Non-linear Vibrations of Beams with Attached Spring-mass System*

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A study is made of the coupled non-linear vibrations of a simply supported beam, with ends restrained elastically in the direction of length of beam, carrying a concentrated mass to which a spring-mass system is attached, and subjected to a transverse periodic force at an arbitrary point under the influence of gravity. The solution is derived for the steady-state vibration in the first mode resonance region with first approximation. Numerical results of the amplitude responses of the concentrated mass on the beam and the mass of the spring-mass system are presented for the principal resonance and the second and third order higherharmonic resonances, and then the occurrence of these resonances, and especially the effect of damping in the spring-mass system on the higherharmonic resonances are investigated. The stability of the steady-state solutions is also investigated for the case of no-damping.

Key Words: Vibration, Non-linear Forced Vibration, Coupled Vibration, Simply Supported Beam, Spring-mass System, Higherharmonic Resonance, Stability

1. Introduction

There are many examples of machines and structures which have the configuration of a beam with heavy weight at an arbitrary point. The damped vibrations and/or the vibration controls of such a system have been studied by many authors. Herman et al.⁽¹⁾ studied the vibrations of a cantilever beam having a concentrated mass which has been supported by a damped spring; Tsujioka⁽²⁾⁽³⁾ analysed the linear vibrations of a beam having a damped spring-mass system and clarified the vibration characteristics; and also Snowdon⁽⁴⁾ and Ohmata⁽⁵⁾ each analysed the linear vibrations of a beam and a mass in the case that a dynamic absorber is attached directly to the beam or attached to a concentrated mass on the beam, and then clarified the control characteristics of vibratory beam. Saito et al.⁽⁶⁾ investigated the principal and higherharmonic resonances of non-linear vibrations of a beam, with ends restrained to move in the longitudinal direction, carrying a concentrated mass.

This paper studies the forced vibrations of a simply supported beam, with ends restrained elastically in the direction of length of beam, carrying a concentrated mass to which a spring-mass system is attached. Due to the end restraint the axial force of the beam is variable with deflection and therefore the system is non-linear. Numerical results are represented for the case of a complete end

restraint, where the non-linear effect is considered great, and the effects of spring-mass system on the occurrence of the principal and higherharmonic resonances and on the non-linear vibration characteristics are investigated. Since we deal only with a comparatively small amplitude, we neglect the effects of large deflection and longitudinal inertia of the mass of beam. In the analysis we assume that the natural frequency of the attached spring-mass system is nearly equal to the fundamental natural frequency of the beam and the region of frequencies studied is limited to that of first mode resonance. Taking the mode function of linear vibrations of a simply supported beam with a concentrated mass as the approximate deflection function in the problem, non-linear ordinary differential equations are derived by applying Galerkin's method to equations of motion, and they are solved by applying the principle of harmonic balance. The stability of solution is also investigated in the case of no-damping and the results are plotted in figures. Although the numerical calculation has not been carried out, the analytical solution derived is of the form which makes it possible to calculate sub-harmonic resonances also.

2. Analysis

Fig.1 shows a simply supported beam, with the right end restrained elastically in the direction of length of beam by a spring of stiffness k_1 , carrying a concentrated mass M at an arbitrary point a to which a spring-mass system (k, m, c) is attached. The non-linear transverse vibrations of such a beam under the influence of gravity are studied. We assume that the concentrated mass M is subjected to a transverse periodic force $F = F_0 \sin \omega t$ and the thickness of beam is small in compari-

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son with its length l .

Taking the coordinate axis as shown in Fig.1 and denoting by $v(x, t)$ the instantaneous deflection of beam, by $w(t)$ the displacement of mass m in the direction of y -axis, and by P the force exerted to beam by spring-mass system through mass M respectively, the equations of motion of beam and mass m are derived as follows:

$$\begin{aligned} EI v_{xxxx} - N v_{xx} + \rho A v_{tt} + M \delta(x-a) v_{tt} \\ - J \delta(x-a) v_{xxxx} + c_0 v_{tt} - P \delta(x-a) \\ - F_0 \delta(x-a) \sin \omega t - \rho A g - M g \delta(x-a) = 0 \end{aligned} \quad (1)$$

$$\begin{aligned} m w_{tt} + k \{w(t) - v(a, t)\} + c \{w(t) - v(a, t)\}_t \\ - m g = 0 \end{aligned} \quad (2)$$

where

$$P = k \{w(t) - v(a, t)\} + c \{w(t) - v(a, t)\}_t \quad (3)$$

and EI is the flexural rigidity of beam, ρA is the mass of beam per unit length, J is the mass moment of inertia of concentrated mass M , c_0 is the coefficient of viscous damping, δ is Dirac's delta function, the subscripts x and t represent differentiation with respect to x and t respectively, and N is the axial force developed due to elastic end restraint (k_1) and is given by

$$\begin{aligned} N = N_0 + \frac{EA'}{2l} \int_0^l (v_x)^2 dx \\ A' = A / \left\{ 1 + \frac{EA}{k_1 l} \right\} \end{aligned} \quad (4)$$

where N_0 is an initial axial tensile force.

The solution can be obtained by solving the equations of motion of beam [Eq.(1)] and mass m [Eq.(2)] simultaneously. The deflection function v is assumed as the sum of a series of a finite number of separated variable terms of the form:

$$v(x, t) = v_0(x) + \sum_{i=1}^q Y_i(x) T_i(t) \quad (5)$$

where v_0 is the static deflection of beam

Fig.1 Geometry of system

under the influence of gravity, and Y_i and T_i are the i th order deflection and time functions of beam, respectively. Now, the main factors that influence the deflection function of beam may be the transverse and rotatory inertias of concentrated mass M and the interactive force of mass-spring system. If we suppose that the rotatory inertia of mass M is not so great in the problem and the concentrated mass on the beam is replaced by the sum of masses M and m , instead of the interactive force acting on the mass M , the deflection function under discussion can be represented approximately by the deflection function of beam which has only a concentrated mass at an arbitrary point. Then, we take the mode function v_i of linear vibration of a simply supported beam having a concentrated mass as the approximate deflection function, which is given by

$$\begin{aligned} v_i(x) = \left\{ \sin \frac{\nu_i(l-a)}{l} / \sin \nu_i \right\} \sin \frac{\nu_i x}{l} \\ - \left\{ \sinh \frac{\nu_i(l-a)}{l} / \sinh \nu_i \right\} \sinh \frac{\nu_i x}{l} \\ - U(x-a) \left\{ \sin \frac{\nu_i(x-a)}{l} - \sinh \frac{\nu_i(x-a)}{l} \right\} \end{aligned} \quad (6)$$

where $U(x-a)$ is a unit step function and ν_i indicate eigenvalues which are roots of the following frequency equation:

$$\begin{aligned} \nu_i \left[\left\{ \sin \frac{\nu_i(l-a)}{l} / \sin \nu_i \right\} \sin \frac{\nu_i a}{l} \right. \\ \left. - \left\{ \sinh \frac{\nu_i(l-a)}{l} / \sinh \nu_i \right\} \sinh \frac{\nu_i a}{l} \right] \\ - \frac{2\rho A l}{M} = 0 \end{aligned} \quad (7)$$

Especially when a concentrated mass is at a supporting end, Eq.(6) is replaced by $Y_i(x) = \sin(i\pi x/l)$.

Now, we assume that the natural frequency of the attached spring-mass system is nearly equal to the fundamental natural frequency of a beam having a concentrated mass, and that the region of frequencies to be analysed is limited to that of first mode resonance, where the effect of spring-mass system is considered great. Within the region of frequencies mentioned above the effect of higher-mode function on the deflection curve is so little that the use of only a fundamental mode function is good enough for analysis.⁽⁸⁾

Thus, taking $q=1$ in Eq.(5), first approximation solution will be sought. The static equilibrium deflection v_0 which represents the effect of gravity is assumed to be represented by Y_1 also. Then Eq.(5) becomes

$$\begin{aligned} v(x, t) = \bar{d} Y_1(x) + Y_1(x) T_1(t) \\ = r \frac{v_1(x)}{v_1(a)} \bar{T}_1(t) \end{aligned} \quad (8)$$

where

$$Y_1(x) = r \frac{v_1(x)}{v_1(a)} \quad (9)$$

$$\bar{T}_1(t) = \bar{d} + T_1(t) \quad (10)$$

and r is the radius of gyration of beam cross section, $T_1(t)$ represents dimensionless vibratory deflection of beam at $x=a$ and $\bar{d}$ represents similarly dimensionless static equilibrium deflection under the influence of gravity at $x=a$. Substituting Eq. (8) into Eq. (2), Eq. (2) becomes

$$\left. \begin{aligned} w_{,tt} + 2c_d w_{,t} + p_m^2 w &= \Phi(t) \\ \Phi(t) &= 2c_d r \bar{T}_{1,t} + p_m^2 r \bar{T}_1 + g \end{aligned} \right\} \quad (11)$$

where $c_d = c/2m$, $p_m^2 = k/m$. The solution of Eq. (11) for the steady-state vibration is obtained as

$$w = \frac{1}{p_d} \int_0^t e^{-c_d(t-t')} \Phi(t') \sin p_d(t-t') dt' \quad (12)$$

where $p_d^2 = p_m^2 - c_d^2$ and the transient-state vibratory terms have been neglected after integration.

Using Eq. (8) and Eq. (12), Eq. (3) is rewritten as follows:

$$\left. \begin{aligned} P &= \frac{k}{p_d} \int_0^t e^{-c_d(t-t')} \Phi(t') \sin p_d(t-t') dt' \\ -kr \bar{T}_1 + \frac{c}{p_d} \left\{ \int_0^t e^{-c_d(t-t')} \Phi(t') \sin p_d(t-t') dt' \right\}_{,t} \\ -cr \bar{T}_{1,t} \end{aligned} \right\} \quad (13)$$

Now, we introduce the following dimensionless quantities:

$$\left. \begin{aligned} f &= \frac{F_0}{\rho A l g}, \quad K = \frac{kr}{\rho A l g}, \quad K_1 = \frac{k_1 l}{EA}, \\ R &= \frac{J}{\rho A l r^2}, \quad S_0 = \frac{N_0}{\rho A l g}, \quad a = \frac{a}{l}, \\ \beta &= \frac{p_d}{p_b} \left[p_d^2 = p_m^2 - (c/2m)^2, \quad p_b^2 = \frac{\pi^4 E r^2}{\rho l^4} \right], \\ \Gamma &= \frac{M}{\rho A l}, \quad r = \frac{m}{\rho A l}, \quad \varepsilon = \frac{p_m}{p_b} \left[p_m^2 = \frac{k}{m} \right], \\ \zeta &= \frac{c_0}{2\rho A p_b}, \quad \mu = \frac{l}{r}, \quad \xi = \frac{g}{r p_b^2}, \quad \sigma = \frac{c}{2m p_b}, \\ \tau &= \frac{\omega t}{n_0} [n_0 = 1, 2, 3, \dots], \quad Q = \frac{\omega}{p_b} \end{aligned} \right\} \quad (14)$$

Applying Galerkin's method to Eq. (1) by using the deflection function of Eq. (9), a non-linear ordinary differential equation in terms of time function $\bar{T}_1$ is obtained as follows:

$$\left. \begin{aligned} \bar{T}_{1,rr} + \left(\frac{2n_0 \phi_1}{Q} \right) \bar{T}_{1,r} + \left(\frac{n_0}{Q} \right)^2 \phi_2 \bar{T}_1 - \left(\frac{n_0}{Q} \right)^2 \phi_3 \bar{T}_1^3 \\ - \left(\frac{n_0}{Q} \right)^2 \Psi(\tau) = 0 \end{aligned} \right\} \quad (15)$$

where

$$\left. \begin{aligned} \phi_1 &= (H_3 \zeta + H_6 \gamma \sigma) \frac{\phi_4}{H_9}, \\ \phi_2 &= \left(\frac{H_1}{\pi^4 H_9} - \frac{S_0 \xi H_2}{\rho H_9} + \frac{K \xi H_8}{H_9} \right) \phi_4, \\ \phi_3 &= \frac{K_1 H_2 H_8 \phi_4}{2\pi^4 H_9 (1 + K_1)}, \quad \phi_4 = 1 / \left(1 + \frac{R H_5}{\mu^2 H_9} \right), \\ \phi_5 &= \frac{\xi H_8 \phi_4}{H_9}, \\ \Psi(\tau) &= \left[\frac{H_7}{H_9} + \Gamma + \gamma + f \sin n_0 \tau \right. \\ &\quad + \frac{n_0 K (\varepsilon^2 - 2\sigma^2)}{\beta Q} \int_0^\tau e^{-(n_0 \sigma / \beta Q)(\tau - \tau')} \\ &\quad \times \left\{ \frac{2\sigma Q}{n_0 \varepsilon^2} \bar{T}_{1,r}(\tau') + \bar{T}_1(\tau') \right\} \\ &\quad \times \sin \left\{ \frac{n_0 \beta}{Q} (\tau - \tau') \right\} d\tau' \\ &\quad + \frac{2n_0 \sigma K}{Q} \int_0^\tau e^{-(n_0 \sigma / \beta Q)(\tau - \tau')} \\ &\quad \times \left\{ \frac{2\sigma Q}{n_0 \varepsilon^2} \bar{T}_{1,r}(\tau') + \bar{T}_1(\tau') \right\} \\ &\quad \times \cos \left\{ \frac{n_0 \beta}{Q} (\tau - \tau') \right\} d\tau' \left. \right] \phi_5 \end{aligned} \right\} \quad (16)$$

and $H_i (i=1 \sim 9)$ are the following dimensionless constants determined by deflection function:

$$\left. \begin{aligned} H_1 &= \frac{l^3}{r} \int_0^1 Y_{1,xxxx} v_1 dx, \quad H_2 = \frac{l}{r} \int_0^1 Y_{1,xx} v_1 dx, \\ H_3 &= \frac{1}{r l} \int_0^1 Y_1 v_1 dx, \quad H_4 = \frac{1}{r} (Y_1 v_1)_{x=a}, \\ H_5 &= -\frac{l^2}{r} (Y_{1,xx} v_1)_{x=a}, \quad H_6 = v_1(a), \\ H_7 &= \frac{1}{l} \int_0^1 v_1 dx, \quad H_8 = \frac{l}{r^2} \int_0^1 Y_{1,x}^2 dx, \\ H_9 &= H_3 + \Gamma H_4 \end{aligned} \right\} \quad (17)$$

The integer n_0 introduced in the analysis is chosen such that the following Eq. (18) expresses a combination of principal and higherharmonic resonances or a combination of principal and subharmonic resonances, and for the former combination n_0 is unity.

The steady-state solution of Eq. (15) is assumed by the following series up to third order higherharmonic terms:

$$\bar{T}_1(\tau) = A_0 + \sum_{n=1}^3 (A_n \cos n\tau + B_n \sin n\tau) \quad (18)$$

Substituting Eq. (18) into Eq. (5) and applying the principle of harmonic balance, the following seven coupled third order algebraic equations for seven unknowns $A_0, A_n, B_n (n=1, 2, 3)$ are obtained:

$$\left. \begin{aligned}
 f_0 &\equiv (\psi_2 - K\psi_5)A_0 - \psi_3 L_0 - \psi_5 \left(\frac{H_7}{H_6} + \gamma + \Gamma \right) = 0, \\
 f_{cn} &\equiv \left[-n^2 + \left(\frac{n_0}{Q} \right)^2 \left[\psi_2 - K\psi_5 \left(\left(\frac{n_0^2 \epsilon^2}{Q^2} - \frac{4n^2 \sigma^2}{\epsilon^2} \right) M_n \right. \right. \right. \\
 &\quad \left. \left. + \frac{4n_0 n \sigma}{Q} N_n \right) \right] A_n \\
 &\quad + \left[\frac{2n_0 n \psi_1}{Q} + K\psi_5 \left(\frac{n_0}{Q} \right)^2 \left(\left(\frac{n_0^2 \epsilon^2}{Q^2} - \frac{4n^2 \sigma^2}{\epsilon^2} \right) N_n \right. \right. \\
 &\quad \left. \left. - \frac{4n_0 n \sigma}{Q} M_n \right) \right] B_n - \psi_3 L_{n+3} \left(\frac{n_0}{Q} \right)^2 = 0, \\
 f_{sn} &\equiv \left[-n^2 + \left(\frac{n_0}{Q} \right)^2 \left[\psi_2 - K\psi_5 \left(\left(\frac{n_0^2 \epsilon^2}{Q^2} - \frac{4n^2 \sigma^2}{\epsilon^2} \right) M_n \right. \right. \right. \\
 &\quad \left. \left. + \frac{4n_0 n \sigma}{Q} N_n \right) \right] B_n \\
 &\quad - \left[\frac{2n_0 n \psi_1}{Q} + K\psi_5 \left(\frac{n_0}{Q} \right)^2 \left(\left(\frac{n_0^2 \epsilon^2}{Q^2} - \frac{4n^2 \sigma^2}{\epsilon^2} \right) N_n \right. \right. \\
 &\quad \left. \left. - \frac{4n_0 n \sigma}{Q} M_n \right) \right] A_n - \psi_5 L_{n+3} \left(\frac{n_0}{Q} \right)^2 \\
 &\quad - \delta_{n0}^n f_{\psi_5} \left(\frac{n_0}{Q} \right)^2 = 0 \quad (n=1, 2, 3)
 \end{aligned} \right\} \dots\dots\dots (19)$$

$$\left. \begin{aligned}
 M_n &= \frac{\left(\frac{n_0 \epsilon}{Q} \right)^2 - n^2}{\left\{ \left(\frac{n_0 \epsilon}{Q} \right)^2 - n^2 \right\}^2 + 4n^2 \left(\frac{n_0 \sigma}{Q} \right)^2}, \\
 N_n &= \frac{\frac{2n_0 n \sigma}{Q}}{\left\{ \left(\frac{n_0 \epsilon}{Q} \right)^2 - n^2 \right\}^2 + 4n^2 \left(\frac{n_0 \sigma}{Q} \right)^2}
 \end{aligned} \right\} \dots\dots\dots (20)$$

where δ_{n0}^n is Kronecker's delta and $L_0 \sim L_6$ are functions of A_0, A_n, B_n ($n=1, 2, 3$), which are given in Appendix.

Equations (19) are solved by using the Newton-Raphson technique for seven unknowns A_0, A_n, B_n . An effective value A_{EM} of dimensionless vibratory amplitude of concentrated mass M is defined as follows:

$$A_{EM} = \left\{ \frac{1}{2} \sum_{n=1}^3 (A_n^2 + B_n^2) \right\}^{1/2} \dots\dots\dots (21)$$

Substituting Eq. (11) and Eq. (18) into Eq. (12) and neglecting the transient-state vibratory terms after integration, the steady-state vibratory solution of mass m of spring-mass system is obtained as follows:

$$\begin{aligned}
 \frac{w}{r} &= \frac{\xi}{\epsilon^2} + A_0 + \left(\frac{n_0 \epsilon}{Q} \right)^2 \sum_{n=1}^3 \left[\left(\left(M_n + \frac{2n_0 n \sigma}{n_0 \epsilon^2} N_n \right) A_n \right. \right. \\
 &\quad \left. \left. - \left(N_n - \frac{2n_0 n \sigma}{n_0 \epsilon^2} M_n \right) B_n \right) \cos n\tau \right. \\
 &\quad + \left(\left(N_n - \frac{2n_0 n \sigma}{n_0 \epsilon^2} M_n \right) A_n \right. \\
 &\quad \left. + \left(M_n + \frac{2n_0 n \sigma}{n_0 \epsilon^2} N_n \right) B_n \right) \sin n\tau \Big] \dots\dots\dots (22)
 \end{aligned}$$

An effective value A_{Em} of dimensionless vibratory amplitude of mass m is also defined as follows:

$$\begin{aligned}
 A_{Em} &= \left\{ \frac{1}{2} \left(\frac{n_0 \epsilon}{Q} \right)^2 \sum_{n=1}^3 \left[\left(\left(M_n + \frac{2n_0 n \sigma}{n_0 \epsilon^2} N_n \right) A_n \right. \right. \right. \\
 &\quad \left. \left. - \left(N_n - \frac{2n_0 n \sigma}{n_0 \epsilon^2} M_n \right) B_n \right)^2 \right. \right. \\
 &\quad + \left(\left(N_n - \frac{2n_0 n \sigma}{n_0 \epsilon^2} M_n \right) A_n \right. \\
 &\quad \left. + \left(M_n + \frac{2n_0 n \sigma}{n_0 \epsilon^2} N_n \right) B_n \right)^2 \Big] \Big\}^{1/2} \dots\dots\dots (23)
 \end{aligned}$$

3. Numerical Results and Discussion

In numerical example we assume that the stiffness k_1 of a spring at a supporting end which is restrained to move in the direction of length of beam is infinite and that the damping of beam can be neglected. Thus, simplifying the boundary conditions, we will study the response of principal and higherharmonic resonances of non-linear coupled vibrations of a beam having a spring-mass system, which has also a damper. The stability of the solutions obtained will be discussed only in no-damping case.

The values of dimensionless quantities with beam and spring-mass system are taken as follows:

$$\left. \begin{aligned}
 n_0 &= 1, \quad f = 1, \quad K_1 = \infty, \quad R = 10^4, \\
 S_0 &= 0, \quad \Gamma = 5, \quad \gamma = 2, \quad \mu = 600 \\
 \xi &= 0.41, \quad \zeta = 0
 \end{aligned} \right\} \dots\dots\dots (24)$$

Taking $\sigma=0$ in two equations of f_0 and f_{c1} in Eqs. (19) and only A_0 among unknowns, and then solving simultaneously these two equations for Q , this Q gives the natural frequency of linear vibrations of a beam having a spring-mass system, which has not a damper, subjected to the axial force developed by static deflection under gravity. Also, taking A_0 and A_1 among unknowns and solving the same two equations for the relation of A_1 to Q as $K=\epsilon=\sigma=0$, we obtain the backbone curves of the non-linear free vibrations of a beam having only a concentrated mass M . We designate these dimensionless linear natural frequencies as $Q_0 (= \omega_0/p_0)$.

Fig. 2 shows the variation of linear natural frequency Q_0 with α varied from 0.01 to 0.5, where α designates the point at which the concentrated mass is located. In Fig. 2 the curve of $\epsilon=0$ corresponds to the case of $k=0$, that is, the beam has only a concentrated mass M , the curve of $\epsilon=\infty$ corresponds to the case of $k=\infty$, that is, the mass of concentrated mass on the beam becomes $(M+m)$, and the two curves of $\epsilon=(Q_0)_{\epsilon=0}$ represent the dimensionless natural frequencies of coupled vibration when the natural frequency of spring-mass system is equal to the natural frequency of the beam in the case of $\epsilon=0$. These two natural frequencies of coupled vibration correspond to the frequencies at the rising points of backbone curves of the principal resonances of non-linear forced vibration.

Fig.3 shows the backbone curves of non-linear free vibration of the beam having only a concentrated mass M , that is $\varepsilon=0$, for various values of α .

In general the backbone curves represent only a hard spring under no gravity, but under the influence of gravity, by the effects of static deflection the backbone curves change from a soft spring at small amplitude to a hard spring at large amplitude, and this tendency becomes more prominent as the concentrated mass M approaches the supporting end, as shown in Fig.3. Observing the profiles of backbone curves in Fig.3, we will study the response of forced coupled vibration of the beam having a spring-mass system in two special cases, one being $\alpha=0.5$ where the concentrated mass is at midspan and the other being $\alpha=0.2$ where the non-linearity is relatively great. Although all unknowns $A_0, A_n, B_n (n=1,2,3)$ are taken into the numerical calculation, A_1, B_2, A_3 are always zero for the steady-state solutions in no-damping case ($\sigma=0$).

Fig.4 shows the response curves of the principal resonance and the second and third order higherharmonic resonances. Fig.4(a) shows the response of the concentrated mass and Fig.4 (b) shows that of the mass m of spring-mass system. The notation $H_i (i=1,2,3)$ given above in Fig.4 is used to indicate the principal resonance and the second and third order higherharmonic resonances of the higher natural frequency of coupled vibration, in order i , and similarly L_i corresponds to that of the lower natural frequency. Two-dot chain curve in Fig.4 is the backbone curve of free vibration of the mass M without spring-mass system(Fig.3), and one-dot chain curves are the backbone curves of the mass M with spring-mass system. The second higherharmonic resonance H_2 near the principal resonance L_1 changes so complicatedly that its details are shown in Fig.5 with expanded abscissa. In Figs.4 and 5 the broken lines on the curve of $\sigma=0$ represent the unstable response. The stability of the solution is investigated by checking the roots of characteristic equations of the system. From Figs.4 and 5 we know the following:

- (1) The inclination of resonance curves of a beam with a spring-mass system becomes more gentle than that of a beam without a spring-mass system (two-dot chain curve).
- (2) The response curves shown in Fig.4 plot the numerical results in the case of $\varepsilon=0.68$, where the natural frequency p_m of spring-mass system is taken equal to the natural frequency of the beam, carrying only the concentrated mass, subjected to an axial force developed by static deflection. In Fig.4 the second order higherharmonic resonance H_2 appears on the left hand side of the principal resonance L_1 , but although not shown in Fig.4, if we take p_m so as to have $\varepsilon=0.88$, H_2 appears just on L_1 , and for a larger ε , H_2 appears on the right hand side of L_1 .
- (3) In the case of no-damping ($\sigma=0$), higherharmonic resonances H_2 and H_3 are stable near the rising point of resonance curve, but unstable at relatively small amplitude and become stable again at large amplitude. The curve H_2 has a loop-like branch and the stability and instability change complicatedly as shown in Fig.5.
- (4) The amplitude of higherharmonic resonance is reduced extremely by relatively small damping. The higherharmonic resonance of mass M ceases to appear as $\sigma>0.05$ (that is about 8% damping ratio of spring-mass system). While the response of mass m , Fig.4(b), not only H_2 but also L_2 appears a little at rising portion of the resonance curves, nevertheless σ will amount to 0.05.
- (5) As shown in Fig.5, although the resonance curve H_2 is represented by only one curve with a large amplitude to $\sigma=0.01$, it is split into two parts of the lower resonance curve and the upper loop-like branch as σ becomes larger than 0.01. When σ becomes as large as 0.0143 or more, the upper loop-like branch disappears and only the lower resonance curve remains. In Fig.5 the lower resonance curve to $\sigma=0.0143$ is omitted because it is very close to the curve to $\sigma=0.014$.
- (6) The response of mass m of spring-mass system has the same characteristics as those of the concentrated mass M , except that the amplitude is relatively large.

Fig.2 Variation of natural frequency with the concentrated mass location α .

Fig.3 Variation of backbone curves of beam having only concentrated mass M .

Fig.6 shows the response curves in the case of $\alpha=0.2$ where the effects of non-linearity are considered to be greater than in the case of $\alpha=0.5$. The notations H_i and L_i given above in Fig.6 have the same meaning as in Fig.4. Except such characteristics that the inclination of response curve is very steep and the change from soft spring to hard spring occurs at relatively small amplitude, the response curves have the same properties as in $\alpha=0.5$ shown in Fig.4.

(a) Response curves of M .

(b) Response curves of m .

Fig.4 Response curves of beam having spring-mass system, ($\alpha=0.5$, $\epsilon=0.68$, $f=1$).

Fig.5 Details of the second order higherharmonic resonance; [Fig.4 (a), H_2].

4. Conclusions

An analytical solution is derived to the coupled non-linear vibrations of a simply supported beam, with ends restrained elastically in the direction of length of a beam, carrying a concentrated mass, to which a spring-mass system is attached and which is subjected to a transverse periodic force, at an arbitrary point under the influence of gravity.

In the analysis, taking the natural frequency of spring-mass system nearly equal to the fundamental frequency of a beam with a concentrated mass M , a solution applicable in the region of first mode resonance, in which the effect of spring-mass system is considered to be very great, is derived.

In numerical examples the amplitudes of response are represented for the principal resonance and the second and third order higherharmonic resonances in the case of a beam completely restrained at ends, where the non-linear properties will appear most clearly. The stability of the steady-state solutions is also investigated only in the case of no-damping.

The results obtained may be summarized as follows:

- (1) The inclination of the resonance curves of the beam with a concentrated mass having a spring-mass system becomes smaller than that of those not having a spring-mass system.
- (2) In the case of no-damping, the stabil-

(a) Response curves of M .

(b) Response curves of m .

Fig.6 Response curves of beam having spring-mass system, ($\alpha=0.2$, $\epsilon=0.83$, $f=1$).

ity of the second and third order higherharmonic resonances of the higher natural frequency of the coupled vibration changes corresponding to the amplitudes as follows: stable at rising, unstable at small amplitude and stable again at large amplitude. Moreover, a loop-like branch appears at large amplitude. The second and third order higherharmonic resonances of the lower natural frequency of the coupled vibration represent a soft spring characteristic up to relatively large amplitude and when the frequency of external force falls from relatively high frequency, a stable response may occur.

(3) The amplitudes of higherharmonic resonances are extremely reduced by the existence of a damping, which is small.

(4) The second order higherharmonic resonance that occurs near the lower principal resonance of coupled vibration varies complicately for various damping ratios.

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Appendix

$$\begin{aligned}
 L_0 &= A_0^3 + \frac{3}{2} \left\{ A_0 \sum_{n=1}^3 (A_n^2 + B_n^2) + A_1(A_2A_3 + B_1B_2 + B_2B_3) + B_1(A_2B_3 - A_3B_2) \right\} + \frac{3}{4} A_2(A_1^2 - B_1^2), \\
 L_1 &= 3A_0(A_0A_1 + A_1A_2 + A_2A_3 + B_1B_2 + B_2B_3) + (3/4)[A_1\{A_1^2 + B_1^2 + 2(A_2^2 + A_3^2 + B_2^2 + B_3^2)\} + A_3(A_1^2 + A_2^2 - B_1^2 - B_2^2) + 2B_3(A_1B_1 + A_2B_2)], \\
 L_2 &= (3/2)[A_0\{2(A_0A_2 + A_1A_3 + B_1B_3) + A_1^2 - B_1^2\} + A_1(A_2A_3 + B_2B_3) + A_2\{A_1^2 + A_3^2 + B_1^2 + B_3^2 + (1/2)(A_2^2 + B_2^2)\} + B_1(A_3B_2 - A_2B_3)], \\
 L_3 &= 3A_0(A_0A_3 + A_1A_2 - B_1B_2) + A_1^3/4 + (3/4)[A_1(A_2^2 - B_1^2 - B_2^2) + 2A_2B_1B_2 + A_3\{2(A_1^2 + A_2^2 + B_1^2 + B_2^2) + A_3^2 + B_3^2\}], \\
 L_4 &= 3A_0(A_0B_1 + A_1B_2 - A_2B_1 + A_2B_3 - A_3B_2) + (3/4)[2A_3(A_2B_2 - A_1B_1) + B_1\{A_1^2 + B_1^2 + 2(A_2^2 + A_3^2 + B_2^2 + B_3^2)\} + B_3(A_1^2 - A_2^2 - B_1^2 + B_2^2)], \\
 L_5 &= 3A_0(A_0B_2 + A_1B_1 + A_1A_3 - A_3B_1) + (3/4)[2A_1(A_2B_3 - A_3B_2) + 2B_1(A_2A_3 + B_2B_3) + B_2\{A_1^2 + B_1^2 + 2(A_2^2 + A_3^2 + B_1^2 + B_3^2)\}], \\
 L_6 &= 3A_0(A_0B_3 + A_1B_2 + A_2B_1) - B_1^3/4 + (3/4)[2A_1A_2B_2 + B_1(A_1^2 - A_2^2 + B_2^2) + B_3\{2(A_1^2 + A_2^2 + B_1^2 + B_2^2) + A_3^2 + B_3^2\}].
 \end{aligned}$$