Vibration Analysis by Substructure Synthesis Method*

(5th Report, Method with identified characteristic matrices)

By Masaaki OOKUMA** and Akio NAGAMATSU***

In this paper, a new substructure synthesis method is proposed. A total structure is divided into some components (substructures). The characteristic matrices (the mass, the damping and the stiffness matrices) of reduced degrees of freedom are identified for each component theoretically or by vibration tests. The identified characteristic matrices of all components are composed to make a reduced equation of motion of the total structure. There are three methods to identify the characteristic matrices of each component, namely Guyan's reduction, an identification from the transfer function and an identification from the modal parameters.

The vibration of a model structure composed of three triangular plates is calculated by substructure synthesis methods, and the results are compared with the calculated result by direct use of the finite element method (FEM). Next, the vibration of an actual crank-shaft of an internal combustion engine is calculated by the proposed method, and the result is compared with the experimental result and the calculated result by the component mode synthesis method (CMS).

Key Words: Vibration, Natural Frequency, Natural Mode, Modal Parameter, Identified Characteristic Matrices, Substructure Synthesis Method

1. Introduction

The authors roughly classified substructure synthesis methods into the methods using the natural modes of components. And the authors proposed the reduced impedance method⁽¹⁾ as a former type of the method and the component mode synthesis method (CMS)^{(2),(3)} as a latter type of the method.

In this paper, another type of the method is proposed. The method is named the characteristic matrix synthesis method. The method uses the characteristic matrices of reduced degrees of freedom of components.

The idea of the characteristic matrix synthesis method is itself not new. The super element method⁽⁴⁾ is one of the methods of this kind. In the super element method, the characteristic matrices calculated by FEM are reduced by Guyan's reduction for use.

The method proposed in this paper is different from the super element method. The characteristic matrices, whose numbers of degrees of freedom are small, are identified from experimental results or from analytical results. There are two types of the methods to identify the

characteristic matrices. The one⁽⁵⁾ uses the transfer function of a component, and the other uses the modal parameters. The latter method is explained in this paper.

After obtaining the characteristic matrices of a component, those matrices can be used like the finite elements of FEM.

The characteristic matrix synthesis method has the following advantages.

(1) All kinds of mathematical methods can be used.

(2) It can cooperate with FEM and other kinds of substructure synthesis methods.

(3) The identified characteristic matrices can be the data in a data-base system since the matrices can represent the dynamic characteristics of a component even in changed boundary condition.

(4) It is easy to use the experimental data for the method.

On the other hand, it has the following problem.

(5) The characteristic matrices can not be identified when the selection of degrees of freedom for the identification is inadequate.

In the present paper, the procedures of the super element method, the method to identify the characteristic matrices using the transfer function and the method using the model parameters are explained. As numerical examples, a simple model structure and an actual crank-shaft are analyzed by the methods. And these three kinds of methods are compared with one another.

2. Explanation of Identification of the Characteristic Matrices

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** Assistant,
Faculty of Engineering,
Tokyo Institute of Technology,
12-1, ookayama 2-chome, Meguro-ku,
Tokyo, 152 JAPAN.

*** Professor,
Faculty of Engineering,
Tokyo Institute of Technology.

2.1 Super element method

Omitting the damping factor in the explanation, the equation of motion of a component is written as

$$-\omega^2 \begin{bmatrix} M_{aa} & M_{ab} \\ M_{ba} & M_{bb} \end{bmatrix} \begin{Bmatrix} x_a \\ x_b \end{Bmatrix} + \begin{bmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{bmatrix} \begin{Bmatrix} x_a \\ x_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_b \end{Bmatrix} \dots\dots\dots (1)$$

where $[M]$ is the mass matrix, $[K]$ is the stiffness matrix. The degrees of freedom of the region a will be removed in the reduction process. The degrees of freedom of the remained region are denoted with b .

Omitting the term of inertia representation in Eq.(1), the displacement $\{x_a\}$ is represented by the displacement $\{x_b\}$.

$$\{x_a\} = -[K_{aa}]^{-1}[K_{ab}]\{x_b\} = [T]\{x_b\} \dots\dots\dots (2)$$

Using the matrix $[T]$ in Eq.(2), the mass matrix and the stiffness matrix are reduced to the same degrees of freedom of region b . The reduced mass matrix $[\bar{M}_{bb}]$ and the stiffness matrix $[\bar{K}_{bb}]$ are written as

$$[\bar{M}_{bb}] = [T]^T I \begin{bmatrix} M_{aa} & M_{ab} \\ M_{ba} & M_{bb} \end{bmatrix} I \begin{Bmatrix} T \\ I \end{Bmatrix} \dots\dots\dots (3)$$

$$[\bar{K}_{bb}] = [T]^T I \begin{bmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{bmatrix} I \begin{Bmatrix} T \\ I \end{Bmatrix} \dots\dots\dots (4)$$

An imaginary substructure represented with $[\bar{M}_{bb}]$ and $[\bar{K}_{bb}]$ is called a 'super element'. Superposing the reduced matrices of super elements (components), a reduced equation of motion of the total structure is obtained and solved. This method can be applied only to the analysis within the low frequency range in which the stiffness contributes much to the dynamic characteristics of the total structure comparing with the inertia influence. Because the term of the inertia is omitted in the reduction process, and no process to correct the approximation of the reduction is used in the super element method. The accuracy of the result by the super element method is much influenced by the selection of the remaining degrees of freedom. To obtain accurate results, not only the degrees of freedom where components connect to one another but also the degrees of freedom where the displacements are large and the masses are large should be retained. Therefore, it is difficult in general to select suitable degrees of freedom.

2.2 Identification using transfer function (response data)

The characteristic matrices are identified using the responses of forced vibration by this method directly. The method is just beginning to be studied.

An equation of motion of a component is written as

$$[K]\{x\} - \omega^2[M]\{x\} + j\omega[C]\{x\} = \{f\} \dots\dots\dots (5)$$

where $[C]$ is a viscous damping matrix.

Measured response $\{x\}$ and forces $\{f\}$ are relative to each other with Eq.(5). Therefore, substituting the measured responses $\{x\}$ and the forces $\{f\}$ in all

measured frequency range into Eq.(5), equation (5) is transformed into an equation (6).

$$[X][K] - [\omega^2][X][M] + j[\omega][X][C] = [F] \dots\dots\dots (6)$$

where $[X]$ is the response matrix on each row of which the response on each frequency is placed. The matrices $[\omega]$ and $[\omega^2]$ are diagonal matrices on the diagonal lines of which angular frequency ω and the square of angular frequency are placed, $[F]$ is the force matrix on each row of which a force vector on a frequency is placed.

Separating its real part and the imaginary part, Eq.(6) is rewritten as

$$\begin{bmatrix} [X_R] - [\omega^2][X_R] - [\omega][X_I] \\ [X_I] - [\omega^2][X_I] + [\omega][X_R] \end{bmatrix} \begin{bmatrix} [K] \\ [M] \\ [C] \end{bmatrix} = \begin{bmatrix} [F_R] \\ [F_I] \end{bmatrix} \dots\dots\dots (7)$$

where suffixes R and I denote the real part and the imaginary part respectively. Considering that the characteristic matrices are symmetric, equation (7) is solved, and the characteristic matrices are obtained. The identified characteristic matrices can be used for substructure synthesis methods.

2.3 Identification using modal parameter

This method is proposed in this paper. Necessary input-data for this method include modal parameters, i.e. natural frequencies and natural modes, of a component. The input-data are prepared in some ways. For example, they can be prepared using experimental data by technique of curve fitting, or they can be obtained in analytical way like FEM. The proposed method is explained here with the input-data (natural frequencies and natural modes) calculated analytically by FEM.

In preparation, a component is modeled with n nodal points, and the mass matrix and the stiffness matrix are obtained by FEM. It is assumed that the number of degrees of freedom of a nodal point is six.

Solving the eigenvalue problem with the characteristic matrices, the natural frequencies and the natural modes from the first order to the p -th are obtained. It is assumed that the natural modes are normalized with the mass matrix.

If all natural modes and natural frequencies are obtained, the inverse matrix of the mode matrix $[\Phi]$ ($6n \times 6n$) should exist. Then, the characteristic matrices can be identified perfectly as

$$[K] = ([\Phi]^{-1})^T [k_i] [\Phi]^{-1} \dots\dots\dots (8)$$

$$[M] = ([\Phi]^{-1})^T [I] [\Phi]^{-1} \dots\dots\dots (9)$$

where $[k_i]$ is the diagonal matrix on the diagonal line of which the modal stiffness $k_i = (2\pi f_i)^2$ is placed, $[I]$ is a

unit matrix whose number of degrees of freedom is $6n$.

We just want the natural frequencies, the natural modes and the transfer functions only between observed nodal points and connecting nodal points in general. And the number of observed nodal points and connecting points is not large generally. Then, it is efficient to use identified characteristic matrices, the number of degrees of freedom of which is small.

It is assumed that the vibration in analyzed frequency range can be represented accurately just with 1st to r -th modal parameters here. A sub-natural mode matrix $[\Psi]$ is made using sub-natural mode vector $\{\psi\}$.

When the number of degrees of freedom of observed nodal points and connecting nodal points is smaller than the number r , the degrees of freedom of some other nodal points are added to the matrix $[\Psi]$ to make the matrix a square matrix. On the other hand, when the number of degrees of freedom of observed nodal points and the connecting nodal points is larger than the number r , some higher order natural modes are added to the matrix to make the matrix a square matrix. It is not always necessary for the added modes to be accurate natural modes so far as they do not have bad influence on the vibration within the analyzed frequency range. It goes without saying that it is the best that the accurate higher order of the natural modes are adopted.

If the inverse matrix $[\Psi]^{-1}$ exists, we get

$$[\bar{K}] = ([\Psi]^{-1})^T [\bar{Q}] [\Psi]^{-1} \quad \dots\dots\dots (10)$$

$$[\bar{M}] = ([\Psi]^{-1})^T [I] [\Psi]^{-1} \quad \dots\dots\dots (11)$$

These are identified characteristic matrices, the number of degrees of freedom of which is small. These matrices can be used as the data of a component, because they represent the dynamic characteristics of the component accurately within the analyzed frequency range.

However, there is a case that the inverse matrix $[\Psi]^{-1}$ can not be obtained. The following are some of the reasons why the matrix $[\Psi]$ is sometimes nearly singular.

(1) A case that only nodal points on node lines or invalid freedom of nodal points are selected.

(2) An unfortunate case that a matrix $[\Psi]$ made with elements of natural modes of selected nodal points is nearly singular.

In these unlucky cases, the number of degrees of remained freedom should be changed or some of the selected points should be exchanged for others.

From a viewpoint of physical consideration, the identified characteristic matrices $[K]$ and $[M]$ must be positive definite matrices. Then, it is proved below that the identified matrices calculated by Eq.(10) and (11) are positive definite (Proof 1) and that the

natural frequencies and the natural modes calculated in the identified matrices by Eq.(10) and (11) correspond to the original values of the structure (Proof 2).

[Proof 1] Since it is assumed that the matrix $[\Psi]$ is positive definite (the inverse matrix $[\Psi]^{-1}$ exists), any vector $\{x\}$ the number of degrees of freedom of which is the same as that of the matrix $[\Psi]$ can be represented with a combination of the columns of the matrix $[\Psi]$ as

$$\{x\} = [\Psi] \{\xi\} \quad \dots\dots\dots (12)$$

where vector $\{\xi\}$ is a coefficient vector. To prove the positive definition of the matrix $[\bar{K}]$ is equivalent to proving that the value of Eq.(13) is always plus.

$$\{x\}^T [\bar{K}] \{x\} \quad \dots\dots\dots (13)$$

Substituting Eq.(10) and Eq.(12) into Eq.(13), we get

$$\begin{aligned} \{x\}^T [\bar{K}] \{x\} &= \{\xi\}^T [\Psi]^T ([\Psi]^{-1})^T [\bar{Q}] [\Psi]^{-1} [\Psi] \{\xi\} \\ &= \{\xi\}^T [\bar{Q}] \{\xi\} \\ &= \sum_{i=1}^s \bar{Q}_i^2 \xi_i^2 > 0 \quad \dots\dots\dots (14) \end{aligned}$$

where s means the number of degrees of freedom of a vector $\{x\}$ and a matrix $[\Psi]$, and ξ_i is the i -th element in a vector $\{\xi\}$. It is found from Eq.(14) that the identified matrix $[\bar{K}]$ is positive definite as far as the vector $\{x\}$ is not a zero vector.

Exchanging $[\bar{K}]$ and $[\bar{Q}]$ to $[\bar{M}]$ and $[I]$ respectively, it is clear that the identified mass matrix $[\bar{M}]$ is positive definite, too.

[Proof 2] Equation (15) of an eigenvalue problem is solved.

$$([\bar{K}] - \lambda [\bar{M}]) \{x\} = \{0\} \quad \dots\dots\dots (15)$$

Substituting Eq.(10), (11) and (12) into Eq.(15), and multiplying Eq.(15) by $[\Psi]^T$ from left side, we get

$$([\bar{Q}] - \lambda [I]) \{\xi\} = \{0\} \quad \dots\dots\dots (16)$$

With Eq.(16), s eigenvalues are obtained. The i -th natural frequency is $\bar{Q}_i/2\pi$, namely f_i .

Substituting $\bar{Q}_i^2$ into λ in Eq.(16), we get $\xi_i = 1$ and $\xi_j (j \neq i) = 0$. The i -th natural mode calculated with the identified characteristic matrices is none other than the i -th column of the matrix $[\Psi]$.

3. Examples

3.1 Simple plate model

A structural model in Fig.1 composed of three triangular plates is analyzed by the three kinds of methods. The bending vibration of the structure is analyzed. The components are connected only with the degrees of freedom of translation.

At the first step, each component is analyzed within lower frequency range than

5000 Hz respectively and its characteristic matrices are identified. Table 1 shows the natural frequencies of the component A in Fig.1. The results in the column 'FEM' are calculated by the super element method. In this analysis, three black nodal points and one white nodal point are used as the remaining nodal points. The column 'CMS' shows the results by CMS. The natural frequencies by the characteristic matrices identified with the modal parameters, which are obtained by FEM, are shown in the column 'MK from modal P.'. The natural frequencies calculated by the characteristic matrices which are identified with the force vibration responses at the 4 nodal points are shown in the column 'MK from response'.

The results of the triangular component B are shown in Table 2. Six nodal points are adopted as the remaining nodal points in this case.

According to Table 1 and Table 2, it is found that the accuracy of higher order of frequency by S.E. becomes worse. Only the first order of the natural frequency is as accurate as the values in other columns. Because Guyan's reduction is adopted in the process for omitting some of degrees of freedom and nothing is done to correct the approximation.

On the other hand, the results by CMS are accurate even though Guyan's reduction is used in a part of the process of CMS. CMS has a process to correct the approximation of the reduction.

The results with the identified characteristic matrices from modal parameters correspond well with the results by FEM. It is natural because the modal parameters are obtained by FEM and used to identify the characteristic matrices. But these matrices, the number of degrees of freedom of which is small, are very useful.

The results with the identified characteristic matrices from transfer functions are accurate, too. It is not necessary for the higher orders of the natural frequencies than the analyzed frequency range to be accurate. Because those represent the residual influence.

Next, the total structure is analyzed superposing the characteristic matrices of the three components. The calculated natural frequencies of the total structure are shown in Table 3.

The results by CMS and by the method using the identified characteristic matrices from modal parameters correspond well with the results by FEM on the whole analyzed frequency range.

The results by the method using the identified characteristic matrices from transfer functions are not very accurate on lower part of the analyzed frequency range. The residual compliance influences the error.

The high order of the natural frequencies cannot be obtained accurately by the super element method. It is similar to the results about each component.

Fig.1 Triangular simple plate model

Table 1 Natural frequencies of component A

Order	FEM	CMS	S.E.	M K from modal P.	M K from response
1	1218	1219	1238	1218	1231
2	4720	4722	5609	4718	4751
3	5211	5228	6147	5205	5201
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4	12058	12114	15994	12031	13876

(Hz)

Table 2 Natural frequencies of component B

Order	FEM	CMS	S.E.	M K from modal P.	M K from response
1	236	236	236	236	236
2	620	620	621	620	620
3	2827	2834	2864	2825	2825
4	4836	4895	4979	4832	4832
5	5321	5335	5599	5318	5317
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6	9062	9600	9635	9055	8195

(Hz)

Table 3 Natural frequencies of triangular model

Order	FEM	CMS	S.E.	M K from modal P.	M K from response
1	621	621	622	631	422
2	976	976	982	981	790
3	2102	2106	2175	2177	2481
4	2942	2943	3136	3232	3066
5	3170	3174	3270	3239	3173
6	4720	4723	5329	4718	4751
7	4720	4723	5603	4718	4751
8	5035	5075	6147	5034	4935
9	5276	5291	6147	5272	5293

(Hz)

3.2 Actual crank-shaft

An actual crank-shaft shown in Fig.2 is analyzed by these methods. Since it is hard to analyze it by direct use of FEM, the crank-shaft is divided into 20 components. The shaft-parts except for the component No.1 are modeled using isoparametric solid elements (2nd) with 51 nodal points 3. The component No.1 is a model with 81 nodal points. The crank-parts are divided using tetra elements with 111 nodal points. The fly-wheel is done using isoparametric thick shell finite elements with 88 nodal points. Assuming the boundary condition that the connecting surfaces never change their shapes, all components are connected with one another. The natural frequencies of the crank-shaft are shown in Table 4. The experimental results are shown in column 'EXP'. The column 'CMS' shows the results by CMS. The results in the column A are obtained with the characteristic

matrices which are identified with the modal parameters (natural frequencies and natural modes) at the degrees of freedom of 4 nodal points (one point in the components No.1, 6, 14 and 19 respectively). The column B shows the results by the analytical process explained in Fig.4. The results in the column C are obtained by the analytical process shown in Fig.5. The column 'S.E.' shows the results by the super element method.

A response (transfer function) of the Y-direction at the nodal point F in Fig.2 is drawn in Fig.6 on the case that the nodal point F is excited by a force (1N). Figure 7 shows the same response, too.

Fig.2 Crank-shaft

Fig.3 Modeling of crank-part of crank-shaft

Table 4 Natural frequencies of simple plate model

Order	EXP.	CMS	A	B	C	S.E.
1	200	203	205	205	209	206
2	245	258	257	262	260	263
3	425	401	385	405	407	417
4	513	500	497	509	497	522
5	541	539	539	553	554	541
6	694	664	656	670	680	736
7	907	859	851	860	863	962
8	1086	1139	1134	1140	1150	1145
9	1131	1102	1103	1103	1097	1182
10	1352	1354	1355	1371	1385	1372
11	1674	1688	1687	1705	1711	2051
12	1714	1484	1484	1486	1491	1702
13	—	1743	1743	1744	1735	2223
14	—	1749	1748	1749	1741	2604

(Hz)

Figure 6 tells that the transfer functions calculated by CMS and Cal.A correspond well with the experimental ones. Figure 7 tells that the results by Cal.A, B and C correspond well with one another and that the results by S.E. is not so accurate as the other ones.

Thus, the identification with the modal parameters can be used with other methods, for example CMS. Using this identification with other methods, complex and large structures can be analyzed highly efficiently and systematically.

Fig.4 Process of Cal.B

Fig.5 Process of Cal.C

Fig.6 Transfer function
Excited point: F (Y-direction)
Measured point: F (Y-direction)

Fig.7 Transfer function
Excited point: F (Y-direction)
Measured point: F (Y-direction)

4. Conclusions

In the present paper, a characteristic matrix synthesis method is proposed and three kinds of identifications for the characteristic matrices are explained. Next, as numerical examples, a simple plate model and an actual crank-shaft are analyzed by a few combinations of the methods. Then it is made clear that the characteristic matrix synthesis method is useful to analyze complex and large structures in cooperation with other methods.

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