (x + \tan \theta_B \cdot y) \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2}
\]

\[
[0 \leq x \leq l]
\]

The second-order differential equation (4) requires two boundary conditions as shown in Eq. (5).

\[
y|_{x=0} = y(0) = 0, \quad \left. \frac{dy}{dx} \right|_{x=l} = y'(l) = 0
\]

In this section, the Runge-Kutta-Gill method as a numerical analysis is adopted to solve the nonlinear equation (4) numerically. In order to facilitate the numerical calculation, the non-dimensional variables \( \xi \), \( \eta \) are introduced.

\[
\xi = \frac{x}{l}, \quad \eta = \frac{y}{l}, \quad \frac{dy}{d\xi} = \frac{dy}{dx} = \tan \theta
\]

Putting \( \sigma \) as a deflection parameter, the basic formula (4) may be rewritten as follows:

\[
\frac{d^2 \eta}{d\xi^2} = -\sigma (\xi + \eta, \frac{d\eta}{d\xi} + \eta) \left[ 1 + \left( \frac{d\eta}{d\xi} \right)^2 \right]^{1/2}
\]

\[
[0 \leq \xi \leq 1]
\]

where,

\[
\sigma = \frac{P\theta^2}{2EI}, \quad \tan \theta_B = \frac{dy}{d\xi}
\]
Corresponding to Eq. (5), the boundary conditions are shown as

$$\eta|_{t=0} = \eta(0) = 0, \quad \frac{d\eta}{ds}|_{t=1} = \eta'(1) = 0 \quad \cdots (8)$$

Now then, assuming \((d\eta/dx)|_{x=0} = 0\) and \((d\eta/dx)|_{x=1} = 0\), the well known analytical solutions in the linear bending theory are given as follows:

$$\begin{align*}
a^2 \frac{d^2 \eta}{dx^2} - a \eta & = -\frac{a}{2} (\xi^2 - 1) \\
\eta & = -\frac{a}{6} \xi^3 (\xi^2 - 3) \quad [0 \leq \xi \leq 1]
\end{align*} \quad \cdots (9)$$

The maximum bending stress \(\sigma_{\text{max}}\) occurring in the outer fibers of the beam at the midspan \(C\) is

$$\sigma_{\text{max}} = \frac{K}{E} \left( \frac{1}{1 + \eta'(0)} \right) \left[ 1 + \eta'(1) \right] \quad \cdots (10)$$

where, \(\xi\) is a distance from the neutral axis of the beam to the outer fibers.

On the other hand, the maximum bending stress in the linear bending theory is expressed as follows:

$$\sigma_{\text{max}} = \frac{K}{E} \left( \frac{1}{1 + \eta'(0)} \right) \quad \cdots (11)$$

2.2 Analytical solution

Introducing the elliptic integrals which are applied to the problem of elastica, the basic nonlinear differential equation (4) governing beam deflections can be solved analytically.

Denoting the arc length of the elastica curve of the beam by \(s\), the relations amongs \(s, R, \theta, x, \phi\) are given by

$$\begin{align*}
\frac{1}{R} &= -\frac{d\theta}{ds}, \quad dy = \tan \theta \\
dy &= ds \cdot \sin \theta, \quad dx = ds \cdot \cos \theta
\end{align*} \quad \cdots (12)$$

From Eqs. (1), (3), and (12), the following equation (13) is derived.

$$\frac{d\theta}{ds} = -\frac{P}{2EI} (x + \tan \theta \cdot y) \quad \{0 \leq x \leq l\} \quad \cdots (13)$$

Differentiating Eq. (13) with respect to \(s\), the following equation (14) is obtained.

$$\begin{align*}
a^2 \frac{d^2 \theta}{ds^2} &= -\frac{P}{2EI} \left( \frac{dx}{ds} + \tan \theta \cdot \frac{dy}{ds} \right) \\
&= -\frac{P}{2EI} \left( \cos \theta + \tan \theta \cdot \sin \theta \right) \\
&= -\frac{P}{2EI} \cos (\theta - \phi) \quad \cdots (14)
\end{align*}$$

Multiplying each side of Eq. (14) by \(2(d\theta/ds)\) and integrating these, the equation (14) is reduced to Eq. (15).

$$\frac{d\theta}{ds} = \frac{P}{EL} \frac{1}{\sin (\phi - \theta) - K} \quad \cdots (15)$$

where, \(K\) is an integral constant.

Considering the boundary conditions that \(\theta = \theta_0\) at the origin \((x = 0, y = 0)\) and \(M_x, (d\theta/ds)\) are zero at the origin, the constant \(K\) becomes zero. Consequently, the equation (15) yields the following equation (16).

$$\frac{d\theta}{ds} = -\sqrt{\frac{P}{EI}} \frac{\sin (\phi - \theta)}{\cos \theta_0} \quad \cdots (16)$$

The negative sign of the right side of Eq. (16) follows from the facts that the deflection angle \(\theta\) decreases with an increase in the arc length \(s\).

From Eqs. (12) and (16), the curvature \(1/R\) is expressed as a function of the deflection angle \(\theta\).

$$\frac{1}{R} = -\frac{d\theta}{ds} = \sqrt{\frac{P}{EI}} \frac{\sin (\phi - \theta)}{\cos \theta_0} \quad \cdots (17)$$

Now, putting as follows in Eq. (16),

$$\sqrt{\frac{P}{EI}} \cos \theta_0 \beta = \phi, \quad \sin (\phi - \theta) = \cos \phi$$

the formulas for the infinitesimal of \(s, \phi, x\) are derived.

$$\begin{align*}
ds &= -\frac{\beta}{\cos \phi} d\phi \\
&= -\sqrt{\frac{2}{1-(1/2)^2}} \sin \phi d\phi \quad \cdots (18) \\
dy &= -\sin \theta \cos \phi d\phi \\
&= -\sqrt{2} (\sin \theta \cdot \sin \phi \cos \phi) \cos \phi d\phi \quad \cdots (19) \\
dx &= -\frac{\beta}{\cos \phi} \cos \phi d\phi \\
&= -\sqrt{2} (\cos \phi \sin \theta \cdot \sin \phi \cos \phi) \cos \phi d\phi \quad \cdots (20)
\end{align*}$$

The domain of integration with respect to \(\phi\) takes from \(\phi = \arccos (\sqrt{\sin \theta_0}) = \pi/2\) to \(\phi = \frac{\pi}{2} \arccos (\sqrt{\sin \theta_0})\) when the deflection angle \(\theta\) is ranging from \(\theta_0 = \theta_\text{a}\) at the end supports A to \(\theta_c = \theta_0\) at the midspan C.

The required quantities such as the arc length, the deflection, and the horizontal distance can be obtained by applying the following Legendre-Jacobi's elliptic integrals (ii) of the first kind \(F(u, \nu)\) and the second kind \(E(u, \nu)\) to the above equations (18), (19), and (20).
\[ F(u, w) = \int_{0}^{w} \frac{d\phi}{\sqrt{1 - u^2 \sin^2 \phi}} \] ... (21)

\[ E(u, w) = \int_{0}^{w} \sqrt{1 - u^2 \sin^2 \phi} \ d\phi \] ... (22)

The arc length \( S \) from the origin \( A \) to the midspan \( C \) is obtained from Eq. (18).

\[ S = \int_{A}^{C} ds = -2 \beta \varphi \int_{0}^{\phi_c} \sqrt{1 - (1/2) \sin^2 \phi} \ d\phi = \sqrt{2} \beta \left( F\left( \frac{\sqrt{2}}{2}, \frac{\pi}{2} \right) - F\left( \frac{\sqrt{2}}{2}, \phi_c \right) \right) \] ... (22)

Similarly, the deflection \( y \) at an arbitrary position \( D \) between the supports \( A \) and \( C \) is expressed as follows from Eq. (19).

\[ y = \int_{0}^{D} dy = 2B \sin \theta_0 \cos \phi \]

\[ -\sqrt{2} \beta \cos \theta_0 \phi\left( \phi \right) \] ... (23)

where,

\[ \phi = \arccos \left( \sqrt{\sin \theta_0} \right) \]

\[ \phi\left( \phi \right) = 2 \left[ F\left( \frac{\sqrt{2}}{2}, \frac{\pi}{2} \right) - F\left( \frac{\sqrt{2}}{2}, \phi \right) \right] \]

Substituting \( \theta = 0, \phi = \phi_c = \arccos \left( \sqrt{\sin \theta_0} \right) \) into Eq. (23), the maximum deflection \( \eta_{\text{max}} \) at the midspan \( C \) is given by

\[ \eta_{\text{max}} = 2B \sin \theta_0 \cos \phi_c \]

\[ -\sqrt{2} \beta \cos \theta_0 \phi\left( \phi \right) \] ... (24)

From Eq. (20), the horizontal distance from \( A \) to \( D \) is expressed in the following equation (25).

\[ x = \int_{0}^{D} dx = 2B \cos \theta_0 \cos \phi \]

\[ + \sqrt{2} \beta \sin \theta_0 \phi\left( \phi \right) \] ... (25)

Setting \( \phi = \phi_c \) in Eq. (25), the half span of beam \( l \) which is equal to the maximum horizontal distance \( x_{\text{max}} \) is described as follows:

\[ l = 2B \cos \theta_0 \cos \phi_c + \sqrt{2} \beta \sin \theta_0 \phi\left( \phi \right) \] ... (26)

The ratios of \( y \), \( x \), and \( S \) to \( l \) are derived as follows, respectively, from Eqs. (22), (23), (25), and (26).

\[ \frac{(y)}{l} = \frac{\sqrt{2} \tan \theta_0 \cos \phi - \phi\left( \phi \right)}{\sqrt{2} \cos \phi_c + \tan \theta_0 \phi\left( \phi \right)} \] ... (27)

\[ \frac{(x)}{l} = \frac{\sqrt{2} \cos \phi + \tan \theta_0 \phi\left( \phi \right)}{\sqrt{2} \cos \phi_c + \tan \theta_0 \phi\left( \phi \right)} \]

\[ \frac{(S)}{l} = 1.8541 - F\left( \frac{\sqrt{2}}{2}, \phi_c \right) \]

\[ \sqrt{2} \cos \theta_0 \cos \phi_c + \sin \theta_0 \phi\left( \phi \right) \] ... (27)

where,

\[ F\left( \frac{\sqrt{2}}{2}, \frac{\pi}{2} \right) = 1.8541 \]

From Eqs. (24) and (26), the nondimensional maximum deflection \( \eta(1) \) and the deflection parameter \( \sigma \) are given by

\[ \eta(1) = \frac{\eta_{\text{max}}}{l} = \frac{\sqrt{2} \tan \theta_0 \cos \phi_c - \phi\left( \phi \right)}{\sqrt{2} \cos \phi_c + \tan \theta_0 \phi\left( \phi \right)} \] ... (29)

\[ \sigma = \frac{P \beta}{2EI} \cos \theta_0 \left( \sqrt{2} \cos \phi_c + \tan \theta_0 \phi\left( \phi \right) \right)^2 \] ... (30)

Putting \( Z \) as a section modulus, the maximum bending stress \( \sigma \) occurring in the outer fibers of the beam at an arbitrary position \( D \) is written as follows:

\[ \sigma = \frac{M}{Z} = \frac{E}{I} \epsilon \] ... (31)

Expressing Eq. (31) as a function of the deflection angle \( \theta \), \( \epsilon \) is reduced to the following equation (32).

\[ \sigma = \sqrt{\frac{EP}{I}} \sin \left( \theta_{\text{a}} - \theta \right) \] ... (32)

In the case of the symmetrical bending, the curvature and the maximum bending stress give the largest and highest values at the midspan \( C \), respectively, and those are calculable from the following equations (33) and (34).

\[ \frac{1}{R_{\text{max}}} = \frac{1}{R_{\text{a}}} = \frac{P \tan \theta_0}{EI} \] ... (33)

\[ \sigma_{\text{max}} = \sigma_{\text{a}} = \sigma \sqrt{\frac{EP}{I}} \tan \theta_0 \] ... (34)

As mentioned above, the representative flexural quantities are obtainable from the analytical solutions besides the numerical analysis as shown in Section 2.1.

2.3 Reduction method

There is another simple method, which is called the reduction method by the author, for estimating the deflection behavior in addition to the complicated theoretical analyses presented in Sections 2.1 and 2.2. Applying the reduction method, the representative flexural quantities such as a maximum deflection, an end slope, a maximum bending stress in large deflection states can be estimated easily from the conventional linear bending theory.

The reduction factors \( k_i \) for the nondimensional maximum deflection, and \( k_i \) for the end slope are defined in the following form.

\[ k_i = \frac{\eta(1)}{\eta(1)} \]

\[ k_i = \frac{\theta_i(1)}{\theta_i(0)} \] ... (35)
where, \( \eta(1) \), \( \eta'(0) \) are the nondimensional maximum deflection, the end slope, respectively, in the large deflection theory and \( \eta(1) \), \( \eta'(0) \) possess the same meanings as in the linear bending theory.

From Eq. (9), \( \eta(1) \) is described as follows:

\[
\eta(1) = \eta_{e+1} = \frac{a}{3} \frac{P t}{6 E I} \tag{36}
\]

Combining Eq. (36) with Eq. (35), the nondimensional maximum deflection \( \eta(1) \) is given by

\[
\eta(1) = \frac{P t}{6 E I} k_1 \tag{37}
\]

and the real maximum deflection \( \eta_{\text{max}} \) is expressed as

\[
\eta_{\text{max}} = \eta(1) \cdot \eta = \frac{P t}{6 E I} k_1 \tag{38}
\]

Similarly, \( \eta'(0) \) is obtained from Eq. (9).

\[
\eta'(0) = \left. \frac{d\eta(t)}{dt} \right|_{t=0} = \frac{a}{2} \frac{P t}{4 E I} \tag{39}
\]

From Eqs. (35) and (39), the end slope \( \eta'(0) \) is written as follows:

\[
\eta'(0) = \tan \theta_0 = \frac{P t}{4 E I} k_2 \tag{40}
\]

Substituting Eq. (40) into Eq. (34), the maximum bending stress \( \sigma_{\text{max}} \) at the midspan is given by

\[
\sigma_{\text{max}} = \frac{P t}{2 f e} \sqrt{k_1} \tag{41}
\]

To sum up, it can be said that the maximum deflection \( \eta_{\text{max}} \), the end slope \( \eta'(0) \), and the maximum bending stress \( \sigma_{\text{max}} \) in large deflection states are obtained by multiplying the maximum deflection \( \left[ \frac{P t}{6 E I} \right] \) in the linear bending theory by the reduction factor \( k_1 \), and by multiplying the end slope \( \left[ \frac{P t}{4 E I} \right] \) in the linear bending theory by the reduction factor \( k_2 \), and by multiplying the maximum bending stress \( \left[ \frac{P t}{2 f e} \right] \) in the linear bending theory by the square root of the reduction factor \( k_1 \), respectively.

In order to apply the reduction method, the values of the reduction factors must be calculated once previously for convenience from Eqs. (42) and (43) based on the exact large deflection theory.

\[
k_1 = \frac{3}{a} \eta(1) = 3 \left( \frac{2}{a} \right) \eta'(0) \frac{\cos \theta_0 \cos \phi_1 - \phi_1}{\cos \theta_0 \left( \frac{2}{a} \cos \phi_0 + \tan \theta_0 \cdot \phi_1 \right)} \tag{42}
\]

Incidentally, the reduction factor \( k_1 \) is also calculable from a variation of the following formula (44) using the relations shown in Eqs. (10), (11), and (13).

\[
k_1 = \left( 1 + \eta'(0) - \eta(1) \right) \tag{43}
\]

Now then, the relation between \( \eta_{\text{max}} \) and \( l \) can be determined from Eq. (20) having elliptic integrals. But it is noticed that considering in the following manner, the \( \eta_{\text{max}} - l \) relation can be known readily without using the elliptic integrals.

Under the conditions of \( y = \eta_{\text{max}} + \theta = 0 \) at the midspan, the following formulas (45) and (46) are obtained from Eq. (13) and from Eq. (16), respectively.

\[
\left. \frac{d\theta}{dt} \right|_{t=0} = -\frac{P}{2 E I} (l + \tan \theta_0 \cdot \eta_{\text{max}}) \tag{45}
\]

\[
\left. \frac{d\theta}{dt} \right|_{t=0} = -\sqrt{\frac{P}{E I}} \tan \theta_0 \tag{46}
\]

Equating Eq. (45) to Eq. (46), the relation between \( \eta_{\text{max}} \) and \( l \) is derived as follows:

\[
\sqrt{\frac{P}{E I}} (l + \tan \theta_0 \cdot \eta_{\text{max}}) = 2 \sqrt{\tan \theta_0} \tag{47}
\]

3. Theoretical Solutions and Experimental Results

A symmetrical three-point bending test is performed to confirm the applicability of the large deflection theory using a compact bending device with a loading system of various small weights. The loading support and both end supports are steel rollers with a certain roundness and the end supports are pivoted on ball bearings to eliminate friction between the beam and the supports.

In the experiment, the deflection angle \( \theta_0 \) at the end support and the maximum deflection \( \eta_{\text{max}} \) at the midspan are measured for several vertical loads \( P \). It is necessary to make a few calibrations because the roundness of the end supports, the thickness and the self-weight of the beam specimen have influence on the deflection measured from the neutral plane of the beam which is a standardized level and on the span length of the beam.

It should be noted that the roundness of the loading support has no effect at all.

The calibrating values \( \Delta \eta_{\text{max}} \) for
$\Delta y_{\text{max}}$ and $\Delta l$ for $l$ can be calculated from the following equations (48) and (49).

$$\Delta y_{\text{max}} = \left( r + \frac{h}{2} \right) \left[ \cos \theta_a - \cos (\theta_a + \theta_b) \right] \cdots (48)$$

$$\Delta l = \left( r + \frac{h}{2} \right) \left[ \sin (\theta_a + \theta_b) - \sin \theta_a \right] \cdots \cdots (49)$$

where, $\theta_a$ is the initial deflection angle at the end supports caused by the self-weight of the beam, $r$ is the radius of the end support rollers and $h$ is the thickness of the beam.

The beam specimen is a thin, polyvinyl chloride (PVC) plate (250 mm in length x 30 mm in width x 0.76 mm in thickness). Young's modulus of specimen $E$ is 3.39 GPa which is obtained experimentally from the following formula (50) based on the linear bending theory.

$$E = \frac{P}{4l} \left( \frac{d}{b} \right)$$

The diameters of the loading support roller and the end support rollers are 10 mm and 30 mm, respectively. The initial setup distance $2l$ between both end supports is 150 mm.

Figure 2 shows the relation between $\eta(1)$ and $\alpha$ obtained from Eq. (7) and (9). In the large deflection theory, the deflection parameter $\alpha$ reaches the maximum value $\alpha_{\text{max}} (= 0.834)$ when $\eta(1)$ is about 0.478 and then the vertical load can not be applied beyond the limit $\alpha_{\text{max}}$ since the beam slips through between both end supports. For $\alpha$ smaller than $\alpha_{\text{max}}$ as is evident from the figure, the nondimensional maximum deflection $\eta(1)$ takes two different values. Namely, it is noticed that there exist two distinct deflections (a small scale deflection and a large scale deflection) for the same vertical load. In a small scale deflection range ($\eta(1) \leq 0.478$), the difference between $\eta(1)$ obtained from the large deflection theory and $\eta(1)$ obtained from the linear bending theory becomes more significant as $\alpha$ increases, and moreover $\eta(1)$ in the large deflection theory is larger about 72% than $\eta(1)$ in the linear bending theory under the conditions of $\alpha = \alpha_{\text{max}}$. On the other hand, the difference widens greatly as $\alpha$ becomes lower in a large scale deflection range ($\eta(1) > 0.478$).

The experimental results shown together in the figure are in good agreement with ones obtained from the exact large deflection theory.

The relation of the end slope $\eta'(0)$ vs. $\alpha$, which is not described here as a graph, shows the same tendency as shown in Fig. 2. Then, it is clear that $\alpha$ takes $\alpha_{\text{max}}$ when $\eta'(0)$ is about 0.793 and that when $\alpha$ is equal to $\alpha_{\text{max}}$, $\eta'(0)$ in the large deflection theory is greater about 90% than $\eta'(0)$ in the linear bending theory.

Figures 3 and 4 show the relations of $\eta(0)$ vs. $\eta(1)$ and of $\theta_b$ vs. $\eta(1)$, respectively. Both $\eta(0)$ and $\theta_b$ in the large deflection theory tend to change curvi-linearly while those in the linear bending theory change linearly. The agreement between the results of the experiment and the exact large deflection theory is quite good in each figure.

Figure 5 shows the relation of $\alpha_{\text{max}}$ vs. $\eta(1)$. $\alpha_{\text{max}}$ becomes larger gradually as $\eta(1)$ increases while $\alpha_{\text{max}}$ obtained from the linear bending theory tends to be parabolic with a peak appearing when $\eta(1)$ is about 0.478. The experimental results coincide with ones from the exact large deflection theory in the figure.

Figure 6 shows the reduction factors $h_1$, $k_1$ vs. $\eta(1)$ on semilogarithmic scales. In other words, the reduction factors are
the scale factors for converting the certain flexural quantities in the linear bending theory to the corresponding flexural quantities in the large deflection theory. In the case of \( \eta(1) = 1 \) as an example, it may be recognized immediately from Fig.6 that the maximum deflection, the end slope, and the maximum bending stress are 7.6 (= \( \lambda_1 \)), 12.7 (= \( \lambda_2 \)), and 3.6 (= \( \lambda_3 \)) times, respectively, which are as large as those calculated from the linear bending theory.

Figure 7 shows the relations of \( \eta(1) \) vs. \( \alpha \) calculated from both the analytical solutions containing elliptic integrals and the numerical solutions using the numerical method (Runge-Kutta-Gill method). As a matter of course, both results obtained by the two different methods agree well with each other. This means that no significant difference is recognized between the two methods. Therefore, concerning a practical problem which is how to choose a proper method for the analysis, it is necessary to consider the labor, cost, and time consumed for calculations and so on besides the required flexural quantities.

Referring to the numerical and analytical method, both have the following advantages and disadvantages. In the numerical method, the representative flexural quantities such as the deflection, the end slope, the deflection angle and

Fig. 4 Relation between the deflection angle \( \theta_e \) at the end support and the nondimensional maximum deflection \( \eta(1) \)

Fig. 5 Relation between the maximum bending stress \( \sigma_{\text{max}} \) at the midspan of beam and the nondimensional maximum deflection \( \eta(1) \)

Fig. 6 Nomograph of the reduction factors \( \lambda_1, \lambda_2 \) for the nondimensional maximum deflection \( \eta(1) \)

Fig. 7 Comparison of the numerical solution with the analytical solution in relation of \( \eta(1) \) vs. \( \alpha \)
the bending stress at an arbitrary position of beam can not be obtained unless the deflection parameter $\alpha$ is known previously. In other words, this means that Young's modulus $E$, the second moment of area $I$, the vertical load $P$, and the initial setup distance $l$ must be presumed. On the other hand, the distance $l$ becomes useless for the analytical method while the deflection angle $\theta$ is required, but it is generally difficult to measure $\theta$ at an arbitrary position. When only the specified flexural quantities at the prescribed position such as the real maximum deflection $\gamma_{\text{max}}$, the maximum bending stress $\sigma_{\text{max}}$ at the midspan of the beam, and the distance $l$ are required, these can be obtained readily from the analytical solutions because it is easy to measure the end slope $\theta$. In use of these theoretical methods, the merits and demerits mentioned above must be taken into account. Meanwhile, it is sufficient to calculate the well known simple linear analytical solutions and to read out the reduction factors from the nomograph in the case of using the reduction method.

It should be noted that the calculations in this report have been carried out by using the microcomputer HZ-808 (Sharp Co., Ltd.).

4. Conclusions

The large deflection problem of a symmetrical three-point bending is analysed.

In deriving the basic formula governing beam deflections, the exact expression of curvature based on the differential geometry is used, and the bending moment produced by the horizontal component, which is ignored in the past analysis, of the reaction force at the end support is taken into account newly.

The derived second-order nonlinear differential equation is solved numerically by using the Runge-Kutta-Gill method. The behavior of the maximum deflection, the end slope, the deflection angle at the end support, and the maximum bending stress are made clear numerically.

On the other hand, it is pointed out that the derived nonlinear differential equation can be solved analytically by applying Legendre-Jacobi's complete and incomplete elliptic integrals of the first and the second kinds. The analytical solutions of the arc length, the maximum deflection, the horizontal distance, and the maximum bending stress are presented as functions of the deflection angle and the results obtained from these analytical solutions are compared with ones from the numerical solutions aforementioned.

Now then, the simple relating equation between the deflection angle at the end support and the maximum deflection at the midspan of the beam is found out without using the elliptic integrals.

A reduction method is proposed to obtain easily the representative flexural quantities such as the maximum deflection, the end slope, and the maximum bending stress in large deflection states from the conventional linear bending theory. Two reduction factors are defined and those are shown graphically as a nomograph.

Moreover, a large flexural experiment of a PVC plate is performed to confirm the applicability of the large deflection theory and the results are compared with those from the theoretical solutions. In the analysis, appropriate calibrations are made for the real maximum deflection and the span length of the beam because the support rollers have a certain definite roundness.

The author would like to acknowledge the experimental assistance of Mr. Ohki and Mr. Okumura who were students in those days and he is grateful to Dr. Iseki, Dr. Tomita, and Dr. Miyata for their helpful discussions.

References