DYNAMIC STRESS ANALYSIS OF A ROTATING SOLID DISC SUBJECTED TO CYCLIC ROTATIONS

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This paper presents an analysis of the dynamic stress in a solid disc subjected to cyclically varying rotations with respect to time, that is,

\[ \ddot{z}(t) = \ddot{z}_0 + \ddot{z}_1 \sin(\ddot{\omega}_2 t) \quad (\ddot{z}_0, \ddot{z}_1, \ddot{\omega}_2: \text{constants}) \]

The radial displacement and stress components are derived from a general solution for a solid disc, which was obtained previously by the author. A dynamic solution leads to the existence of the two kinds of resonance circular frequencies. For \( \ddot{z}_1 > 0 \) and \( \ddot{z}_0 = 0 \), the resonance circular frequency \( \ddot{\omega}_2 \) agrees with half of \( \lambda_n \), which corresponds to the eigen-circular frequencies for in-plane vibrations of rotating discs. For \( \ddot{z}_1 > 0 \) and \( \ddot{z}_0 > 0 \), \( \ddot{\omega}_2 \) agrees in both cases, \( \lambda_n \) and \( \lambda_n/2 \). It is finally discussed how the circular frequency \( \ddot{\omega}_2 \) influences the stress components and their distributions.

Key Words: Elasticity, Structural Analysis, Rotating Disc, Dynamic Stress, Cyclic Rotation

1. Introduction

To estimate stresses exactly in a disc subjected to rapidly changing rotations, it is insufficient to make a quasi-static analysis of it. A dynamic analysis is required. So far, the dynamic analysis has been carried out for discs which are subjected to the rotations in a ramp, step and exponentially increasing process with respect to time \( t \). The obtained results show the dynamic behavior of the disc is similar to that of a spring-mass oscillating system. This fact proves the existence of natural frequencies for the rotating discs. Meanwhile, the natural circular frequencies of the in-plane vibrations for the rotating discs have been studied by Bhuta et al. \( (4) \), Brunell \( (5) \), Amada \( (6) \) and Ohtaki \( (7) \).

When the disc is subjected to cyclically varying rotations, the stresses occurring in the disc are considerably different from the case of the monotonically increasing or decreasing rotations. Chowdhury et al. \( (8) \) and Kikuchi et al. \( (9) \) studied on this problem, but they did not discuss the numerical computations.

By using a general solution obtained previously by the author, this paper presents a dynamic stress analysis of a rotating solid disc subjected to cyclically varying rotations. Especially, when the circular frequency of the rotating disc approaches the natural frequency, the dynamic stresses in the disc are discussed in detail.

2. Statement of Problem and its Solution

A solid disc with radius \( a \) is rotating at the angular speed \( \omega(t) \) as shown in Fig.1.

![Fig.1 Rotating disc](image)

Under several assumptions, the equation of motion in the radial direction is given by

\[ \frac{\partial^2 \ddot{u}}{\partial t^2} + \frac{\partial}{\partial r} \left( \frac{\partial \ddot{u}}{\partial r} \right) = \frac{\sigma}{1-\nu} \frac{\partial^2 \ddot{u}}{\partial r^2} \]

in a dimensionless form. Let \( \ddot{u} \) and \( \ddot{r} \) be the radial displacement and time, respectively. The dimensionless stress components are represented by

\[ \ddot{\sigma}_t = \frac{1}{1-\nu} \left[ \ddot{\sigma}_r + \ddot{\sigma}_\theta \right] \quad (2a) \]

\[ \ddot{\sigma}_\theta = \frac{1}{1-\nu} \left[ \ddot{\sigma}_r - \ddot{\sigma}_\theta \right] \quad (2b) \]
where $v$ is Poisson's ratio. The dimensionless quantities are defined by
\[ \bar{u} = u/a, \bar{t} = t/a, \bar{\sigma} = \sigma/E, \bar{\omega} = (a/\pi)\bar{\omega}, \bar{f} = f(a) \]
where the constant $a$ is given by
\[ a^2 = (1 - v^2)\mu/E \]
Assuming that there is no external force on the outer periphery of the disc, the boundary condition is given by
\[ \bar{u} = 0 \text{ at } \bar{r} = 1 \]
\[ \bar{u} = \bar{\sigma} = 0 \text{ at } \bar{t} = 0 \]
which stands for the state of rest.

Under the conditions (5) and (6), a solution to Eq. (1) is given by
\[ \bar{\omega}(\bar{r}, \bar{t}) = 4(1 + \nu) \int_0^1 \frac{j(\lambda_\nu f)}{\Phi(\lambda_\nu, \nu)} \int_0^\pi \bar{\omega}(\bar{r} - \bar{\xi}) \sin \bar{\xi} \, d\bar{\xi} \, d\bar{\tau} \]
where the eigenvalue $\lambda_\nu$ is the $n$th root satisfying the following characteristic equation,
\[ \lambda_\nu J_\nu(\lambda_\nu) + 2 \nu J_\nu(\lambda_\nu) - \lambda_\nu J_\nu(\lambda_\nu) = 0 \]
\[ \lambda_\nu = 0, \quad n = 1, 2, \ldots, \infty \]
The function $\Phi(\lambda_\nu, \nu)$ in Eq. (7) is given by
\[ \Phi(\lambda_\nu, \nu) = -\frac{1}{2} \frac{4}{2} \frac{\lambda_\nu J_\nu(\lambda_\nu)}{\Phi(\lambda_\nu, \nu)} \]
The stress components $\bar{\sigma}_r, \bar{\sigma}_0$ are derived from Eq. (2) as follows:
\[ \bar{\sigma}_r = -\frac{2}{1 + \nu} \int_0^\pi \frac{j(\lambda_\nu f)}{\Phi(\lambda_\nu, \nu)} \int_0^\pi \bar{\omega}(\bar{r} - \bar{\xi}) \sin \bar{\xi} \, d\bar{\xi} \, d\bar{\tau} \]
\[ \bar{\sigma}_0 = -\frac{2}{1 + \nu} \int_0^\pi \frac{j(\lambda_\nu f)}{\Phi(\lambda_\nu, \nu)} \int_0^\pi \bar{\omega}(\bar{r} - \bar{\xi}) \sin \bar{\xi} \, d\bar{\xi} \, d\bar{\tau} \]

3. Dynamic Stresses for a Disc Subjected to Cyclic Rotations

The variable rotations of the disc are specified by the cyclic process, as shown in Fig. 2.

\[ \bar{\omega}(t) = \bar{\Omega}_0 + \bar{\Omega}_1 \sin \bar{\omega}_1 t \]

where $\bar{\Omega}_0$ is the average angular speed. The quantities in Eq. (11) are transformed into non-dimensional ones by Eq. (3). Substituting Eq. (11) into Eq. (7), the displacement $\bar{u}$ is written as
\[ \bar{u} = \bar{u}_0 + \bar{u}_1 + \bar{u}_2 \]
where $\bar{u}_0$ is due to the step rotation $\bar{\Omega}_0$, $\bar{u}_1$ due to $\bar{\Omega}_1$ and the cyclic rotation, and $\bar{u}_2$ only due to the cyclic rotation, which are represented by
\[ \bar{u}_0 = 4(1 + \nu) \frac{\lambda_\nu J_\nu(\lambda_\nu)}{\Phi(\lambda_\nu, \nu)} \]
\[ \bar{u}_1 = 4(1 + \nu) \frac{\lambda_\nu J_\nu(\lambda_\nu)}{\Phi(\lambda_\nu, \nu)} \frac{\sin \bar{\omega}_1 t + \sin \bar{\omega}_1 \bar{\omega}_1}{\lambda_\nu + \bar{\omega}_1} \]
\[ \frac{\sin \bar{\omega}_1 t - \sin \bar{\omega}_1 \bar{\omega}_1}{\lambda_\nu - \bar{\omega}_1} \]
\[ \bar{u}_2 = 4(1 + \nu) \frac{\lambda_\nu J_\nu(\lambda_\nu)}{\Phi(\lambda_\nu, \nu)} \frac{1 - \cos \bar{\omega}_1 t}{\lambda_\nu + \bar{\omega}_1} \]
\[ \frac{\cos \bar{\omega}_1 t - \cos \bar{\omega}_1 \bar{\omega}_1}{\lambda_\nu - \bar{\omega}_1} \]

Fig. 2 Rotating process of cyclic variation

The stress components $\bar{\sigma}_r, \bar{\sigma}_0, \bar{\sigma}_\theta$ are given by
\[ \bar{\sigma}_r = \bar{\sigma}_0 + \bar{\sigma}_1 + \bar{\sigma}_2 \]
\[ \bar{\sigma}_0 = \frac{2}{1 + \nu} \int_0^\pi \frac{j(\lambda_\nu f)}{\Phi(\lambda_\nu, \nu)} \]
\[ \bar{\sigma}_1 = \frac{2}{1 + \nu} \int_0^\pi \frac{j(\lambda_\nu f)}{\Phi(\lambda_\nu, \nu)} (\lambda_\nu f - \lambda_\nu f) \]
\[ \bar{\sigma}_2 = \frac{2}{1 + \nu} \int_0^\pi \frac{j(\lambda_\nu f)}{\Phi(\lambda_\nu, \nu)} (\lambda_\nu f - \lambda_\nu f) \]

The non-dimensional quantities $\bar{\Omega}_0, \bar{\Omega}_1, \bar{\Omega}_2$ in Eqs. (13)-(17) are defined by
\[ \bar{\Omega}_1 = (a/\pi)\bar{\Omega}_0, \bar{\Omega}_1 = (a/\pi)\bar{\Omega}_0, \bar{\Omega}_1 = (a/\pi)\bar{\Omega}_0 \]

For $\bar{\Omega}_0 = 0$ and $\bar{\Omega}_1 = 0$, twice the circular frequency $\lambda_n$, which agrees with $\lambda_n$, becomes a resonance frequency which makes the stress and displacement blow up. For $\bar{\Omega}_0 = 0$ and $\bar{\Omega}_1 = 0$, there are two kinds of resonance frequencies, which are $\lambda_n$, $\lambda_n$, $\lambda_n$. The stresses and displacement are proportional to $1/\lambda_n$, $2\lambda_n$, $3\lambda_n$ as $\lambda_n$ comes close to $\lambda_n$ for $\bar{\Omega}_0 = 0$ and $\bar{\Omega}_1 = 0$. They are proportional to $1/\lambda_n$, $2\lambda_n$, $3\lambda_n$ as $\lambda_n$ and $\lambda_n$ approach $\lambda_n$ and $\lambda_n$ respectively for $\bar{\Omega}_0 = 0$ and $\bar{\Omega}_1 = 0$.

The characteristic value $\lambda_n$ given by
\[ \lambda_n = \lambda_n, \lambda_n, \lambda_n \]

(6), (10),
An increasing of inertial moment brings about a divergence of the stress and displacement, which is caused by radial deformation. Bhuta et al. (4) called this phenomenon "static resonance". Since Eq. (8) contains Poisson's ratio $\nu$, the value of $\lambda_n$ may depend on $\nu$. The influence of $\nu$ on $\lambda_n$ is shown in Fig. 3 for $n=1$ to 5, where the symbol $\bar{\omega}$ in

$$\bar{\omega}_n = n \sqrt{\frac{E}{\rho}}$$

(18)

which means that the critical angular speed $\omega$ decreases with the radius $a$ and increases with $\sqrt{E/\rho}$. The heavier is the disc material, the lower is the critical angular speed $\omega$.

The detailed descriptions of this problem are given in the papers (4) to (7).

4. Computational Results

The numerical computations are carried out for a disc with the radius $a=0.15m$. The disc material is S45C whose properties are $E=205,950MPa(2.1x10^5kg/cm^2)$, $\nu=0.3$, $\rho=7.702x10^3$ kg/m$^3(0.786x10^{-6}kg/mm^2)$, $\omega=0.278$. The eigenvalues $\lambda_n$ (n=1 to 10) from Eq. (8) are listed in Table 1.

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Table 1 Eigenvalues $\lambda_n$ by Eq. (8)

Instead of the non-dimensional $\bar{\omega}$, time $t$ with dimension is employed to grasp the phenomena easily. The dynamic stress changes sinuously with respect to time after start of rotation. Its maximum amplitude is attained at the second cycle, at $t=1.65x10^{-4}s$ and its minimum at fourth cycle. After the same varying processes as the previous one is repeated. A half of the maximum amplitude of the dynamic stress $\sigma_d$ for $\bar{\omega}_2=0.8$

$$\sigma_d = \frac{\rho^2}{8} (3+3\nu-1) \dot{\theta}^2$$

(19)

$$\sigma_d = \frac{\rho^2}{8} (\frac{3(1+\nu)}{1-\nu}) \dot{\theta}^2$$

instead of the non-dimensional $\bar{\dot{\theta}}$, time $t$ with dimension is employed to grasp the phenomena easily. The dynamic stress changes sinuously with respect to time after start of rotation. Its maximum amplitude is attained at the second cycle, at $t=1.65x10^{-4}s$ and its minimum at fourth cycle. After the same varying processes as the previous one is repeated. A half of the maximum amplitude of the dynamic stress $\sigma_d$

$$\sigma_{d,max}$$

The time-response of the dynamic stress $\sigma_d = \sigma_{d,max}$ at $t=0$ is shown in Fig. 5 for $\bar{\omega}_2=0.98$

Fig. 5 Time-response of dynamic stress for $\bar{\omega}_2=0.8$

case we have $\bar{\omega}_2=0.8$, with respect to time for $\bar{\omega}_2=0.8$. The dotted line indicates the quasi-static stress which is given by

$$\sigma_{t} = \frac{\rho^2}{8} (\frac{3(1+\nu)}{1-\nu}) \dot{\theta}^2$$

The time-response of the dynamic stress $\sigma_d$ for $\bar{\omega}_2=0.8$

in the range of $t=0 - 4.5x10^{-4}s$. The amplitude of a sinusoidally varying stress increases linearly with time in this interval. But its increasing rate decreases gradually with time elapsed and the maximum amplitude takes place at the 13th cycle, at $t=11.2x10^{-4}s$. After that, the amplitude decreases and attains its minimum at $t=22.6x10^{-4}s$. This changing process is repeated. As shown in this figure, the time-response of the dynamic stress causes "beat" oscillation. The ratio of the maximum dynamic stress $\sigma_{d,max}$ to the quasi-static stress $\sigma_{t,max}$ reaches 14.55. This phenomenon is caused as twice the circular frequency $\bar{\omega}_2$ comes closer to the lowest natural frequency $\lambda_1=2.035$ for the in-plane vibrations of the rotating
Fig. 5 Time-response of dynamic stress for $\bar{\omega}_2=0.90$

disc.

Fig. 6 shows a similar time-response for $\bar{\omega}_2=1.0$ to Fig. 5, in the interval $t=0-4.5 \times 10^{-4}$ s. The cyclic variation of the dynamic stress becomes larger than in the case of Fig. 5. The maximum amplitude occurs at $t=25.85 \times 10^{-4}$ s and the value of $\omega_{r,\text{max}}$ is 31 times $\omega_{r,\text{max}}$. It can be seen that the amplitude of "the beat oscillation" becomes larger and its period is longer as $2\bar{\omega}_2$ approaches the 1st resonance frequency $\lambda_1$.

The results in Fig. 4-6 show that the dynamic stress at the center of the disc varies sinusoidally and its amplitude also changes sinusoidally, which is called "beat oscillation", as $2\bar{\omega}_2$ comes close to the resonance frequency $\lambda_1$. A typical variation of the dynamic stress is illustrated in Fig. 7. The quantity $\omega_{r,\text{max}}$ is defined as half the amplitude of the dynamic stress-variation and $\omega_{r,\text{max}}$ is its maximum value and $\omega_{r,\text{max}}$ the maximum value of the quasi-static stress.

Fig. 8 shows how the circular frequency $\bar{\omega}_2$ in Eq. (11) influences the period and amplitude. The ordinate is $\omega_{r,\text{max}}$ and the abscissa is time after start of rotation. It can be understood that the amplitude of "the beat oscillation" considerably increases as $2\bar{\omega}_2$ approaches $\lambda_1$. The period of "the beat oscillation" becomes $1.29 \times 10^{-7}$ s for $\bar{\omega}_2=0.95$ and $4.99 \times 10^{-3}$ s for $\bar{\omega}_2=1.0$, and the ratio between those values is approximately 3.9.

The relation between $\omega_{r,\text{max}}$ and $\bar{\omega}_2$ is shown in Fig. 9. The ordinate of this figure is the ratio $\eta$ of the maximum dynamic stress $\omega_{r,\text{max}}$ to the quasi-static one $\omega_{r,\text{max}}$ and the abscissa is the circular frequency $\bar{\omega}_2$. Half of the 1st resonant frequency $\lambda_{1/2}=1.0175$ is plotted on that axis. The circles in this figure denote the computed results and the dotted line does the value of $1/|\lambda_{1/2}|$ which may have large influence.
on the dynamic stress. For \( \tilde{N}_1 = 0.98 \), the
value of \( 1/|\lambda_1 - 2\tilde{N}_1| \) gives 2.30 whose
difference is approximately 13.2% as compared with
\( \eta = 2.65 \). For \( \tilde{N}_2 = 0.98 \), the values of \( \eta \) and
\( 1/|\lambda_1 - 2\tilde{N}_2| \) are 14.55 and 13.33 respectively,
the difference between which is approximately
8.4%. For the case of \( \tilde{N}_2 = 1.0 \), we have \( \eta =
31.02 \) and \( 1/|\lambda_1 - 2\tilde{N}_2| = 28.54 \). The difference
between them reduces to 7.9%.

The discussions done so far are focused
on the dynamic stress at the center of the
disc. It will be presented below how the

\[ \tilde{N}_1 = 1.0 \]

\[ \tilde{N}_2 = 1.0 \]

stress variation. The increasing trend of the
stress distribution in the negative region
is similar to that in Fig. 11 (a). This result
shows that the stress variation in the region
except for the center is also a sinusoidal
motion.

5. Conclusions

Adopting the cyclically varying rotation,
\[ \tilde{Q}(\tilde{t}) = \tilde{Q}_0 + \tilde{Q}_1 \sin(\tilde{\omega}_1 \tilde{t}) \]
and analyzing the dynamic stresses which were
derived from a general solution obtained
previously by the author, the following
conclusions have been obtained:

(1) The dynamic stress and displacement con-
sist of the terms influenced by \( \tilde{Q}_0 \),
\( \tilde{Q}_1 \), and \( \tilde{Q}_n \) only.

(2) For \( \tilde{N}_2 = 0 \), \( \tilde{N}_1 = 1.0 \), the resonant circular
frequency \( \tilde{\omega}_2 \) agrees with half the natu-
ral circular frequency \( \tilde{\omega}_n \) of the
in-plane vibrations.
As 2 \( \tilde{N}_2 \) approaches \( \lambda_n \), the stress and
displacement grow in accordance with
\( 1/|\lambda_n - 2\tilde{N}_2| \).

(3) For \( \tilde{N}_2 = 0 \), \( \tilde{N}_1 = 1.0 \), there are two kinds of
resonance frequencies which agree with
both \( \tilde{\omega}_n \) and \( \tilde{\omega}_n / 2 \).
As \( \tilde{N}_2 \) comes closer to \( \tilde{\omega}_n / 2 \), the
stress and displacement become propor-
tional to \( 1/|\lambda_n - 2\tilde{N}_2| \).

(4) The dynamic stress varies sinusoidally
after start of rotation and its ampli-
tude changes also in the sinusoidal
motion, which is called "beat oscilla-
tion."

(5) The ratio between the maximum dynamic
stress \( \tilde{S}_{\text{max}} \) and the quasi-static
stress \( \tilde{S}_{\text{static}} \) at the center of the
disc increases with \( \tilde{N}_2 \).

The dynamic stress grows in the formula

\[ \tilde{S}_{\text{max}} = \tilde{S}_{\text{static}} \]

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of $1/\sqrt{\alpha_1 - \delta_2}$ as $\delta_2$ approaches $\lambda_1/2$.

(6) The period of "the heat oscillation" becomes longer as $\delta_2$ approaches $\lambda_1/2$.

References

or


or


