Discrete-vortex Simulation of a Two-dimensional Turbulent Jet*

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A discrete-vortex method is used to simulate a two-dimensional turbulent free jet. The analysis is made to predict the development of the flow, the mean velocity distribution, the turbulent intensity distribution, and the pressure distribution of an asymmetric and a symmetric jet with respect to the jet center line. The results of the asymmetric jet agree relatively well with available experimental data. It is found that the vortex clusters are nearly symmetric with respect to the center line upstream of the merging point of the shear layers but the vortex clusters arrange in a pattern similar to a von Karman vortex street downstream of the merging point.

Key Words : Numerical Analysis, Two-dimensional Turbulent Jet, Discrete-vortex Method, Turbulent Mixing, Coherent Structure, Cavitation

1. Introduction

Cavitation in submerged jets is considered to occur in the low-pressure cores of turbulent eddies generated in the shear zone[1]. In order to understand inception and development of cavitation in submerged jets, it is necessary to clarify the generation and the development of turbulent eddies and the turbulent pressure fluctuation in submerged jets. A number of experimental investigations have been made about a large-scale structure in turbulent jets in connection with the aerodynamic sound generation[2]. Recently, Antonia et al.[3] measured space-time correlations of longitudinal and normal velocity fluctuations and of temperature fluctuations of a turbulent plane jet. Their measurements supported a vortex street pattern of large structures similar to a two-dimensional von Karman vortex street downstream of the merging point of the jet mixing layers. Close to the jet exit, their measurements indicated symmetrical structures with respect to the jet center line.

Recent applications of a discrete-vortex method to turbulent plane mixing layers[4-6] and unsteady separated flow behind a normal plate[7] have shown its usefulness for the simulation of large scale structures in turbulent shear flows. Beavers et al.[8] calculated the motion of two infinite parallel vortex sheets to simulate the growth

of vortices in shear layers of a two-dimensional jet. Davis et al.[9] simulated the development of a starting vortex of an axisymmetric circular jet. Acton[10] simulated the upstream portion of a circular jet with double rows of ring vortices and numerically demonstrated the roll up of a vortex sheet into large axisymmetric structures. Oler et al.[11] proposed a vortex street model of a flow in the similarity region of a two-dimensional free turbulent jet. The flow was simulated by superposition of Rankin vortices arranged to represent a vortex street. The mean velocity components and the magnitude of Reynolds stress were properly determined by the model. However, there has been made no theoretical research on the development of large scale structures of a two-dimensional jet. The pressure fluctuation in the mixing zone of jets is little known.

In the present investigation, a discrete vortex model is used to simulate a two-dimensional turbulent free jet. The generation and the development of large scale structures, the mean velocity distribution, the turbulent intensity distribution and the pressure fluctuation which is important to understand cavitation in submerged jets are analyzed numerically.

2. Nomenclature

C_p : pressure coefficient
 P : pressure
 P_∞ : pressure at downstream infinity
 T : non-dimensional time
 U_0 : velocity in the slit at upstream infinity

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b : half width of the slit
 f : complex velocity potential $=\phi+i\psi$
 t : time
 w : complex velocity $=u-iv$
 u, v : x and y components of velocity
 z : coordinate $=x+iy$
 Γ_{pk}, Γ_{qk} : circulations of vortex points
 ζ : coordinate $=\xi+i\eta$
 ρ : density of fluid
 ϕ : velocity potential
 ψ : stream function

3. A Discrete-vortex Model of a Two-dimensional Jet

The model proposed here is one of a two-dimensional jet discharged from a slit as shown in Figure 1. The model approximates the shear layers shed from the exit corners, points B and E in the z -plane, by arrays of line vortices. The velocity of any vortex is the sum of the two-dimensional irrotational potential flow discharged from the slit and the velocity induced at the vortex position by all other vortices. In order to calculate the velocity of any one of the vortices, the z -plane is transformed into an upper half plane (ζ -plane) shown in Figure 2. By using the theorem of Schwarz and Christoffel, the transformation from the z -plane into the ζ -plane is given by

$$z = -\frac{2bi}{\pi} \left\{ \sqrt{\zeta^2 - 1} - i \ln \left(\frac{i + \sqrt{\zeta^2 - 1}}{\zeta} \right) \right\} \quad (1)$$

The exit corners $z = \pm bi$ and $x = -\infty$ are transformed to $\zeta = \mp 1$ and $\zeta = 0$, respectively.

The complex velocity potential f_1 for the irrotational flow discharged from the slit is given by

$$f_1 = \frac{2bU_0}{\pi} \ln \zeta \quad (2)$$

where b and U_0 are the half width of the slit and the velocity in the slit at upstream infinity, respectively. If the number of the vortex elements introduced into the flow field from the vicinity of the points B and E is $(N+1)$, the complex velocity potential f_2 of a flow induced by the vortices is given by

$$f_2 = -\sum_{k=0}^N \frac{i\Gamma_{pk}}{2\pi} \ln(\zeta - \zeta_{pk}) + \sum_{k=0}^N \frac{i\Gamma_{qk}}{2\pi} \ln(\zeta - \bar{\zeta}_{qk}) \quad (3)$$

where Γ_{pk} and ζ_{pk} represent respectively the circulation and the location of k -th p -vortex, Γ_{qk} and $\bar{\zeta}_{qk}$ the circulation and the location of k -th q -vortex; an overbar indicates a complex conjugate. The complex velocity potential f for the total flow field is the sum of Eqs. (2) and (3).

Fig. 1 z -plane

Fig. 2 ζ -plane

$$f = f_1 + f_2 \quad (4)$$

The Kutta condition is used to determine the circulations of newly introduced vortices. In order to satisfy the Kutta condition at the exit corners $z = \pm bi$, there must be stagnation points at $\zeta = \pm 1$ in the ζ -plane. Thus we get

$$(df/d\zeta)_{\zeta=\pm 1} = 0 \quad (5)$$

By using Eqs. (2), (3) and (5), the circulations of newly introduced vortices Γ_{p0} and Γ_{q0} can be determined by the following equations.

$$\begin{aligned} & -\frac{i\Gamma_{p0}}{2\pi} \left(\frac{1}{1-\zeta_{p0}} - \frac{1}{1-\bar{\zeta}_{p0}} \right) \\ & -\frac{i\Gamma_{q0}}{2\pi} \left(\frac{1}{1-\zeta_{q0}} - \frac{1}{1-\bar{\zeta}_{q0}} \right) \\ & = -\frac{2bU_0}{\pi} + \sum_{k=1}^N \frac{i\Gamma_{pk}}{2\pi} \\ & \quad \times \left(\frac{1}{1-\zeta_{pk}} - \frac{1}{1-\bar{\zeta}_{pk}} \right) \\ & \quad + \sum_{k=1}^N \frac{i\Gamma_{qk}}{2\pi} \left(\frac{1}{1-\zeta_{qk}} - \frac{1}{1-\bar{\zeta}_{qk}} \right) \\ & -\frac{i\Gamma_{p0}}{2\pi} \left(\frac{1}{1+\zeta_{p0}} - \frac{1}{1+\bar{\zeta}_{p0}} \right) \\ & -\frac{i\Gamma_{q0}}{2\pi} \left(\frac{1}{1+\zeta_{q0}} - \frac{1}{1+\bar{\zeta}_{q0}} \right) \\ & = -\frac{2bU_0}{\pi} + \sum_{k=1}^N \frac{i\Gamma_{pk}}{2\pi} \left(\frac{1}{1+\zeta_{pk}} - \frac{1}{1+\bar{\zeta}_{pk}} \right) \\ & \quad + \sum_{k=1}^N \frac{i\Gamma_{qk}}{2\pi} \left(\frac{1}{1+\zeta_{qk}} - \frac{1}{1+\bar{\zeta}_{qk}} \right) \quad (6) \end{aligned}$$

where ζ_{p0} and ζ_{q0} are the vortices-introducing points in the ζ -plane. The vortices are introduced into the flow field at $z = \delta \pm bi$ in the z -plane.

The velocity field in the z -plane is given by

$$w = u - iv = (df/d\zeta) \cdot (d\zeta/dz) \quad (7)$$

where u and v are the velocity com-

ponents in the x - and the y -directions, respectively. The velocity $w(z_n)$ at a vortex point can be obtained by [12]

$$w(z_n) = \left(\frac{df_n}{dz} \right)_{z=z_n} \cdot \left(\frac{dz}{dz} \right)_{z=z_n} - \frac{i\Gamma_n}{2\pi} \cdot \frac{g''(z_n)}{g'(z_n)} \quad (8)$$

where

$$f_n(\zeta) = f(\zeta) + \frac{i\Gamma_n}{2\pi} \ln(\zeta - \zeta_n) \quad (9)$$

when the transformation between the ζ -plane and the z -plane is given by

$$\zeta = g(z) \quad (10)$$

The scheme for calculating the time development of the system is

$$z_n(t + \Delta t) = z_n(t) + \overline{w(z_n)} \cdot \Delta t \quad (11)$$

where t and Δt are time and the time interval, respectively.

The pressure p is calculated from the Bernoulli equation

$$C_p = \frac{P - P_\infty}{(\rho U_\infty^2 / 2)} = -\frac{1}{U_\infty^2} \left| \frac{df}{dz} \right|^2 - \frac{2}{U_\infty^2} \frac{\partial \phi}{\partial t} \quad (12)$$

where C_p , P_∞ and ρ represent respectively the pressure coefficient, the pressure at downstream infinity and the density of fluid and ϕ implies the velocity potential. The unsteady term $\partial \phi / \partial t$ can be evaluated from

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial t} \{ \text{Real}(f) \}$$

$$\begin{aligned} &= \text{Real} \left(\sum_{k=0}^N \frac{i\Gamma_{pk}}{2\pi} \cdot \frac{1}{\zeta - \zeta_{pk}} \cdot \frac{\partial \zeta_{pk}}{\partial t} \right. \\ &\quad - \sum_{k=0}^N \frac{i\Gamma_{pk}}{2\pi} \cdot \frac{1}{\zeta - \zeta_{pk}} \cdot \frac{\partial \zeta_{pk}}{\partial t} \\ &\quad + \sum_{k=0}^N \frac{i\Gamma_{qk}}{2\pi} \cdot \frac{1}{\zeta - \zeta_{qk}} \cdot \frac{\partial \zeta_{qk}}{\partial t} \\ &\quad \left. - \sum_{k=0}^N \frac{i\Gamma_{qk}}{2\pi} \cdot \frac{1}{\zeta - \zeta_{qk}} \cdot \frac{\partial \zeta_{qk}}{\partial t} \right) \quad (13) \end{aligned}$$

where

$$\frac{\partial \zeta_{pk}}{\partial t} = \frac{\pi i}{2b} \cdot \frac{\zeta_{pk}}{\sqrt{\zeta_{pk}^2 - 1}} \cdot \left(\frac{df}{dz} \right)_{z=z_{pk}} \quad (14)$$

$$\frac{\partial \zeta_{qk}}{\partial t} = \frac{\pi i}{2b} \cdot \frac{\zeta_{qk}}{\sqrt{\zeta_{qk}^2 - 1}} \cdot \left(\frac{df}{dz} \right)_{z=z_{qk}} \quad (15)$$

Because of the absence of viscosity, it is probable that some vortices come close to one another and develop unreasonably large velocities at each position. This is avoided by coalescing such vortices with separation of less than $\delta r_1 (= 0.05b)$ to combine these vortices into an equivalent single vortex.

In order to decrease the computational time, the starting vortices are combined into an equivalent pair of vortices whose circulations are the sum of individual circulations and whose positions are the centers of the clusters. Exploratory calculations with or without a combination of the starting vortices showed that the development of a jet was not noticeably affected.

Referring to reference [13], the values of the time step $\Delta T (= \Delta t U_\infty / b)$ and the time interval for the introduction of new vortices are

Fig. 3 Development of starting vortices (asymmetric flow with respect to x -axis)

taken as 0.2 and 0.4 ($=2\Delta T$), respectively. Preliminary calculations showed that the development of a flow by $\Delta T=0.2$ was not different appreciably from that by $\Delta T=0.1$. The value of δ for vortex introducing points is set at $0.05b$.

Calculations are made for symmetric and asymmetric flows about the x -axis. For the asymmetric flow, an artificial disturbance [12] determined by equation (16) is used.

$$\begin{cases} f_p = \frac{2}{\pi} \cdot 0.1 \cdot \xi \cdot \sin\left(\frac{\pi T}{3}\right), & 0 \leq T \leq 3 \\ f_p = 0, & T > 3 \end{cases} \dots (16)$$

where T is non-dimensional time defined by tU_0/b .

4. Numerical Results and Discussion

Figure 3 shows the development of the starting vortices of the asymmetric jet up to $T=20$. The artificial disturbance determined by Eq. (16) was applied to the model jet. After the flow starts at $T=0$, the shear layers shed from the exit corners begin to roll up and the starting vortices are formed. The

Fig. 4 Distributions of vortices (symmetric flow with respect to x -axis)

Fig. 5 Time histories of u on x -axis (symmetric flow with respect to x -axis)

Fig. 6 Time histories of u on x -axis (asymmetric flow with respect to x -axis)

pressure distributions are also shown in Figure 3. The low pressure regions around the centers of the starting vortices and the high pressure region downstream of the vortices widen with the growth of the vortices. The development of the starting vortices of the symmetric jet is similar to Figure 3.

The vortex distributions of the symmetric and the asymmetric jets at $T=110$ are shown in Figures 4 and 11, respectively. After the formation of

the starting vortices, the instabilities grow on the initial shear layers and small eddies are formed. These eddies interact on each other and grow as they move downstream. In the case of the asymmetric jet, the distribution of the vortices is nearly symmetric till the two shear layers merge at $x/b=10-20$. Downstream of the merging point, large eddies are formed and the symmetry of the flow is lost. Recently, Antonia et al.[3] measured space-

Fig. 7 Variation of $\bar{u}$ on x -axis

Fig. 8 Variation of jet width

Fig. 9 Mean velocity profiles
(asymmetric flow with respect to x -axis)

Fig. 10 Turbulence intensity distributions
(asymmetric flow with respect to x -axis)

time correlations of velocity fluctuations and of temperature fluctuations of a plane jet. They pointed out that counter-rotating structures appearing alternately on opposite sides of the jet centerline located in the region downstream of the merging point of the mixing layers and similar structures which were symmetric with respect to the center line did in the mixing layers near the jet exit. The present calculation for the asymmetric jet explains their measurements.

The time histories of the velocity component u on the jet axis at different locations are shown in Figures 5 and 6 for the symmetric and the asymmetric jets, respectively. The first peak of u in each position is due to the passage of the starting vortices. The amplitude and the period of the time history

of u are small in the vicinity of the jet exit. It is noteworthy that the period of u of the asymmetric jet increases rapidly between $x=12$ and 15. This is due to the confluence of the two mixing layers and the formation of the large eddies as mentioned before. In the case of the symmetric jet, the period of u increases monotonically with x .

Figure 7 shows the streamwise variations of the longitudinal mean velocity $\bar{u}$ on the x -axis. In this figure, $\bar{u}$ is normalized by the mean velocity, $\bar{U}_0$ at $x/b=2$. The value of $\bar{U}_0$ is $1.207U_0$. The time mean velocities are the average values from $T=70$ to 160. The measurements by Hussain and Thompson[14] are also shown in Figure 8. The measurement and the mean velocity of the asymmetric jet sharply drop at $x/b \approx 11$ ($x/2b \approx 5.5$). The agreement between the calculation for the asymmetric jet and the measurement is good. On the contrary, the mean velocity of the symmetric jet only slightly decreases with an increasing distance from the jet exit.

The streamwise variations of the half width of the jets are shown in Figure 8. The half width B is defined as the width corresponding to the transverse location of $\bar{u}_{max}/2$ in the mean velocity profile. The calculated half width are smaller than the experimental results. The half width of the asymmetric jet and the measurement sharply increase at $x/b \approx 11$ and the gradient of the width of the calculated asymmetric jet is much the same as that of the measurements. The behaviors of the streamwise variations of the mean

Fig. 11 Time histories of pressure on x -axis (asymmetric flow with respect to x -axis)

Fig. 12 Vortex and pressure distributions (asymmetric flow with respect to x -axis)

velocity on the x -axis and the half width of the asymmetric jet can be explained from the confluence of the mixing layers and the formation of large eddies at $x/b=10-20$.

Figures 9 and 10 show the mean velocity profiles and the longitudinal turbulence intensities of the asymmetric jet, respectively. The distortions of the mean velocity profiles and the turbulence intensity distributions may be due to the shortage of the time interval for calculating the mean values. The measurement by Hussain and Thompson are also shown in these figures. The agreement between calculation and measurement is relatively good.

Figure 11 shows the time histories of the pressure on the x -axis. Figure 12 shows the distributions of the vortices and the pressure distributions of the asymmetric jet at $T=106$, 108 and 110. At this time stage, the flow is considered to be steady statistically. With the formation of large eddies downstream of the end of the potential core, low pressure regions grow rapidly. Yamazaki et al.[15] pointed out from their observation of cavitation in a circular jet that cavitation initiated in the region downstream of the end of the potential core, $6D \sim 9D$ (D is the exit diameter of the nozzle). This may correspond to the formation of large eddies downstream of the end of the potential core as is shown for the two-dimensional jet by the present calculation.

5. Conclusions

By using a discrete vortex model, the development of vortical structure, the mean velocity distribution, the turbulent intensity distribution and the pressure fluctuation of a two-dimensional free jet are calculated. Main specific points of interest are as follows;

- (1) Calculated mean velocity and turbulent intensity distributions of an asymmetric jet agree relatively well with experimental results.
- (2) Symmetrical structures with respect to the jet axis exist in the flow development region. Downstream of the merging point of the jet mixing layers, the symmetry of the flow is lost and an asymmetric vor-

tex street pattern similar to a von Karman vortex street appears.

(3) With the formation of large eddies downstream of the end of the potential core, a low pressure region grows rapidly.

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6. References

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