Oscillatory Flow through a Convergent Tube*

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This paper deals with the laminar oscillatory flow of an incompressible fluid through a slightly convergent tube. The velocity distributions in the convergent tube, whose convergent angles are in the range of 2-4°, are measured by a Laser Doppler Velocimeter. The flow in the convergent tube is numerically analysed by a finite difference method (SOR) without neglecting the convective terms in the Navier-Stokes equation. Experimental results agreed well with the numerical ones. The velocity profile of a laminar oscillatory flow in the convergent tubes is quite different from the velocity profile in a straight tube of circular cross section, even if the convergent angle is very small. The instantaneous pressure distribution in the convergent tube is as uniform at each cross section as in a straight circular tube.

Key Words : Fluid Mechanics, Unsteady Flow, Convergent Tube, Oscillatory Flow, Laser Doppler Velocimeter, Finite Difference Method

1. Introduction

Recently unsteady pipe flow has been increasingly investigated by many researchers and the behavior of unsteady pipe flow has been clarified gradually. In the field of medicine and physiology the flow through stenosis has been investigated for long time in order to clear up the cause of aneurysm.

On the oscillatory flow through straight tube, Nakano et al.⁽¹⁾⁻⁽³⁾ obtained the numerical and approximate solutions for laminar oscillatory flow in inlet region and compared them with the experimental results. Ohmi et al.⁽⁴⁾ published the theoretical and experimental results on turbulent oscillatory flow through straight tube. Investigations on the steady flow through stenosis, velocity distributions and pressure distributions, which are obtained theoretically, numerically or experimentally, were reported by Forester and Young⁽⁵⁾, Tanahashi et al.⁽⁶⁾ and Lee and Fung⁽⁷⁾. In most of conventional research⁽⁸⁾ on the flow through the pipe whose cross-sectional area changes continuously such as convergent tube and stenosis, the velocity distribution is approximated with that in straight tube. Takahashi et al.⁽⁹⁾ reported on the analytical relation between flow rate and pressure difference of oscillatory flow through the convergent tube, assuming the stream line to be linear and considering the effect of convective acceleration term.

The behavior of the flow through the pipe, whose cross-sectional area changes continuously, seems to be different from

that through the straight tube with constant cross-sectional area. This paper deals with the laminar oscillatory flow of an incompressible fluid through a slightly convergent tube. The velocity distributions in the convergent tube are measured by a Laser Doppler Velocimeter and compared with numerical results obtained by a finite difference method without neglecting the convective terms in Navier-Stokes equation. With these considerations the behavior of oscillatory flow through a convergent tube will be explained.

2. Nomenclature

F : non-dimensional frequency ($= r_0^2 f / \nu$)
 f : frequency
 L : non-dimensional length of convergent tube ($= l / r_0$)
 P : non-dimensional pressure ($= p / (\rho u_0^2)$)
 p : pressure
 Q : non-dimensional flow rate ($= q / q_0$)
 q : instantaneous flow rate
 q_0 : mean flow rate
 R : non-dimensional radial coordinate ($= r / r_0$)
 $\bar{R}$: normalized radial coordinate ($= R / R_w$)
 R_w : non-dimensional radius of convergent tube ($= r_w / r_0$)
 r_0 : radius of straight tube
 r_w : radius of convergent tube
 Re : mean Reynolds number ($= 2 r_0 u_0 / \nu$)
 St : Strouhal number ($= 2 r_0 f / u_0$)
 t : time
 U : non-dimensional axial velocity ($= u / u_0$)
 $\bar{U}$: normalized axial velocity ($= U R_w^2$)
 u_0 : mean axial velocity in straight tube ($= q_0 / (\pi r_0^2)$)
 $\bar{V}$: non-dimensional radial velocity ($= v / u_0$)
 $\bar{V}$: normalized radial velocity ($= V R_w^2$)
 z : non-dimensional axial coordinate ($= z / r_0$)
 θ : converging angle
 θ_d : diverging angle
 β : amplitude of oscillatory flow rate
 ϕ : non-dimensional stream function
 ω : non-dimensional vorticity
 τ : non-dimensional time ($= ft$)

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3. Experimental Apparatus

Fig. 1 shows a schematic diagram of the experimental apparatus. Convergent tube and straight tube between upstream tank and convergent tube are made of transparent acrylic. The flow rate through the tube is controlled by the spool type control orifice at the downstream end. The opening of the control orifice is a slit of 9×1 mm. By driving the spool sinusoidally through an eccentric cam, the opening area as well as flow rate is varied sinusoidally. The eccentric cam is driven by a servo motor through a reduction gear. By adjusting the rotational speed of the servo motor and the eccentricity of the cam, the frequency and amplitude of flow oscillation can be set. The displacement of spool can be measured by an eddy current type transducer connected to the spool at the other side of the cam.

Axial velocities at points in horizontal plane including the tube axis are measured by Laser Doppler Velocimeter System (LDV). To prevent the effect of deflection of the laser beam at the wall of convergent tube, both sides of the convergent tube are filled with water. Then only by displacing the LDV normal to the tube axis the velocity distribution at the section can be measured. Table 1 shows the dimensions of convergent tubes used. Velocity distributions are measured at relatively the same positions, which are shown in Fig. 2, for every convergent tube. Also the velocity distribution in the straight tube is measured at a section 63mm upstream of the inlet of convergent tube. White paint is added only a little as the seeding material.

4 Numerical Calculation

It is assumed that the flow is laminar and axially symmetric and the fluid is incompressible. The flow rate varies sinusoidally.

The vorticity and the stream function of the axially symmetric flow shown in Fig. 3 are given by

$$\omega = \partial V / \partial Z - \partial U / \partial R \quad \dots\dots\dots (1)$$

$$U = (\partial \psi / \partial R) / R, \quad V = -(\partial \psi / \partial Z) / R \quad \dots\dots\dots (2)$$

Taking the shape of venturi tube into consideration, the coordinates are transformed by the following equations.

$$\xi = \tanh(AZ^*), \quad \eta = \tanh(BR) \quad \dots\dots (3)$$

where $Z^* = Z - (Z_c + L)$ for $Z \leq L$ and $Z^* = Z + (Z_c - L)$ for $Z > L$. The constants A and B are determined corresponding with the convergent angle θ and the diverging angle θ_d of venturi tube. By this transformation the axial section $-\infty \leq Z \leq +\infty$ is mapped on to the section $-1 \leq \xi \leq +1$ and the wall of venturi tube is mapped on to the lines $\eta = \pm 1$. The dimensionless time is defined as

$$\tau = ft \quad \dots\dots\dots (4)$$

After these transformations the equation of vorticity and the vorticity transport equation are given as follows

$$\omega = -\frac{C_a(\xi)}{C_h(\eta)} \left\{ -2A\xi \frac{\partial \psi}{\partial \xi} + C_a(\xi) \frac{\partial^2 \psi}{\partial \xi^2} \right\} + \frac{C_b(\eta)}{C_h(\eta)} \frac{\partial \psi}{\partial \eta} - \frac{C_b(\eta)}{C_h(\eta)} \left\{ -2B\eta \frac{\partial \psi}{\partial \eta} + C_b(\eta) \frac{\partial^2 \psi}{\partial \eta^2} \right\} \quad \dots\dots\dots (5)$$

$$\begin{aligned} & \frac{2}{R_*} F C_h(\eta) \frac{\partial \omega}{\partial \tau} + \frac{C_a(\xi)}{C_h(\eta)} \omega \frac{\partial \psi}{\partial \xi} - C_a(\xi) C_b(\eta) \frac{\partial \psi}{\partial \xi} \frac{\partial \omega}{\partial \eta} + C_a(\xi) C_b(\eta) \frac{\partial \psi}{\partial \eta} \frac{\partial \omega}{\partial \xi} \\ &= \frac{2}{R_*} \left[C_h(\eta) C_a(\xi) \left\{ -2A\xi \frac{\partial \omega}{\partial \xi} + C_a(\xi) \frac{\partial^2 \omega}{\partial \xi^2} \right\} + C_h(\eta) C_b(\eta) \left\{ -2B\eta \frac{\partial \omega}{\partial \eta} + C_b(\eta) \frac{\partial^2 \omega}{\partial \eta^2} \right\} \right. \\ & \left. + C_b(\eta) \frac{\partial \omega}{\partial \eta} - \frac{\omega}{C_h(\eta)} \right] \quad \dots\dots\dots (6) \end{aligned}$$

where

$$C_a(\xi) = (1 - \xi^2)A,$$

$$C_b(\eta) = (1 - \eta^2)B,$$

$$C_h(\eta) = (\tanh^{-1} \eta) / B$$

Fig. 1 Schematic diagram of experimental apparatus

Table 1 Dimensions of venturi tube

r_0 mm	r_c mm	θ °	l mm	θ_d °
10.5	3.5	2.0	200	2.0
10.5	3.5	3.0	134	2.0
10.5	3.5	4.0	100	2.0

Fig. 2 Measuring sections

$\bar{\psi}^{(n)}$ and $\bar{\omega}^{(n)}$ denote the values computed from the difference equations and $\psi^{(n-1)}$ and $\omega^{(n-1)}$ the values obtained at the $(n-1)$ th iteration. The differences h_1 , h_2 and h_3 used in this paper are 0.025, 0.025 and 0.05 respectively. The relaxation factors used are the values between 0.05 and 1.0. In the calculation, for each time step between $\tau = 0.05$ and $\tau = 1.0$ the values of ψ and ω at all points on plane are calculated from eqs. (9)-(12). When the calculated values of ψ and ω at $\tau = 1.0$ coincide with the values at $\tau = 0$, the solution is considered a convergent solution. The calculation starts with the value $\lambda = 0$. Once the convergent solution is obtained, the value of λ is increased a little. In this paper the convergent solution at $\lambda = 0.5$ is considered a final solution because the calculation does not converge when $\lambda = 1.0$.

For the calculation of steady flow, initial guesses were zero both for ψ and ω . For oscillatory flow initial guesses were the final solution for a flow with lower Reynolds number and lower oscillation frequency.

When the following conditions are satisfied, solutions are considered to have converged.

$$|(\psi_{\tau=0} - \psi_{\tau=1.0}) / \psi_{\tau=1.0}| < 5 \times 10^{-3}$$

$$|(\omega_{\tau=0} - \omega_{\tau=1.0}) / \omega_{\tau=1.0}| < 5 \times 10^{-3}$$

Since in the region near the center axis of tube the absolute values of ψ and ω are very small, it is not easy to satisfy the conditions shown above. Then the following equations are used as criteria for convergence.

$$|\psi_{\tau=0} - \psi_{\tau=1.0}| < 5 \times 10^{-5}$$

$$|\omega_{\tau=0} - \omega_{\tau=1.0}| < 5 \times 10^{-5}$$

Using the difference equations of eqs. (2), (7) and (8), the values of U , V , $\partial P / \partial \xi$ and $\partial P / \partial \eta$ were calculated from the values of ψ and ω obtained according to the procedure shown above. By the calculation of pressure distributions the pressures at the point $Z=0$ and $R=0$ were used as reference pressure and an integration was carried out at first along the center line and then from center line to the wall using a trapezoidal formula.

5 Results

Fig. 6 shows an example of the streamline for steady flow which is obtained from numerical calculation. Excluding the flows whose convergent angle is 2° and whose mean Reynolds number is less than 100, all numerical results show that the flow separates in diverging section. In the flow shown in Fig. 6, a separation

Fig. 6 Streamlines for steady flow

occurs near the end of diverging section, so that the separation does not appear in the figure. Even if the separation occurs in diverging section, streamlines in converging section are linear regardless of convergent angle and Reynolds number.

Fig. 7 shows the axial velocity distribution for steady flow. Radial coordinate is referred to the radius of straight tube r_0 and axial velocity by mean velocity in straight tube u_0 . Solid lines in the figure show the velocity distributions obtained from numerical calculations and the mark $\circ$ the measured velocity. Both results agree very well except the results at $Z/L=0.81$.

Fig. 8 shows the axial velocity distribution for oscillatory flow. The marks of measured velocity distributions correspond with the marks shown in Fig. 9. Numerically calculated velocity distributions

Fig. 7 Axial velocity distribution for steady flow

Fig. 8 Axial velocity distribution for oscillatory flow

Fig. 9 Flow rate oscillation

agree very well with measured ones.

In Fig. 10 the velocity distributions for steady flow are compared with the average velocity distributions during a period. It is seen from this figure that the distribution of time average velocities agrees with that for steady flow of same Reynolds number and same convergent angle regardless of frequency and amplitude of flow rate oscillation. The difference between the time average velocity and the instantaneous velocity is defined as the velocity variation ΔU . Fig. 11 shows the distribution of ΔU for the case $\theta=2^\circ$ and $Re=500$. For different convergent angles and Reynolds numbers velocity variations obtained from numerical calculation agree very well with measured ones.

It can be concluded from Figs. 7-11 that the numerical calculation explained in Section 4 is accurate enough. Then only the numerical results will be treated hereafter. In order to compare the velocity distributions at different axial positions, the distribution is expressed with new variables, $\bar{R}=R/R_w$, $\bar{U}=U/R_w^2$.

Fig. 12 shows the transition of velocity profiles in convergent tube for steady flow. The parabolical velocity distribution in straight tube drawn with solid line approaches the uniform velocity dis-

Fig. 10 Distribution of axial time mean velocities

Fig. 11 Distribution of axial variational velocities

tribution as the flow goes downstream. With an increase of convergent angle and Reynolds number the tendency becomes more remarkable.

Fig. 13 shows the relation between the distribution of the velocity variation $\Delta \bar{U} = \Delta U \times R_w^2$ and Strouhal number St . In the figure the distributions of $\Delta \bar{U}$ in upstream straight tube are also shown. In general the velocity variation in straight tube is characterized with Strouhal number St , that is, with an increase of St , the distribution of the velocity variation approaches uniform piston shape. It is seen from Fig. 13 that the distribution of the velocity variation in divergent tube has the same tendency. At each Strouhal number, as the flow goes downstream, the distribution of velocity variation $\Delta \bar{U}$ reduces the concavity near the center line and the convexity near the wall. The distribution of $\Delta \bar{U}$ near the wall, however, is similar to that in straight tube at the same Strouhal number. That is, in convergent tube due to the convective effect $\Delta \bar{U}$ is homogenized and approaches uniform distribution, but in the region near the wall $\Delta \bar{U}$ depends on Strouhal number. Then the distribution near the wall is very similar to that in straight tube. Fig. 14

Fig. 12 Transition of axial velocity distribution for steady flow

Fig. 13 Transition of axial velocity distribution ($\theta=2^\circ$)

shows the effect of convergent angle on the distribution of velocity variation at the section $Z/L=0.81$. It is seen from this figure that when Strouhal number is same, the distribution of $\Delta\bar{U}$ is nearly independent of the convergent angle.

The distributions of radial velocities are shown in Figs. 15-17. Since in straight tube the radial velocity at any point is zero, only the distributions at three sections in convergent tube are shown in figures. Fig. 15 shows that as the flow goes downstream the radial velocity in convergent tube increases, but the shape of distribution changes little.

Similarly to the axial velocity, the distribution of time average radial velocities coincides with the radial velocity

Fig. 14 Effect of θ on axial variational velocity ($Z/L=0.81$, $Re=200$, $St=0.63$)

Fig. 15 Distribution of radial velocities

Fig. 16 Distribution of radial time mean velocities

Fig. 17 Distribution of radial variational velocities ($\theta=2^\circ$)

distribution for steady flow at same convergent angle and same Reynolds number. The shape of radial velocity distribution is almost independent of Reynolds number, but depends on the convergent angle.

The radial position, where the radial velocity variation $\Delta\bar{V}$ shows its maximum, approaches the wall as Strouhal number increases. This corresponds to the fact that as Strouhal number increases, the distribution of $\Delta\bar{U}$ approaches the uniform distribution.

Fig. 18 Pressure distribution for steady flow

(a) $\theta = 2.0^\circ$ $Re = 304$ $\beta = 0.24$

Fig. 19 Pressure distribution for oscillatory flow

Figs. 18 and 19 show the calculated pressure distributions. It is clear from Fig. 18, which shows the pressure distribution of steady flow, that the pressure at each section in convergent tube is almost uniform. Fig. 19 shows the pressure distribution of oscillatory flow. The instants $\tau=0.25$ and $\tau=0.75$ correspond to the time when the flow rate is maximum and when minimum respectively. At $\tau=0.5$, when the flow is in deceleration, through the whole pipe line the pressure gradient is positive, that is the pressure at downstream point is higher than that at upstream. But the region near the downstream end of convergent tube the pressure gradient is negative because of the velocity increase. The numerical results for other Reynolds numbers and other frequencies show almost the same pressure distribution. Then it can be concluded that the pressure distribution at any section in convergent tube is uniform even in oscillatory flow.

6 Conclusions

The laminar oscillatory flow of an incompressible fluid through a slightly convergent tube is investigated numerically and experimentally. The flow in the convergent tube is numerically analysed by a finite difference method without neglecting the convective term in Navier-Stokes equation. The velocity distributions are measured by a Laser Doppler Velocimeter. Experimental results agreed well with the numerical ones. In the range of investigation, that is $\theta=2\sim4^\circ$, $R_e=100\sim500$, $S_r=0.08\sim2.61$, $\beta=0.14\sim0.24$, the following conclusions can be obtained.

- (1) The time mean velocity distribution of oscillatory flow in convergent tube coincides with the distribution of steady flows for same convergent angle and same Reynolds number. The profile of the velocity distribution is quite different from the profile in straight tube of circular cross-section, even if the convergent angle is very small.
- (2) The profile of velocity distribution in convergent tube approaches the uniform velocity distribution because of the effect of convergence as the flow goes down-

stream. This tendency becomes more remarkable as the convergent angle and Reynolds number increase.

(3) The distribution of velocity variation in oscillatory flow, which indicates the difference between the instantaneous velocity and time mean velocity, depends on Strouhal number. Similarly to the velocity variation of oscillatory flow in straight tube, the distribution of velocity variation approaches the uniform distribution except for the region near the wall.

(4) In both cases of steady and oscillatory flows, the pressure distributions in convergent tube are almost uniform on each cross-section regardless of convergent angle, Reynolds number and Strouhal number.

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