Vibration of a Bowed String*

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By Seiichiro MAEZAWA** and Katsuhiro TEMMA***

Steady self-excited vibrations of a bowed string with flexural rigidity and internal friction subject to small air resistance are analysed by Fourier series solutions by means of series transformation method.

The existence of small flexural rigidity brings about a large increase of the magnitude and fluctuation of the necessary bow force to maintain vibration.

Small internal friction has an effect of mitigating these bad influences of flexural rigidity.

Key Words : Theory of Vibration, Self-excited Vibration of Continuous Body, Bowed String, Flexural Rigidity and Internal Friction

1. Introduction

The authors have been for more than ten years endeavoring to analyse steady vibrations of a bowed string of string instrument, considering them as sustained self-excited vibrations proper. Namely, they are treated as results of balance between negative damping due to characteristic of dry frictional force acting between the string and the bow, on the one hand, and positive damping due to air resistance, internal friction of the string and sound energy delivery through the bridge and so on, on the other.

The method of analysis is a series transformation method introduced nearly twenty years ago by one of the authors to obtain a Fourier series solution for steady forced vibration in a piecewise-linear system ⁽¹⁾.

In the first stages of our study, the frictional force characteristic is simplified to a piecewise-linear one as shown in Fig.1, where the dynamic friction is constant and equal to that corresponding to the mean relative velocity during the slip interval ⁽²⁾⁽³⁾. Afterwards, our theory is extended to be able to include the general frictional force characteristic as shown in Fig.2 ⁽⁴⁾⁽⁵⁾. At the same time, internal friction of the string ⁽⁴⁾⁽⁷⁾, sound energy delivery through the bridge ⁽⁸⁾⁽⁹⁾ and damping due to bowing action ⁽⁷⁾⁽⁹⁾ are added to air resistance ⁽⁵⁾⁽⁶⁾ as various causes of positive damping. Meanwhile, the influence of flexural rigidity, torsional compliance of the string and compliance of bowing action are taken into consideration ⁽⁴⁾⁽⁷⁾⁽⁸⁾⁽⁹⁾⁻⁽¹¹⁾.

Fig.1 Frictional Force Characteristic with Dynamic Friction Constant

Fig.2 General Frictional Force Characteristic

The gist of our method of analysis is the recognition of the double facts, that, considering a whole violin including strings, bridge and body with its air volume as a linear vibrating system, the displacement of the bowed point is a linear response at this point of this system to the periodic force input of bow force, on the one hand, while, on the other, this bow force and the displacement of the bow point are related to each other according to the frictional force characteristic between the bow and the string. The linear response is a superposition of frequency responses to harmonic components of the periodic bow force.

The combination of these two conditions yields an infinite set of simultaneous equations to determine the unknown Fourier coefficients of the bow force.

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** Professor, Faculty of Physical sciences and Engineering, Meisei University, Hodokubo 337, Hino, Tokyo, 191 Japan.

*** Assistant Professor, Kisarazu National College of Technology, Kiyomidai-higashi 2, Kisarazu, Chiba, 292 Japan.

Two papers by McIntyre, Woodhouse and Schumacher published in 1979⁽¹²⁾⁽¹³⁾, treating also transient vibrations, start from the same recognition as ours. Their convolution of impulse responses corresponds to our superposition of frequency responses. However, approaches are entirely different. As to steady vibrations, each approach may have its advantage. This is a justification of our preparing this paper.

In the 1st report⁽⁵⁾⁽⁶⁾, our analysis deals with a perfectly flexible string subject to air resistance as the only positive damping. In this paper, the effects of flexural rigidity and internal friction of the string are also taken into consideration.

Hitherto, the effect of the flexural rigidity of a string is pointed out as the cause of inharmonicity of higher modes of its natural vibration. However, its effect on the necessary bow force to maintain steady self-excited vibration is scarcely discussed. The inharmonicity of higher modes of natural vibration brings about some detuning from resonance of higher harmonics of a sustained self-excited vibration, which has a saw-toothed waveform and so abounds with many strong higher harmonics. As a result, the small necessary bow force to maintain vibration in a perfectly flexible string is greatly magnified by the existence of rather small flexural rigidity, as shown in Chap.5.

This bad effect of flexural rigidity is more or less mitigated by the effect of internal friction and of damping due to bowing action which raise the height of the non-resonant part of the frequency response curve at the bowed point, as shown also in Chap.5.

Fig.3 Bowed String and Bow

2. Equations of Motion and Their Formal Solutions

The point S, whose distance from the left end A is ξ , of the string AB of length l with both ends fixed is bowed by a bow running perpendicularly to the string with a constant velocity v_0 (Fig.3). We assume that the width of the bow is nil and so the bowed point is a mathematical point. The distance $\bar{\xi}$ of the point S from the right end B is assumed to be greater than ξ , namely $\xi \leq \bar{\xi}$. This assumption does not detract any generality from our analysis.

We present here the important notations to be used in the following, referring to Fig.3.

- l : length of the string,
- $\xi, \bar{\xi}$: distances of the bowed point from the left end A and from the right end B, where $\xi \leq \bar{\xi}$,
- $x, \bar{x}$: distance of an arbitrary point P on the left half of the string from A and distance of an arbitrary point $\bar{P}$ on the right half from B,
- $y(x, t), \bar{y}(\bar{x}, t)$: displacements of the points P and $\bar{P}$ from their equilibrium positions,
- μ : line density of the string,
- T : tension of the string,
- $a = \sqrt{T/\mu}$: propagation velocity of lateral wave,
- $\nu = \pi a/l$: undamped natural circular frequency (fundamental tone),
- EI : flexural rigidity of the string,
- $\bar{\tau}$: coefficient of internal friction,
- $2c$: coefficient of air resistance,
- $\epsilon = (2\pi/l)\sqrt{EI/T}$: ratio of flexural rigidity,
- $\gamma = c/\nu$: damping ratio of air resistance,
- $\tau = \bar{\tau}\nu$: damping ratio of internal friction,
- t : time,
- ω : circular frequency of sustained self-excited vibration,
- $\Omega = \omega/\nu$: frequency ratio,
- $\theta = \omega t$: dimensionless time with its origin at the midpoint of the stick interval,
- $\phi = \theta - \pi$: dimensionless time with its origin at the midpoint of the slip interval,
- θ_0, ϕ_0 : lengths of the stick interval and the slip interval,
- $\theta_0/(2\pi)$: stick interval ratio,
- $\beta = \phi_0/2\pi$: slip interval ratio,
- v_0 : velocity of the bow (constant),
- v : relative velocity of the bow to the bowed point,
- $f(v)$: frictional force corresponding to v .

The basic assumptions for deriving the equations of motion are as follows:

- (1) both ends of the string are fixed,
- (2) the width of the bow is nil,
- (3) compliance and damping of bowing action are represented by two spring-dashpot systems (Fig.3),
- (4) torsion of the string is not considered,
- (5) frictional force characteristic between the bow and the string is a general one as shown in Fig.2, which may include jumps AB and A'B', and
- (6) characteristic of internal friction of the string material follows Voigt model.

The equations of motion for the left half and for the right half of the string will be

$$\left. \begin{aligned} \frac{\partial^2 y}{\partial t^2} + 2c \frac{\partial y}{\partial t} + \frac{EI}{\mu} \frac{\partial^4 y}{\partial x^4} + \bar{\tau} \frac{EI}{\mu} \frac{\partial^3 y}{\partial x^3 \partial t} \\ = a^2 \frac{\partial^2 y}{\partial x^2} \\ \frac{\partial^2 \bar{y}}{\partial t^2} + 2c \frac{\partial \bar{y}}{\partial t} + \frac{EI}{\mu} \frac{\partial^4 \bar{y}}{\partial \bar{x}^4} + \bar{\tau} \frac{EI}{\mu} \frac{\partial^3 \bar{y}}{\partial \bar{x}^3 \partial t} \\ = a^2 \frac{\partial^2 \bar{y}}{\partial \bar{x}^2} \end{aligned} \right\} \dots\dots\dots (1)$$

The boundary conditions at the left end A and at the right end B are written as

$$y_{x=0} = \bar{y}_{x=0} = 0, \quad \partial^2 y / \partial x^2|_{x=0} = \partial^2 \bar{y} / \partial \bar{x}^2|_{\bar{x}=0} = 0 \quad (2)$$

and at the bowed point they give

$$\left. \begin{aligned} y_{x=\xi} &= \bar{y}_{\bar{x}=\bar{\xi}}, \quad \partial y / \partial x|_{x=\xi} = -\partial \bar{y} / \partial \bar{x}|_{\bar{x}=\bar{\xi}} \\ \partial^2 y / \partial x^2|_{x=\xi} &= \partial^2 \bar{y} / \partial \bar{x}^2|_{\bar{x}=\bar{\xi}} \end{aligned} \right\} \quad (3)$$

$$-EI(1 + \bar{\tau} \partial / \partial t)(\partial^3 y / \partial x^3|_{x=\xi} + \partial^3 \bar{y} / \partial \bar{x}^3|_{\bar{x}=\bar{\xi}}) = B(\theta) = f(v_0 - \dot{\eta}_\xi) \quad (4)$$

where $B(\theta)$ is the frictional force exerted by the bow on the bowed point, which we have called the bow force, and

$$\dot{\eta}_\xi = \partial y / \partial t|_{x=\xi} = \partial \bar{y} / \partial t|_{\bar{x}=\bar{\xi}} \quad (5)$$

is the velocity of the bowed point (Fig. 3).

Following the foregoing papers (4) (5), we consider the simplest type of vibration, namely the type of vibration whose whole period consists of one stick interval and one slip interval.

We divide the bow force $B(\theta)$ into two components. Namely, the first is a constant frictional force $f(v_0/\beta)$, which corresponds to the mean relative velocity during the slip interval when the slope of static friction characteristic h is infinitely steep. The second is an additional bow force $G(\theta)$, which is the necessary bow force to maintain steady vibration overcoming air resistance, flexural rigidity, internal friction and sound energy delivery through the bridge and so on. This additional bow force $G(\theta)$ is equal to zero for a perfectly flexible string subject to no damping. Thus, we have

$$B(\theta) = f(v_0/\beta) + G(\theta) \quad (6)$$

Correspondingly, the displacement of the whole string is divided into two, namely

$$\left. \begin{aligned} y(x, t) &= y_s(x) + \eta(x, t) \\ \bar{y}(\bar{x}, t) &= \bar{y}_s(\bar{x}) + \bar{\eta}(\bar{x}, t) \end{aligned} \right\} \quad (7)$$

where $y_s(x)$ and $\bar{y}_s(\bar{x})$ are static deflections due to the constant force $f(v_0/\beta)$ on the bowed point of the left half and that of the right half and they are given by

$$\left. \begin{aligned} y_s(x) &= \frac{f(v_0/\beta)}{T} \left(\frac{\xi x}{l} - \frac{\sinh \lambda x \sinh \lambda \xi}{\lambda \sinh \lambda l} \right) \\ \bar{y}_s(\bar{x}) &= \frac{f(v_0/\beta)}{T} \left(\frac{\xi \bar{x}}{l} - \frac{\sinh \lambda \bar{x} \sinh \lambda \xi}{\lambda \sinh \lambda l} \right) \\ \lambda &= \sqrt{\frac{T}{EI}} \end{aligned} \right\} \quad (8)$$

The vibrational displacement $\eta(x, t)$ and $\bar{\eta}(\bar{x}, t)$ have their differential equations and boundary conditions of the same form as those Eqs. (1), (2) and (3) for $y(x, t)$ and $\bar{y}(\bar{x}, t)$. The condition corresponding to Eq. (4) has two distinct forms according as it belongs to the stick interval or to the slip interval. Thus, we have

$$\begin{aligned} -EI(1 + \bar{\tau} \partial / \partial t)(\partial^3 \eta / \partial x^3|_{x=\xi} + \partial^3 \bar{\eta} / \partial \bar{x}^3|_{\bar{x}=\bar{\xi}}) \\ = G(\theta) = h(v_0 - \dot{\eta}_\xi) - f(v_0/\beta) \end{aligned} \quad (9)$$

and

$$\begin{aligned} -EI(1 + \bar{\tau} \partial / \partial t)(\partial^3 \eta / \partial x^3|_{x=\xi} + \partial^3 \bar{\eta} / \partial \bar{x}^3|_{\bar{x}=\bar{\xi}}) \\ = G(\theta) = -(df/dv)|_{v=v_0/\beta} \left(\frac{1-\beta}{\beta} v_0 + \dot{\eta}_\xi \right) \end{aligned}$$

$$+ \frac{1}{2!} (d^2 f / dv^2)|_{v=v_0/\beta} \left(\frac{1-\beta}{\beta} v_0 + \dot{\eta}_\xi \right)^2 + \dots \quad (10)$$

In the following, all the expressions for displacements, velocities and forces are made dimensionless by taking v_0/ν , v_0 and $\{Tl/(\xi \bar{\xi})\} v_0/\nu$ as their units. We do not distinguish them by special notations.

A Fourier series for the additional bow force $G(\theta)$ can be written as

$$\begin{aligned} G(\theta) &= \sum_{n=0}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \\ &= R \sum_{n=0}^{\infty} (a_n - j b_n) e^{jn\theta}, \quad \theta = \omega t, \\ b_0 &= 0, \quad j = \sqrt{-1} \end{aligned} \quad (11)$$

where the letter R denotes the real part of the complex quantity that follows and the Fourier coefficients a_0 , a_n and b_n ($n = 1, 2, \dots$) are unknown at present to be determined afterwards.

Then, we can write formal solutions for $\eta(x, t)$ and $\bar{\eta}(\bar{x}, t)$ as the superpositions of frequency responses to harmonic components of the force input $G(\theta)$, namely

$$\left. \begin{aligned} \eta(x, t) &= R \sum_{n=0}^{\infty} N_n(x) (a_n - j b_n) e^{j(n\theta - \omega x/l)} \\ \bar{\eta}(\bar{x}, t) &= R \sum_{n=0}^{\infty} \bar{N}_n(\bar{x}) (a_n - j b_n) e^{j(n\theta - \omega \bar{x}/l)} \end{aligned} \right\} \quad (12)$$

where the frequency transfer functions are given by

$$\left. \begin{aligned} N_n(x) e^{-j\omega x/l} \\ &= N'_n(x) e^{-j\omega x/l} - N''_n(x) e^{-j\omega x/l} \\ \bar{N}_n(\bar{x}) e^{-j\omega \bar{x}/l} \\ &= \bar{N}'_n(\bar{x}) e^{-j\omega \bar{x}/l} - \bar{N}''_n(\bar{x}) e^{-j\omega \bar{x}/l} \end{aligned} \right\} \quad (13)$$

Especially, for the bowed point $x = \xi$ or $\bar{x} = \bar{\xi}$, we have

$$\eta_\xi = \eta(\xi, t) = R \sum_{n=0}^{\infty} M_n (a_n - j b_n) e^{j(n\theta - \omega \xi/l)} \quad (14)$$

where

$$M_n e^{-j\omega \xi/l} = M'_n e^{-j\omega \xi/l} - M''_n e^{-j\omega \xi/l} \quad (15)$$

$$\left. \begin{aligned} M'_n e^{-j\omega \xi/l} &= \frac{l^2}{\xi \bar{\xi}} \frac{1}{\sqrt{Z_n}} \\ &\times \frac{\sin \left(\frac{\bar{\omega}_n - j \bar{C}_n}{a} \xi \right) \sin \left(\frac{\bar{\omega}_n - j \bar{C}_n}{a} \bar{\xi} \right)}{\frac{\bar{\omega}_n - j \bar{C}_n}{a} l \sin \left(\frac{\bar{\omega}_n - j \bar{C}_n}{a} l \right)} \\ M''_n e^{-j\omega \xi/l} &= \frac{l^2}{\xi \bar{\xi}} \frac{1}{\sqrt{Z_n}} \\ &\times \frac{\sinh \left(\frac{\bar{\alpha}_n - j \bar{\beta}_n}{a} \xi \right) \sinh \left(\frac{\bar{\alpha}_n - j \bar{\beta}_n}{a} \bar{\xi} \right)}{\frac{\bar{\alpha}_n - j \bar{\beta}_n}{a} l \sinh \left(\frac{\bar{\alpha}_n - j \bar{\beta}_n}{a} l \right)} \end{aligned} \right\} \quad (16)$$

$$\left. \begin{aligned} \frac{\bar{\omega}_n - j \bar{C}_n}{a} \\ \frac{\bar{\alpha}_n - j \bar{\beta}_n}{a} \end{aligned} \right\} = \frac{2\pi}{\varepsilon} \left(\frac{\sqrt{Z_n} \mp 1}{2(1 + j n Q \tau)} \right)^{1/2} \quad (17)$$

$$Z_n = 1 + \varepsilon^2 (n^2 Q^2 - 2 j n Q \tau) (1 + j n Q \tau) \quad (18)$$

$$\tau = \frac{c}{\nu}, \quad \varepsilon = \frac{2\pi}{l} \sqrt{\frac{EI}{T}}, \quad \tau = \bar{\tau} \nu, \quad Q = \frac{\omega}{\nu} \quad (19)$$

and the signs $\sqrt{\quad}$ and $\{\quad\}^{1/2}$ for the square root of a complex quantity $A + jB$ ($A \geq 0$) mean

$$\sqrt{A+jB} = \frac{1}{\sqrt{2}} \left\{ \sqrt{A^2+B^2+A} + j \frac{B}{\sqrt{A^2+B^2+A}} \right\} \dots\dots\dots (20)$$

And the expressions in the right sides of Eqs.(13) for $N_n(x)e^{-j\mu_n x}$ and $\bar{N}_n(\bar{x})e^{-j\bar{\mu}_n \bar{x}}$ are given by substituting x for ξ for the former and $\bar{x}$ for $\bar{\xi}$ for the latter in the expressions in Eqs.(16), except in the term $l^2/(\xi\bar{\xi})$.

These formal solutions together satisfy the left halves of the boundary conditions (9) and (10).

3. Fourier Series Solution by Series Transformation

Our remaining task is to determine the Fourier coefficients a_n , $\bar{a}_n$ and b_n ($n=1,2,\dots$) so that they satisfy the right halves of the boundary conditions (9) and (10). To accomplish this task we resort to a series transformation method⁽¹⁾. Since the transformation process for our present task is quite similar to that in the 1st report⁽⁵⁾ except the differences of the expressions for the transfer functions $N_n(x)e^{-j\mu_n x}$, $\bar{N}_n(\bar{x})e^{-j\bar{\mu}_n \bar{x}}$ and $M_n e^{-j\theta_n}$, we give here only the results of its application.

To begin with, we divide the expression for the additional bow force $G(\theta)$ into two, namely a Fourier series $K(\theta)$ which is valid only in the stick interval and a Fourier series $H(\psi)$ which is valid only in the slip interval as follows:

$$G(\theta) = K(\theta) + H(\psi) \dots\dots\dots (21)$$

$$\left. \begin{aligned} K(\theta) &= \sum_{n=0}^{\infty} (\alpha_{2m+1} \cos \mu_n \theta + \beta_{2m+1} \sin \mu_n \theta), \quad -\theta_0/2 < \theta < \theta_0/2; \quad K(\theta) = 0, \quad \theta_0/2 < \theta < 2\pi - \theta_0/2 \\ H(\psi) &= \sum_{n=0}^{\infty} (\bar{\alpha}_{2m+1} \cos \bar{\mu}_n \psi + \bar{\beta}_{2m+1} \sin \bar{\mu}_n \psi), \quad -\psi_0/2 < \psi < \psi_0/2; \quad H(\psi) = 0, \quad \psi_0/2 < \psi < 2\pi - \psi_0/2 \end{aligned} \right\} \dots\dots\dots (22)$$

where

$$\mu_n = (2m+1)\pi/\theta_0, \quad \bar{\mu}_n = (2m+1)\pi/\psi_0 \quad (m=0,1,2,\dots) \dots\dots\dots (23)$$

Fourier coefficients a_n , $\bar{a}_n$ and b_n ($n=1,2,\dots$) for the old series can be calculated from those for the new series α_{2m+1} , β_{2m+1} , $\bar{\alpha}_{2m+1}$ and $\bar{\beta}_{2m+1}$ ($m=0,1,2,\dots$) by means of the following transformation formulas

$$\left. \begin{aligned} a_n &= \frac{1}{1+\delta_{n0}} \left\{ \frac{2}{\pi} \cos \frac{n\theta_0}{2} \sum_{m=0}^{\infty} (-1)^m \frac{\mu_m}{\mu_m^2 - n^2} \alpha_{2m+1} + (-1)^n \frac{2}{\pi} \cos \frac{n\psi_0}{2} \sum_{m=0}^{\infty} (-1)^m \frac{\bar{\mu}_m}{\bar{\mu}_m^2 - n^2} \bar{\alpha}_{2m+1} \right\} \\ b_n &= \frac{2}{\pi} \cos \frac{n\theta_0}{2} \sum_{m=0}^{\infty} (-1)^m \frac{n}{\mu_m^2 - n^2} \beta_{2m+1} + (-1)^n \frac{2}{\pi} \cos \frac{n\psi_0}{2} \sum_{m=0}^{\infty} (-1)^m \frac{n}{\bar{\mu}_m^2 - n^2} \bar{\beta}_{2m+1} \end{aligned} \right\} \dots\dots\dots (24)$$

where δ_{n0} is Kronecker's delta notation, and it is equal to unity for $n=0$ and zero for $n \neq 0$. The old expression derived from Eq.(14) for the velocity of the bowed point in the stick interval, namely

$$\dot{\eta}_e = \sum_{n=1}^{\infty} n \Omega M_n ((a_n \sin \phi_n + b_n \cos \phi_n) \cos n\theta - (a_n \cos \phi_n - b_n \sin \phi_n) \sin n\theta) \dots\dots\dots (25)$$

is transformed using $\mu_k = (2k+1)\pi/\theta_0$ ($k=0,1,2,\dots$) into a new one for the stick interval as follows

$$\begin{aligned} \dot{\eta}_e &= \sum_{k=0}^{\infty} \left\{ \sum_{m=0}^{\infty} (C_{km} \alpha_{2m+1} + D_{km} \beta_{2m+1} + \bar{H}_{km} \bar{\alpha}_{2m+1} + \bar{I}_{km} \bar{\beta}_{2m+1}) \right\} \cos \mu_k \theta \\ &+ \sum_{k=0}^{\infty} \left\{ \sum_{m=0}^{\infty} (E_{km} \alpha_{2m+1} + F_{km} \beta_{2m+1} + \bar{J}_{km} \bar{\alpha}_{2m+1} + \bar{K}_{km} \bar{\beta}_{2m+1}) \right\} \sin \mu_k \theta \end{aligned} \dots\dots\dots (26)$$

The harmonic balance between the left side and the right side of the remaining right half of the boundary condition (9) for the stick interval, after introducing this new expressions for $\dot{\eta}_e$, results in an infinite set of simultaneous linear equations for α_{2m+1} , β_{2m+1} , $\bar{\alpha}_{2m+1}$ and $\bar{\beta}_{2m+1}$ ($m=0,1,2,\dots$) as an infinite number of unknowns, namely

$$\left. \begin{aligned} \sum_{m=0}^{\infty} \left\{ \left(C_{km} + \frac{l^2}{\xi \bar{\xi}} \delta_{km} \right) \alpha_{2m+1} + D_{km} \beta_{2m+1} + \bar{H}_{km} \bar{\alpha}_{2m+1} + \bar{I}_{km} \bar{\beta}_{2m+1} \right\} &= \left\{ 1 - f \left(\frac{v_0}{\beta} \right) / (\hbar v_0) \right\} \frac{4}{\pi} \frac{(-1)^k}{2k+1} \\ \sum_{m=0}^{\infty} \left\{ E_{km} \alpha_{2m+1} + \left(F_{km} + \frac{l^2}{\xi \bar{\xi}} \delta_{km} \right) \beta_{2m+1} + \bar{J}_{km} \bar{\alpha}_{2m+1} + \bar{K}_{km} \bar{\beta}_{2m+1} \right\} &= 0, \quad \sigma = \frac{T}{\pi \hbar a}, \quad (k=0,1,2,\dots) \end{aligned} \right\} \dots\dots\dots (27)$$

where δ_{km} is Kronecker's delta notation and is equal to unity for $k=m$ and zero for $k \neq m$. The coefficients of this set of equations C_{km} etc. and $\bar{H}_{km}$ etc. ($k,m=0,1,2,\dots$) are given by the sums of the following infinite series:

$$\left. \begin{aligned} C_{km} &= \frac{4}{\pi \theta_0} (-1)^{k+m} \Omega \mu_k \mu_m \sum_{n=1}^{\infty} \frac{n(1+\cos n\theta_0)}{(\mu_k^2 - n^2)(\mu_m^2 - n^2)} M_n \sin \phi_n \\ D_{km} &= -E_{mk} = \frac{4}{\pi \theta_0} (-1)^{k+m} \Omega \mu_k \sum_{n=1}^{\infty} \frac{n^2(1+\cos n\theta_0)}{(\mu_k^2 - n^2)(\mu_m^2 - n^2)} M_n \cos \phi_n \\ F_{km} &= \frac{4}{\pi \theta_0} (-1)^{k+m} \Omega \sum_{n=1}^{\infty} \frac{n^3(1+\cos n\theta_0)}{(\mu_k^2 - n^2)(\mu_m^2 - n^2)} M_n \sin \phi_n \\ \bar{H}_{km} &= \frac{4}{\pi \theta_0} (-1)^{k+m} \Omega \mu_k \bar{\mu}_m \sum_{n=1}^{\infty} \frac{n(1+\cos n\theta_0)}{(\mu_k^2 - n^2)(\bar{\mu}_m^2 - n^2)} M_n \sin \phi_n \\ \bar{I}_{km} &= \frac{4}{\pi \theta_0} (-1)^{k+m} \Omega \mu_k \sum_{n=1}^{\infty} \frac{n^2(1+\cos n\theta_0)}{(\mu_k^2 - n^2)(\bar{\mu}_m^2 - n^2)} M_n \cos \phi_n \\ \bar{J}_{km} &= -\frac{4}{\pi \theta_0} (-1)^{k+m} \Omega \bar{\mu}_m \sum_{n=1}^{\infty} \frac{n^2(1+\cos n\theta_0)}{(\mu_k^2 - n^2)(\bar{\mu}_m^2 - n^2)} M_n \cos \phi_n \\ \bar{K}_{km} &= \frac{4}{\pi \theta_0} (-1)^{k+m} \Omega \sum_{n=1}^{\infty} \frac{n^3(1+\cos n\theta_0)}{(\mu_k^2 - n^2)(\bar{\mu}_m^2 - n^2)} M_n \sin \phi_n \end{aligned} \right\} \dots\dots\dots (28)$$

($k, m=0,1,2,\dots$)

where we take the limit of both numerator and denominator tending to zero when any term in the summation Σ becomes an indefinite form $0/0$.

Similarly, a new series expression for the velocity $\dot{\eta}_s$ of the bowed point in the slip interval is obtained by performing substituting $\theta \rightarrow \phi$, $\theta_0 \rightarrow \phi_0$ and $\mu_s \rightarrow \bar{\mu}_s$ and by writing $\bar{C}_{km}$ etc. and $\bar{H}_{km}$ etc. instead of C_{km} etc. and H_{km} etc. in the expression (26), where $\bar{C}_{km}$ etc. and $\bar{H}_{km}$ etc. can be obtained from C_{km} etc. and H_{km} etc. by performing an exchange $\theta_0 \rightarrow \phi_0$ in the expressions (28). The harmonic balance between the left and the right sides of the right half of Eq.(10), after introducing this new expression for $\dot{\eta}_s$ in the slip interval, yields another infinite set of simultaneous, in general, nonlinear equations for our Fourier coefficients α_{2m+1} , β_{2m+1} , $\bar{\alpha}_{2m+1}$ and $\bar{\beta}_{2m+1}$ ($m=0, 1, 2, \dots$) as an infinite number of unknowns.

In the first report, we have written this set with the terms up to the second order of the Taylor expansion of $f(v)$ of frictional force characteristic function. In this paper, we discard terms above the first order as small correction terms to be estimated afterwards. The errors committed by this piecewise-linearization are extremely small when, as usual, the recoiling velocity of the bowed point during the slip interval is almost constant. In this case the representative point stays close to the point P in Fig.2. Thus we are also given a linear infinite set of equations in the slip interval as

$$\left. \begin{aligned} \sum_{m=0}^{\infty} \{ -A' H_{km} \alpha_{2m+1} - A' I_{km} \beta_{2m+1} + (\delta_{km} - A' \bar{C}_{km}) \bar{\alpha}_{2m+1} - A' \bar{D}_{km} \bar{\beta}_{2m+1} \} &= A' \frac{1-\beta}{\beta} \frac{4}{\pi} \frac{(-1)^k}{2k+1} \\ \sum_{m=0}^{\infty} \{ -A' J_{km} \alpha_{2m+1} - A' K_{km} \beta_{2m+1} - A' \bar{E}_{km} \bar{\alpha}_{2m+1} + (\delta_{km} - A' \bar{F}_{km}) \bar{\beta}_{2m+1} \} &= 0 \quad (k=0, 1, 2, \dots) \end{aligned} \right\} \dots\dots\dots (29)$$

where

$$A' = - \left(\frac{df}{dv} \right)_{v=v_0/\beta} \left(\frac{\xi \xi}{l^2} \right) \left(\frac{\pi a}{T} \right) \dots\dots (30)$$

For a bowing with definite ξ/l and v_0 by a given bow on a given string, namely with definite ε , τ , γ , μ , ν , a , T , $f(v)$ and h , all the coefficients C_{km} etc., $\bar{C}_{km}$ etc., H_{km} etc. and $\bar{H}_{km}$ etc. and constants in these sets of simultaneous equations (27) and (29) are functions of two parameters, namely the frequency ratio $Q = \omega/\nu$ and the stick interval ratio $\theta_0/(2\pi)$.

Numerical computations for a solution start with these parameters assumed to have some definite values. The solution thus obtained is a mathematically possible one, but is not necessarily a physically possible one, which must satisfy the actual switching-over conditions between the two intervals. If the frictional force characteristic has no hysteresis as shown in Fig.2, these conditions are given by

$$G(\theta_0/2) = G(-\theta_0/2) = F_s - f(v_0/\beta) \dots\dots (31)$$

To determine the combination of Q and $\theta_0/(2\pi)$ satisfying these conditions, we may resort to a graphical method as shown in Fig.11 in Chap.5 when high accuracy is not demanded.

Furthermore, we have to satisfy a compatibility condition that the necessary bow force $B(\theta)$ in the stick interval should not exceed in the absolute value the maximum static friction F_s , namely, for $-\theta_0/2 < \theta < \theta_0/2$

$$-F_s - f(v_0/\beta) < G(\theta) < F_s - f(v_0/\beta) \dots\dots (32)$$

If the solution obtained above does not satisfy this condition, it must be discarded as an incompatible one.

Before closing this chapter, we add several brief remarks on our method.

As seen from Eqs.(21), (22) and (23), we utilize Fourier series of odd number terms for our transformed series. Series of even number cosine terms and odd number sine terms can also be adopted for them⁽⁹⁾. The answer to the question, which one of these series is better, depends on the convergency of itself and of the infinite series for the coefficients C_{km} etc., $\bar{C}_{km}$ etc., H_{km} etc. and $\bar{H}_{km}$ etc.. The transformation formulas and the infinite set of

equations for the latter are quite similar to those given above for the former and may be omitted in this paper.

While the gist of our method is in the solution of two infinite sets of simultaneous linear solutions, except in the case where a closed solution can be analytically obtained as in the 1st report⁽⁵⁾, we must resort to solving truncated sets of finite number of equations. The convergency problem arises also here. Whether or not the sufficient conditions generally available for the existence of a unique solution and for the convergence of solutions of truncated equations to this true solution are guaranteed in our example is not known to the authors now. This point will be a subject of our future study.

The additional bow force $G(\theta)$ may have discontinuities at the switching-over points between the stick interval and the slip interval. Thus the Fourier series expressing it may manifest Gibbs phenomena. In order to avoid them and to accelerate convergence, we adopt Lanczos' smoothing procedure. This procedure consists in averaging the values of triangular series of finite number of terms over the interval $(\theta - \delta, \theta + \delta)$ and replace its value at θ by this average. For example, the n -th coefficients a_n and b_n in the expression (11) for $G(\theta)$ are multiplied by the factor $\sin n\delta/(n\delta)$. Accordingly, a_n and b_n in the expression (14) for $\dot{\eta}_s$ are also multiplied by the same factor. Real examples are shown in Chap.5.

4. Compliance and Damping of Bowing Acting

Bowing action by a bow made of animal hairs with human hand is accompanied with some compliance and damping. Moreover, the finite width of the bow, which is neglected in this paper, is a source of damping.

We simplify these effects by providing the bow with two spring-dashpot systems (Fig.3). Total spring constant and damping coefficient may be put equal to K and C respectively. If we represent by δ the vibrational component of the lag

by which the bowed point falls behind the bow, we have

$$C\delta + K\delta = G(\theta) \quad (33)$$

and $\dot{\eta}_e$ in Eqs. (9) and (10) must be replaced by $\dot{\eta}_e + \dot{\delta}$.

As a result, we add to the frequency transfer function (15), the following term

$$\left. \begin{aligned} \bar{M}_n e^{-i\delta} &= \frac{\delta}{1 + 2j\zeta n\Omega} \\ \delta &= \frac{Tl}{K\xi\xi}, \quad \zeta = \frac{C\nu}{2K} \end{aligned} \right\} \quad (34)$$

Fortunately, the additions to the coefficients C_{nm} etc., H_{nm} etc., C_{km} etc. and H_{km} etc. can be obtained in closed form in spite of the expressions as infinite series⁽¹⁶⁾.

5. Flexural Rigidity and Internal Friction

The frequency ω_n of the n -th mode of natural vibration of a perfectly flexible string is almost exactly n times the fundamental undamped natural frequency $\nu = \pi a/l$, even if it is subject to small air resistance γ ($\omega_n/\nu = \sqrt{n^2 - \gamma^2} \approx n - \gamma^2/(2n)$). Steady vibration of a bowed string has a sawtoothed waveform and contains many strong higher harmonics. However, owing to this harmonicity of higher modes of natural vibration, the necessary bow force to maintain steady vibration is extremely small because of almost exact resonance of higher harmonic components.

If small flexural rigidity ($\epsilon > 0$) exists, the frequency of the n -th mode of natural vibration becomes $n(1 + n^2\epsilon^2/4)^{1/2}$ times the fundamental undamped natural frequency ν . Then, there arise many higher harmonics of steady vibration which suffer considerable detuning from resonance. As a result, the necessary bow force to maintain vibration becomes far greater in magnitude and far richer in fluctuation than that for a perfectly flexible string. This situation can be seen from Fig. 4 and Fig. 5 (pay attention to their ordinate scales).

Fig. 4 gives the necessary additional bow force $G(\theta)$ at Helmholtz point ($\Omega = 1.0$ and $\theta_0/(2\pi) = 8/9$) of a perfectly flexible string ($\epsilon = r = 0$) subject to small air resistance ($\gamma = 0.001$) bowed at $\xi/l = 1/9$. Fig. 5 gives $G(\theta)$ at Helmholtz point of a slightly stiff string with small internal friction ($\epsilon = 0.015$ and $r = 0.01$) (for open E string we have $\epsilon = 0.0153$). The other parameters are the same as those in Fig. 4. The existence of a small flexural rigidity magnifies the necessary additional bow force by more than ten times and gives rise to severe fluctuation. Fig. 6 gives the velocity $\dot{\eta}_e$ of the bowed point corresponding to Fig. 5. All quantities marked with # are results of Lanczos' smoothing procedure, where δ specifying the interval $(\theta - \delta, \theta + \delta)$ for averaging is taken equal to $\pi/(2K+1)$ corresponding to the truncation number $2K+1$ ($K=50$) of the infinite set of equations.

Helmholtz point $\Omega = 1.0$ and $\theta_0/(2\pi) = 1 - \xi/l$ is chosen as the reference point of comparison because general features of $G(\theta)$ do not change with small changes of Ω and $\theta_0/(2\pi)$ in its vicinity, except changes

in heights of sharp spikes at both ends of the stick interval.

Now, what is the cause of this severe fluctuations of $G(\theta)$? To answer this question, we make the following considerations. Let us consider the necessary additional bow force to maintain Helmholtz' mode of vibration, which is described in our dimensionless form for the displacement of the bowed point by

$$\eta_e = \frac{2l}{\pi\xi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin \frac{n\pi\xi}{l} \sin n\theta \quad (35)$$

The Fourier coefficients a_n and b_n ($n=1, 2, \dots$) for $G(\theta)$ to maintain this mode of vibration can be calculated from Eq. (14) as follows:

$$\left. \begin{aligned} a_n &= \frac{\sin \phi_n}{M_n} \frac{2l}{\pi\xi} (-1)^{n-1} \frac{1}{n^2} \sin \frac{n\pi\xi}{l} \\ b_n &= \frac{\cos \phi_n}{M_n} \frac{2l}{\pi\xi} (-1)^{n-1} \frac{1}{n^2} \sin \frac{n\pi\xi}{l} \end{aligned} \right\} \quad (n=1, 2, \dots) \quad (36)$$

They reproduce Fig. 4 for a perfectly flexible string subject to small air resistance $\gamma = 0.001$. And they give Fig. 7 for a stiff

Fig. 4 Necessary Additional Bow Force for Perfectly Flexible String

Fig. 5 Necessary Additional Bow Force for Stiff String

Fig. 6 Velocity of Bowed Point of Stiff String

string with internal friction having the same parameters as those of Fig.5. It is needless to say that Fig.7 can not coincide exactly with Fig.5, since as seen from Fig.6 the actual waveform of velocity at the bowed point differs from that of Helmholtz' mode especially during the slip interval. Moreover, if we increase n indefinitely, a_n and b_n do not converge. In Fig. 7 n is bounded by 50. Notwithstanding these conditions, the resemblance of Fig.7 to Fig.5 is conspicuous.

Furthermore, Fig.9 gives the necessary bow force $G\#(\theta)$ for the vibration of waveform shown in Fig.8, which is Helmholtz' mode with rounded corners and is described by the formula with $\delta = \pi/20$

$$\eta_n = \frac{2l}{\pi\xi} \sum_{n=1}^{\infty} \frac{\sin n\delta}{n\delta} \frac{(-1)^{n-1}}{n^2} \sin \frac{n\pi\xi}{l} \sin n\theta \quad \dots\dots\dots(37)$$

From these figures, we know that the great increase in magnitude and severe fluctuations of $G(\theta)$ are mainly due to the sharpness of switching-over between the stick interval and the slip interval. The peculiar construction of the bow that it is made of animal hairs and has a considerable width is adopted, it seems, to supply cushioning and damping actions. These actions blunt the sharp corners of waveform and reduce the increase of magnitude and mitigate the fluctuations of the necessary bow force. We have numerical results showing that $G(\theta)$ is reduced to one half when we add to the frequency transfer functions of the string itself Eq.(15) those due to bowing acting Eq.(34) with $\bar{\sigma}=0.11536$ and $\xi=0.05$.

And it is needless to say that the main object of a wound string is to decrease its flexural rigidity while keeping the necessary line density for given pitch tuning (11).

Lastly, concerning the effect of internal friction of the string we have Fig. 10, which gives the necessary additional bow force $G(\theta)$ for a string with no internal friction ($r=0$) while having other parameters and bowing conditions equal to those in Fig.5. From this figure we see that the existence of small internal friction also serves to mitigate the bad effects of flexural rigidity.

6. Conditions of Capture and Release

If conditions of capture and release of the string by the bow are given by Eqs. (31), we resort to a graphical method as shown in Fig.11 to determine two parameters Q and $\theta_0/(2\pi)$ satisfying them, when great accuracy is not demanded. We have only to determine a point with ordinate $F_s - f(v_0/\beta)$ on the locus curve $G(\theta_0/2) = G(-\theta_0/2)$. Fig.12 gives $G(\theta)$ for the combination of the two parameters obtained in this manner. Two sharp spikes at the start and at the end of the stick interval called by Schelleng "rabbit ears" (14)(15) have equal heights 1.0. Fig.11 and Fig.12 are based on numerical computations utilizing Fourier series with even number cosine terms and

odd number sine terms mentioned in Chap.3 and taking the effect of compliance and damping of bowing action into consideration.

Fig.7 Necessary Additional Bow Force for Stiff String to Maintain Helmholtz' Mode of Vibration

Fig.8 Displacement Waveform of Helmholtz' Mode with Rounded Corners

Fig.9 Necessary Additional Bow Force for Stiff String to Maintain Vibration Mode in Fig.8

Fig.10 Necessary Additional Bow Force for Stiff String without Internal Friction

Fig.11 Graphical Solution for Conditions of Capture and Release

Fig.12 Additional Bow Force with "Rabbit Ears" ($\gamma=0.001$, $\epsilon=0.01$, $r=0.01$, $\bar{\sigma}=0.115$, 36 , $h=\infty$, $\zeta=0.05$, $G^*(-\theta_0/2)=G^*(\theta_0/2)=1.0$, $\xi/l=1/9$, $\Omega=1.0021$, $\theta_0=317.07^\circ$)

7. Conclusions

Steady self-excited vibrations of a bowed string with flexural rigidity and internal friction subject to small air resistance are analyzed by Fourier series solutions obtained by means of series transformation method.

Main results are as follows:

(1) The existence of small flexural rigidity brings about a large increase of the maximum value and severe fluctuations of the necessary bow force to maintain vibration.

(2) The sharpness of switching-over between the stick interval and the slip interval affects the severeness of the fluctuations strongly.

(3) Small internal friction and compliance and damping of bowing action have an effect of mitigating bad influence of flexural rigidity.

(4) Sharp spikes of bow force at the start and at the end of the stick interval, so-called "rabbit ears", are confirmed by theory.

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