Lateral Vibrations of an Asymmetrical Shaft Driven by a Universal Joint

(2nd Report, Generation Mechanism of Unstable Vibrations)*

By Hiroshi OTA** and Masayoshi KATO***

An asymmetrical shaft driven by a universal joint has a concentrated mass at the middle, and is assumed to have no distributed mass. The generation mechanism of unstable vibrations is explained by considering the dynamical energy of the shaft system, and the relationship between the natural angular frequencies and the rotating speed at which instability occurs. As for the unstable vibration near the major critical speed, the angular velocity fluctuation of the shaft changes the time rate of growth in the unstable vibration. The asymmetry of a flat shaft determines each of the rotating speeds from which three unstable regions expand at about half the major critical speed. The average input energy supplied during one revolution of the shaft increases monotonously for the S-type unstable vibration, while it has both positive and negative values for the D-type unstable vibration.

Key Words: Vibration of Rotating Body, Universal Joint, Asymmetrical Shaft, Natural Angular Frequency, Increase in Rate of Energy, Generation Mechanism of Unstable Vibrations

1. Introduction

The angular velocity fluctuation based on the mechanism of a universal joint changes the behavior of an asymmetrical shaft driven by the universal joint. Near the major critical speed the unstable region is extended slightly in its width. At about half the major critical speed, the driven shaft generates a different kind of vibration⁽¹⁾. It is well known that an unstable vibration occurs⁽²⁾⁽³⁾ when the asymmetrical shaft rotates with a constant angular velocity, and the generation mechanism of the vibration can be explained⁽⁴⁾⁻⁽⁶⁾.

This report presents physical explanations for the unstable vibrations of the asymmetrical shaft driven by a universal joint and also for their generation mechanism. For this purpose, some studies are made of the relationship between the natural angular frequency and the unstable region. Then, an increase in the rate of mechanical energy is determined.

2. Natural Angular Frequency and Unstable Regions

A universal joint drives an asymmetrical shaft with a concentrated mass m at the middle point S. One end of the asymmetrical shaft is supported simply by the

universal joint, and the other end by a self-aligning double-row ball bearing. When the shaft vibrates laterally, the mass point S always exists on xy plane of a stationary rectangular co-ordinate system $O-xyz$ shown in Fig. 1. Point A is the center of the universal joint, and y_a axes are perpendicular to the plane which contains both the center line Az_{a1} of the drive shaft and the center line Az of the driven shaft. θ pin (AQ), one of two pins forming the cross-pin of the universal joint, is a pin which rotates with the driven yoke, and angle θ represents the revolutionary angle of the θ pin from y_a axis. Let the joint angle be α_a , and let the maximum and minimum values of the rigidity concerning deflection of the asymmetrical shaft be k_1 and k_2 , respec-

Fig. 1 Cross-pin, stationary rectangular co-ordinate system $O-xyz$, rotational rectangular co-ordinate system $O-x'y'z$, and restoring force F ($k_1 > k_2$, this figure is viewed from z direction, and ξ is negative.)

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** Professor, Faculty of Engineering, Nagoya University, (Furo-cho, Chikusa-ku, Nagoya, Japan).

*** Assistant, Faculty of Engineering, Nagoya University.

tively. Using the rms value $p = \sqrt{(k_1 + k_2)/(2m)}$ of natural angular frequencies, we show the deflection x, y at the point S by a non-dimensional complex number Z ,

$$Z = X + iY = e^{-i\beta}(x + iy)/\delta_{st} \dots (1)$$

in which $i = \sqrt{-1}$, $\delta_{st} = g/p^2$, g is the acceleration by gravity, and β the angle between the θ pin direction and the maximum rigidity direction S1 (y'). The drive shaft is assumed to rotate with a constant angular velocity ω , and ψ pin (AP) is assumed to lie in the xz plane at time $t=0$.

Considering the small quantity up to the second order both of an asymmetry

$$\Delta = (k_1 - k_2)/(k_1 + k_2) \dots (2)$$

and of a coefficient of the angular velocity fluctuation,

$$\epsilon = \cos \alpha_a \sum_{k=1}^{\infty} 2k C_{k-1} \left(\frac{\sin \alpha_a}{2} \right)^{2k} \dots (3)$$

we obtain the following equation of motion concerning the mass point S⁽¹⁾:

$$\ddot{Z} + Z = \Delta \bar{Z} (e^{i2\Omega\tau} + \epsilon e^{i4\Omega\tau} - \epsilon) \dots (4)$$

in which $\tau = pt$, $\Omega = \omega/p$. $\bar{Z}$ is a conjugate number of Z , and dot notation means differentiation with respect to non-dimensional time τ .

For $\epsilon=0$, an exact solution of Eq. (4) exists, and this solution is given in the form⁽¹⁾

$$Z = ae^{i(p_0\tau + \varphi)} + a^* e^{i((2\Omega - p_0)\tau - \varphi)} \dots (5)$$

In Eq. (5), $p_0 = p_{0j}$ ($j=1 \sim 4$) are given by the following equations⁽⁴⁾.

$$\left. \begin{aligned} p_{01} &= \Omega + \Omega_1, & p_{02} &= \Omega - \Omega_1 \\ p_{03} &= \Omega + \Omega_2, & p_{04} &= \Omega - \Omega_2 \end{aligned} \right\} \dots (6)$$

where,

$$\left. \begin{aligned} \Omega_1 &= \sqrt{\Omega^2 + 1 + \sqrt{4\Omega^2 + \Delta^2}} \\ \Omega_2 &= \sqrt{\Omega^2 + 1 - \sqrt{4\Omega^2 + \Delta^2}} \end{aligned} \right\} \dots (7)$$

The frequencies p_{01} and p_{02} are always real numbers, while p_{03} and p_{04} become complex numbers when the following inequality holds.

$$\Omega^2 + 1 < \sqrt{4\Omega^2 + \Delta^2} \dots (8)$$

Then the shaft system generates an unstable vibration whirling with angular velocity Ω in addition to the steady state vibrations with the natural angular frequencies p_{01}, p_{02} .

The second approximation by the asymptotic method⁽¹⁾ gives the following results about the unstable conditions of the shaft system.

When $\Omega \neq 1$,

$$(i) F_1 < 0 \dots (9)$$

$$(ii) F_1 > 0, G_1 < 0, G_1^2 - 4F_1 > 0 \dots (10)$$

When $\Omega = 1/2$,

$$(i) F_2 < 0 \dots (11)$$

$$(ii) G_2^2 - 4F_2 < 0 \dots (12)$$

where,

$$\left. \begin{aligned} F_1 &= \left(d_1 d_1^* - \frac{\epsilon^2 \Delta^2}{4Q^2} \right)^2 - \frac{\Delta^2 d_1^{*2}}{4Q^2} \\ G_1 &= d_1^2 + d_1^{*2} - \frac{\Delta^2}{4Q^2} (1 - 2\epsilon^2) \\ d_1 &= \frac{1}{2Q} \left\{ 1 - Q^2 - \frac{(1 - Q^2)^2 - \Delta^2}{4Q^2} \right\} \\ d_1^* &= \frac{1}{2Q} \left\{ 1 - Q^2 - \frac{(1 - Q^2)^2 - \Delta^2/2}{4Q^2} \right\} \\ F_2 &= \left(d_2 d_2^* - \frac{\epsilon^2 \Delta^2}{16Q^2} \right)^2 - \frac{\epsilon^2 \Delta^2 d_2^{*2}}{16Q^2} \\ G_2 &= d_2^2 + d_2^{*2} + \frac{\epsilon^2 \Delta^2}{16Q^2} \\ d_2 &= \frac{1}{4Q} \left\{ 1 - 4Q^2 - \frac{(1 - 4Q^2)^2 + 4\Delta^2}{16Q^2} \right\} \\ d_2^* &= \frac{1}{4Q} \left\{ 1 - 4Q^2 - \frac{(1 - 4Q^2)^2 - 4\Delta^2/3}{16Q^2} \right\} \end{aligned} \right\} \dots (13)$$

When $\epsilon=0$, it is easily verified that Eq. (10) does not hold. Now Eq. (8) is assumed to give an exact result about the unstable region for $\epsilon=0$, and we consider the unstable condition (9). Putting $\epsilon=0$ in Eq. (9) yields

$$\frac{4Q^2 F_1}{d_1^{*2}} = \left\{ 1 - Q^2 - \frac{(1 - Q^2)^2 - \Delta^2}{4Q^2} \right\}^2 - \Delta^2 < 0 \dots (14)$$

The first approximation gives the relation $(1 - \Omega^2)^2 - \Delta^2 < 0$, which coincides with inequality (8). On the other hand, inequality (14) which is supposed to have had its degree of accuracy increased by the second approximation does not apparently coincide with Eq. (8). But Eq. (14) and Eq. (8) are confirmed to coincide by transformation of the equation. Accordingly, Eq. (9) degenerates to Eq. (8) when $\epsilon \rightarrow 0$. Thus, Eq. (9) gives the condition for generation of the unstable vibration when $\Omega \neq 1$ and $\epsilon \neq 0$.

When $\Omega = 1/2$, the unstable regions appear near the rotating speeds which satisfy $F_2=0$ and $G_2^2 - 4F_2=0$ derived by putting $\epsilon=0$ in Eqs. (11) and (12).

$$\left. \begin{aligned} \Omega_A^2 &= (1 - \Delta^2)/4 \\ \Omega_B^2 &= (1 - \Delta^2/3)/4 \\ \Omega_C^2 &= (1 + \Delta^2/3)/4 \end{aligned} \right\} \dots (15)$$

These values Ω_A, Ω_B , and Ω_C coincide with the abscissas of three points A, B, and C lying on the base line $\epsilon=0$ in Fig. 2.

Substituting $Z = X + iY$ into equation of motion (4), and separating real and imaginary parts, we obtain the following equations.

$$\left. \begin{aligned} \ddot{X} + (1 + \epsilon\Delta)X &= \Delta(X \cos 2\Omega\tau + Y \sin 2\Omega\tau) \\ &+ \epsilon\Delta(X \cos 4\Omega\tau + Y \sin 4\Omega\tau) \\ \ddot{Y} + (1 - \epsilon\Delta)Y &= \Delta(X \sin 2\Omega\tau - Y \cos 2\Omega\tau) \\ &+ \epsilon\Delta(X \sin 4\Omega\tau - Y \cos 4\Omega\tau) \end{aligned} \right\} \dots (16)$$

Equation (16) shows that the asymmetrical shaft system driven by a universal joint is equivalent to a dynamic superposition of the following three systems:

A mass system supported by the springs with such unequal rigidities as $1+\epsilon\Delta$ in X direction and $1-\epsilon\Delta$ in Y direction, and two asymmetries Δ and $\epsilon\Delta$ rotating with constant angular velocities Ω and 2Ω , respectively.

This equivalent model system is shown in Fig. 3. The rotating shaft system is considered to have vibrational characteristics similar to the asymmetrical shaft supported by the pedestals with directionally unequal rigidities. Consequently, the whirling locus of the shaft becomes generally elliptic. The shaft system has the natural angular frequencies $\pm p_{0j}$ ($j=1\sim 4$), and it produces vibrations with frequencies $4\Omega \pm p_{0j}$ which are caused by the second term of the right hand side of Eq. (16), and by both shaft asymmetry and angular velocity fluctuation. The angular velocity $\Omega=\Omega_A$ is a resonant point where frequencies $4\Omega-p_{03}$ and p_{03} meet, and $\Omega=\Omega_c$ is a resonant point of $4\Omega+p_{02}$ and $-p_{02}$. The angular velocity $\Omega=\Omega_B$ is a resonant point of $4\Omega+p_{02}$ and p_{03} , and also of $4\Omega-p_{03}$ and $-p_{02}$. Hence, Ω_A , Ω_B , and Ω_c are the solutions of $2\Omega-p_{03}=0$, $4\Omega+p_{02}-p_{03}=0$, and $2\Omega+p_{02}=0$, respectively.

The second approximate solution by the asymptotic method^{*1} yields the simultaneous differential equations concerned with amplitudes A , A^* and phases ϕ , ϕ^* of the vibrations whirling forward and backward. These equations are transformed into linear differential equations concerning U , V , U^* , and V^* as follows:

$$\left. \begin{aligned} U &= A \cos \phi, \quad V = A \sin \phi \\ U^* &= A^* \cos \phi^*, \quad V^* = A^* \sin \phi^* \end{aligned} \right\} \dots\dots (17)$$

If $\mu \pm i\lambda$ are solutions of the characteristic equation derived from the linear differential equations, then the general form of $Ae^{i\phi} = U + iV$ is given by

$$Ae^{i\phi} = e^{\mu\tau} \{ A_1 e^{i(\lambda\tau + \phi_1)} + A_2 e^{i(-\lambda\tau + \phi_2)} \} \dots\dots (18)$$

Also, $A^*e^{i\phi^*}$ is expressed similarly. Then λ represents the natural angular frequency in the rotating co-ordinate system.

According to the asymptotic method,

Fig. 2 Unstable region

the vibration recognized in the first approximation is regarded as the natural vibration of the system. The natural angular frequencies p_s viewed from the stationary co-ordinate system are determined as follows. When $\Omega \neq 1$,

$$p_s = \Omega \pm \lambda, -\Omega \pm \lambda \dots\dots (19)$$

because the natural vibration is $Z = Ae^{i(\Omega\tau + \phi)} + A^*e^{i(-\Omega\tau + \phi^*)}$. When $\Omega = 1/2$,

$$p_s = 2\Omega \pm \lambda, -2\Omega \pm \lambda \dots\dots (20)$$

because the natural vibration is $Z = Ae^{i(2\Omega\tau + \phi)} + A^*e^{i(-2\Omega\tau + \phi^*)}$.

Especially for $\epsilon=0$, the following result is obtained. When $\Omega \neq 1$,

$$\left. \begin{aligned} \dot{A} &= -\{\Delta A / (2\Omega)\} \sin 2\phi \\ \dot{\phi} &= d_1 - \{\Delta / (2\Omega)\} \cos 2\phi \\ A^* &= 0, \quad \phi^* = -d_1^* \end{aligned} \right\} \dots\dots (21)$$

is derived. For the unstable condition $D_1 = d_1^2 - \{\Delta / (2\Omega)\}^2 < 0$, $\lambda=0$ holds, and we have

$$p_s = \Omega, -\Omega - d_1^* \dots\dots (22)$$

For the stable condition $D_1 > 0$, $\lambda = \sqrt{D_1}$ holds,

$$p_s = \Omega \pm \sqrt{D_1}, -\Omega - d_1^* \dots\dots (23)$$

The frequency $-\Omega - d_1^*$ corresponds to p_{02} in Eq. (6), and $\Omega + \sqrt{D_1}$ corresponds to p_{03} . Consequently, natural vibrations with frequencies p_{02} , p_{03} , and $2\Omega - p_{03} = p_{04}$ exist. The vibration of p_{01} is not contained here because it is considered a small one of

$$-\{\Delta A^* / (8\Omega^2)\} e^{i(3\Omega\tau - \phi^*)}$$

, which is recognized in the second approximation and is excited by a natural vibration of p_{02} .

When $\Omega = 1/2$ and $\epsilon=0$, we obtain

$$\dot{A} = 0, \quad \dot{\phi} = d_2, \quad A^* = 0, \quad \dot{\phi}^* = -d_2^* \dots\dots (24)$$

and

$$p_s = 2\Omega + d_2, -2\Omega - d_2^* \dots\dots (25)$$

The values p_s of Eq. (25) correspond to p_{03} and p_{02} of Eq. (6). Accordingly, the natural vibrations of p_{02} and p_{03} exist near this rotating speed. The vibrations of p_{01} and p_{04} are not included here, because they are both small vibrations as well as $-\{\Delta A^* / (8\Omega^2)\} e^{i(3\Omega\tau - \phi^*)}$ when $\Omega \neq 1$.

Fig. 3 Equivalent model system

*1 Equations (16) and (17) in reference (1)

Furthermore, when μ is a negative value, this represents "the negative damping coefficient." We mentioned this in the previous report⁽¹⁾.

Figures 4(a), (b) show the numerical results about the natural angular frequencies λ . These figures present the curves of d_1 , $\sqrt{d_1}$, d_2 , and d_2^* , which are the values of λ when $\varepsilon=0$. The abscissas of the points A, B, and C in Fig. 2 coincide with those of the intersecting points A, B, and C at which $d_2=0$, $d_2^*=-d_2$, and $d_2^*=0$ hold in Fig. 4(b), respectively. The values of p_0 in Eq. (6) coincide well with p_s in Eqs. (22), (23), and (25).

3. Energy Supplied to Rotating Shaft System

This chapter presents an explanation for the energy supplied to a rotating shaft system which generates an unstable vibration.

The following non-dimensional quantities are used in Eqs. (29)~(34).

$$\left. \begin{aligned} F'_2 &= F_2/(mg), \quad M'_\theta = M_\theta/(mg\delta_{st}) \\ W' &= W/(mg\delta_{st}\dot{p}) \\ T' &= T/(mg\delta_{st}), \quad V' = V/(mg\delta_{st}) \end{aligned} \right\} \dots\dots (26)$$

Primes which show non-dimensional quantities are omitted hereafter.

As shown in Fig. 1, let ξ be an angle of deflection OS measured counterclockwise from the direction S2 (x') with shaft minimum rigidity. Then, the following equations hold.

$$\left. \begin{aligned} \sin \xi &= (-X \sin \theta + Y \cos \theta)/|Z| \\ \cos \xi &= (X \cos \theta + Y \sin \theta)/|Z| \end{aligned} \right\} \dots\dots (27)$$

Using the following relation

$$\theta + \xi = \tan^{-1}(Y/X) \dots\dots (28)$$

, we obtain the whirling component F_2 of

(a) $\Omega \neq 1$

(b) $\Omega \approx 1/2$

Fig. 4 Natural angular frequencies ($\varepsilon=0.1$, $\Delta=0.1$)

the restoring force F of the asymmetrical shaft (F_1 represents the centripetal component)*2.

$$F_2 = -\Delta|Z|\sin 2\xi \dots\dots (29)$$

The whirling force F_2 is generated by the drive torque M_θ transmitted to the driven shaft by the universal joint. Therefore the following equation holds.

$$M_\theta = -\Delta|Z|^2 \sin 2\xi \dots\dots (30)$$

Using Eqs. (27), (30), and the following relations

$$e^{i2\theta} = e^{i2\Omega\tau} + \varepsilon e^{i4\Omega\tau} - \varepsilon + \dots$$

$$\dot{\theta} = \Omega + 2\varepsilon\Omega \cos 2\Omega\tau + \dots$$

, we obtain the supplied energy per unit time W ,

$$\begin{aligned} W &= M_\theta \dot{\theta} \\ &= -\Delta\Omega \operatorname{Im}[Z^2(e^{-i2\Omega\tau} + 2\varepsilon e^{-i4\Omega\tau})] \dots\dots (31) \end{aligned}$$

or

$$\frac{W}{\Delta\Omega|Z|^2} = -(1 + 2\varepsilon \cos 2\Omega\tau) \sin 2\xi \dots\dots (32)$$

where, the symbol $\operatorname{Im}[\ ]$ shows an imaginary part of [a complex number]. The value of Eq. (32) for $\Omega \neq 1$ should differ considerably from that for $\Omega \approx 1/2$, because ξ depends

Fig. 5 Increase in rate of energy during growth of unstable vibration ($\Omega \neq 1$, $\Delta=0.1$)

*2 Refer to Fig. 2 in reference (5).

on the rotating angular velocity of the shaft and the whirling mode.

On the other hand, kinetic and potential energies T and V of the shaft system are given by the following equations:

$$\left. \begin{aligned} T &= (\dot{X}^2 + \dot{Y}^2)/2 \\ V &= (1-\Delta)(X^2 + Y^2)/2 \\ &\quad + \Delta(X \sin \Theta - Y \cos \Theta)^2 \end{aligned} \right\} \dots\dots\dots (33)$$

Calculating the increase in the rate of mechanical energy of the shaft system $\dot{T} + \dot{V}$ from Eq. (33), we obtain

$$\begin{aligned} \dot{T} + \dot{V} &= \dot{X}(\dot{X} + X - \Delta(X \cos 2\Theta + Y \sin 2\Theta)) \\ &\quad + \dot{Y}(\dot{Y} + Y - \Delta(X \sin 2\Theta - Y \cos 2\Theta)) \\ &\quad + \Delta\dot{\Theta}((X^2 - Y^2) \sin 2\Theta - 2XY \cos 2\Theta) \end{aligned} \dots\dots\dots (34)$$

The first and second terms in the right-hand side of Eq. (34) are equated to zero from Lagrange's equations of motion⁽¹⁾. Thus it is confirmed that the increase in the rate of mechanical energy of the shaft system $\dot{T} + \dot{V}$ coincides with $\dot{W}$ of Eq. (31).

Figures 5~7 show numerical results of calculation on the increase in the rate of the energy $\dot{W}/(\Delta\Omega|Z|^2)$ and on the angle $\xi = \angle x^*OS$ when an unstable vibration occurs. In these figures initial values are always taken as $U=V=U^*=V^*=0.005$.

When $\Omega \neq 1$, the results are shown in Fig. 5, with solid or broken lines for $\epsilon = 0.1$ or 0 (with or without fluctuation in angular velocity). Here $\dot{W}$ is divided by $|Z|^2$ to examine the increase in the rate of energy for the same shaft deflection. The rotating speeds $\Omega = 0.949$ and 1.048 are near the boundaries in the unstable region. When the shaft rotates once, the average value of $\dot{W}/(\Delta\Omega|Z|^2)$ for $\epsilon = 0.1$ is larger

Fig. 6 Increase in rate of energy during growth of S-type unstable vibration ($A > A^*$)

than that for $\epsilon = 0$. When $\Omega = 1$, the average value for $\epsilon = 0$ is larger. This fact corresponds to the analytical results concerning the negative damping coefficient μ shown in the previous study⁽¹⁾.

When $\xi = -\pi/4$, energy is supplied at the maximum rate. When $\Omega = 1$, the angle ξ takes somewhat different values from $-\pi/4$, because the shaft whirls along a spiral line and fluctuates in angular velocity. Consequently, it is more difficult for the shaft system to obtain energy than when $\epsilon = 0$. When $\Omega = 0.949$ and 1.048 , energy enters the shaft system more easily than when $\epsilon = 0$, because the angle ξ takes some values nearer to $-\pi/4$ than when $\epsilon = 0$.

Figures 6 and 7 show the increase in the rate of energy during a half rotation of the shaft when $\Omega \neq 1/2$. Figure 6 shows the results for the S-type unstable vibration which occurs in the unstable region extending from the point A in Fig. 2. Here $A > A^*$ always holds. Figure 7 shows the results for the D-type unstable vibration which occurs in the unstable region extending from the point B in Fig. 2, and for $A < A^*$. Unlike the case of $\Omega \neq 1$, the energy can go not only into but also out of the shaft system.

The change of ξ in Fig. 6 differs from that in Fig. 7. This is due to a

Fig. 7 Increase in rate of energy during growth of D-type unstable vibration ($A < A^*$)

difference of modes in which the shaft whirls. The value $W/(\Delta\Omega|Z|^2)$ corresponding to angle ξ shows that it is determined only by the sign of $-\sin 2\xi$ whether the energy goes into or out of the shaft system.

In the unstable region extending from the point C in Fig. 2, $A < A^*$ holds, and a result similar to Fig. 7 is obtained. The analytical results by the second approximation in the previous report⁽¹⁾ give the following equations.

When $\Omega \neq 1$,

$$\left. \begin{aligned} \frac{d}{d\tau}(A^2 + A^{*2}) &= -\frac{\Delta A^2}{Q} \sin 2\phi \\ \cos 2\phi &= \frac{2Q}{\Delta} \left(d_1 - \frac{A^{*2}}{A^2} d_1^* \right) \end{aligned} \right\} \dots\dots\dots (35)$$

When $\Omega \neq 1/2$,

$$\left. \begin{aligned} \frac{d}{d\tau}(A^2 + A^{*2}) &= -\frac{\epsilon \Delta A^2}{2Q^2} \sin 2\phi \\ \dot{\phi} &= d_2 - \frac{\epsilon \Delta}{4Q} \cos 2\phi - \frac{A^{*2}}{A^2} (d_2^* + \dot{\phi}^*) \end{aligned} \right\} \dots\dots\dots (36)$$

Whichever the case is, it is seen that $d(A^2 + A^{*2})/d\tau > 0$, namely, the increase in energy of the shaft system occurs under $\sin 2\xi < 0$. Consequently, when the shaft generates the S-type unstable vibration, the locus of the representative point on the plane (U, V) ought to exist in the second or fourth quadrant. When the shaft generates the D-type unstable vibration at $\Omega \neq 1/2$, the representative point on the plane (U, V) may exist in the first or third quadrant too, and the energy of the shaft system may decrease its average during one revolution of the shaft.

We calculate the average value of W during one revolution of the drive shaft $\tau_1 = N\pi/\Omega \sim \tau_2 = (N+2)\pi/\Omega$ (N is an arbitrary positive integer):

$$W_m = \frac{Q}{2\pi} \int_{\tau_1}^{\tau_2} W d\tau$$

The following equations are obtained with an accuracy of the second order of small quantities ϵ, Δ .

$$\left. \begin{aligned} W_m &= \Delta[\eta_1]_{\text{mean}} \quad (Q \neq 1) \\ W_m &= \Delta[\eta_2]_{\text{mean}} \quad (Q \neq 1/2) \end{aligned} \right\} \dots\dots\dots (37)$$

where,

$$\left. \begin{aligned} \eta_1 &= -QA^2 \sin 2\phi - \Delta AA^* \sin(\phi - \phi^*) \\ &\quad + \frac{1-Q^2}{4Q} A^{*2} \sin 2\phi^* \\ \eta_2 &= \left(\frac{1-4Q^2}{4Q} - 2\epsilon Q \right) A^2 \sin 2\phi \\ &\quad + \frac{1-4Q^2}{12Q} A^{*2} \sin 2\phi^* \end{aligned} \right\} \dots\dots\dots (38)$$

When the rotating shaft generates the S-type unstable vibration, η_1 and η_2 take values proportional to A^2 or A^{*2} . Accordingly, the average energy supplied to the shaft changes in accordance with $e^{2\mu\tau}$ (μ is the negative damping coefficient).

When the shaft generates a D-type unstable vibration ($\Omega \neq 1/2$), it is expected

Fig. 8 Increase in rate of energy during growth of D-type unstable vibration

that η_2 will change in a complex manner. Figure 8 shows the numerical result of η_2 in this case. The initial values are taken as $U=V=U^*=V^*=0.005$. According to Fig. 8, the average value of the increase in the rate of energy sometimes becomes negative during about 7 revolutions of the drive shaft.

4. Conclusions

In this report, physical considerations were made of an asymmetrical shaft driven by a universal joint, and the generation mechanism of unstable vibration was explained. The results obtained may be summarized as follows:

(1) When the shaft rotates near the major critical speed, the angular velocity fluctuation decreases the growth of unstable vibration near the middle of the unstable region. This phenomenon relates to the fact that angle ξ becomes more remote from $-\pi/4$ due to angular velocity fluctuation. Near the boundaries of the unstable region, the angular velocity fluctuation increases the growth of unstable vibration because angle ξ comes closer to $-\pi/4$.

(2) When the shaft rotates near half the major critical speed, the unstable regions appear separately at three rotating speeds of Eq. (15), as determined only by the asymmetry Δ .

(3) The energy supplied to the asymmetrical shaft has components which fluctuate periodically. The average energy supplied during one shaft revolution increases monotonously in proportion to the second power of amplitude when an S-type unstable vibration occurs. But it fluctuates periodically, and sometimes takes a negative value during some shaft revolutions.

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