Plastic Deformation of Metallic Materials with Elliptic Inclusions

By Takeji ABE**, Shigeru NAGAKI*** and Takaaki HAYASHI*

Plastic deformation of metallic material with elliptic inclusion is studied using the rigid-plastic finite element method. A plane model of material with elliptic inclusion is assumed. The effects of the aspect ratio, the yield stress ratio and the volume fraction of the inclusion on the deformation behaviour are studied. The results show that the deformation mode is much affected by the aspect ratio of the model. The relationship between the deformation mode and the applied stress necessary for the deformation is discussed. The effect of crystal slip is also studied by incorporating the Bishop-Hill theory of crystal slip into the rigid-plastic finite element analysis.

Key Words: Plasticity, Inclusion, Inhomogeneous Material, Deformation, Strain Distribution, Rigid-Plastic FEM

1. Introduction

It is important to clarify the elastic-plastic deformation behaviour of materials with inclusions, in order to discuss the deformation or the strength of metallic materials, such as metals with other phases or two-phase alloys(1)-(15). It is also important to study the mechanical behaviour of composite materials. Fundamental data obtained on those problems are expected to be useful for the material design of alloys or composite materials.

J. D. Eshelby(1)-(3) first analysed the deformation of elliptic inclusion imbedded in an infinite elastic body. Since then, his analysis has been applied to various problems(4)-(8).

M. F. Ashby(9)(10) proposed a theory of precipitation hardening of metals, in which the misfit strain due to inclusions is assumed to be replaced by dislocations. His idea has been applied to the explanation of the strain hardening of metals with small particles. The flow stress of multi-phase material has been discussed by Y. Tomita et al(11)-(15).

Analysis based on the upper bound theory in plasticity was also applied to the deformation of a rigid-plastic inclusion imbedded in a rigid-plastic matrix, where elastic deformation is ignored(16)(17). It was pointed out that the volume fraction and the shape of inclusion affect the deformation behaviour of the material.

The results about plastic deformation mentioned above are based on some simplified assumptions. It seems necessary to obtain calculated results based on more detailed analysis.

The inhomogeneous plastic deformation of a two-dimensional material with circular or elliptic rigid-plastic inclusions in a rigid-plastic matrix is analysed in the present paper. The rigid-plastic finite element method (R-P FEM)(18)-(22) based on the upper bound theory in plasticity is mainly used. Bishop-Hill theory(23)(24) is partly used in order to take the crystal slips of fcc metals into account(25)(26), which will be called the rigid-slip-plastic finite element method (R-S-P FEM).

2. Analysis

Rigid-plastic finite element method based on the upper bound theory in plasticity is adopted in the following analysis. The method was first formulated by S. Kobayashi et al(18)(19) and W. Klie et al(20), in which the incompressibility condition is satisfied with the Lagrangian multiplier(27). Namely, it is known from the variational principle that the value of the following function takes the minimum value for the correct solution of velocity field,

\[ \Phi = \int \sigma \varepsilon \, \text{d}V - \int a \phi_0 \, \text{d}V - \int \sigma \varepsilon \, \text{d}V \]

where \( \sigma \) is the equivalent stress, \( \phi \) the Lagrangian multiplier, \( T \) the force prescribed on the surface \( S \) of the body \( V \), and \( u \) is the admissible velocity field. The relation \( a \, u = 0 \) represents the condition of incompressibility. The equivalent strain rate \( \varepsilon \) is given by

* Received 31st January, 1985
** Professor, Department of Applied Mechanics, Okayama University, Tsushima Naka 3-1-1, Okayama, 700 Japan.
 *** Lecturer, ditto.
 + Sanyo Electric Co., Yasuhachi-cho, Yasuhachi-gun, Gifu Prefecture, 503-01 Japan.
The rigid-plastic finite element method is a method of numerical calculation in which a discretization technique similar to the conventional finite element method is adopted in order to minimize Eq.(1).

Bishop-Hill theory is used to incorporate crystal slips into rigid-plastic finite element analysis. According to the Bishop-Hill theory, the plastic work rate for unit volume of a face-centered cubic metal deforming with slips is given by

\[ W = -B \varepsilon_t + A \dot{\varepsilon}_t + 2F \dot{\varepsilon}_{pl} + 2G \dot{\varepsilon}_{iso} + 2H \dot{\varepsilon}_{vp} \]

where \( B, A, F, G, \) and \( H \) are constants determined from the combination of slip systems \( (23)(24) \). The actually operating slip systems are given as a combination of slip systems which gives the maximum value of \( W \) in Eq.(3). Equations (1) and (3) are combined in the rigid-plastic analysis \( (25)(26) \).

3. Model and Method of Calculation

The model of inhomogeneous material used in the present study is a plane model with inclusions placed regularly as shown in Fig.1. The applied force is assumed to be uniaxial in x-direction. The plane stress state is also supposed in the following analysis (it is not necessary, therefore, to consider the second term of the right hand side of Eq.(1)). The total region of the model material can be represented with a rectangular region OACB owing to the symmetry of the geometrical configuration. It is also possible to assume that the boundary lines OACB remain straight after deformation. As for the boundary conditions, OB is fixed in x-direction and the displacement corresponding to the mean strain \( \varepsilon_t \) is given in x-direction at the boundary AC. The boundary OA is fixed in y-direction and the displacement corresponding to the strain \( -\varepsilon/2 \) is given in y-direction at the boundary BC. Instead of the strain rate used in the previous chapter, only the strain will be used in the following, considering a small time step.

The region OACB is divided into 252 triangular elements with 143 nodes. The equivalent strain \( \varepsilon_{eq} \) becomes very small and the term \( 1/\varepsilon_{eq} \) diverges when a rigid area appears during deformation. When \( \varepsilon_{eq} \) is smaller than a certain value, a small amount of strain is added in order to avoid divergence of calculation, which was proposed by K. Oeakada, J. Nakano and K. Mori \( (28) \).

In order to study the effect of the shape of the inclusions, the aspect ratio \( R \) is defined by

\[ R = \frac{b}{a} \]

where \( a \) and \( b \) are the lengths of the unit cell OACB in x and y directions, respectively, as shown in Fig.1.

The inclusions are assumed to be harder than the matrix in the present study. The yield stresses of the inclusion and the matrix are assumed to be \( \sigma_1 \) and \( \sigma_2 \), respectively. They are substituted into \( \sigma_{eq} \) in Eq.(1). The yield stress ratio \( a \) is defined by

\[ a = \frac{\sigma_1}{\sigma_2} \]

The yield stress ratio \( a \) is mainly assumed to be 1.5 in the following numerical examples. The volume fraction of inclusions is supposed to be \( \psi = 0.1 \sim 0.3 \).

R. Hill \( (29)(31) \) proposed a strain concentration tensor \( A_{\varepsilon} \) to describe the deformation behavior of inhomogeneous materials. It is defined by

\[ \varepsilon_{eq} = A_{\varepsilon\varepsilon} \varepsilon_{\varepsilon} \]

where \( \varepsilon_{eq} \) and \( \varepsilon_{\varepsilon} \) are the strain component for an arbitrary point and the
mean strain, respectively. Although the tensor \( A_m \) in Eq. (6) is defined for respective components of strain, the mean strain concentration coefficients \( A_1 \) and \( A_m \), for the inclusion and the matrix, respectively, are defined with the equivalent strain as follows.

\[
\begin{align*}
\langle \varepsilon_{1\text{m}} \rangle &= A_1 \varepsilon_m \\
\langle \varepsilon_{\text{m}} \rangle &= A_m \varepsilon_m
\end{align*}
\]  

(7)

\( \langle \varepsilon_{1\text{m}} \rangle \) and \( \langle \varepsilon_{\text{m}} \rangle \) are the mean equivalent strains for inclusion and matrix, respectively. \( \varepsilon_m \) is the mean equivalent strain for the whole material. The coefficients \( A_1 \) and \( A_m \) are suitable for considering the averaged behavior of inclusion and matrix based on plastic work.

A parameter \( \chi \) expressing the multi-axiality of strain distribution in the material is defined by

\[
\chi = \frac{\varepsilon_1}{\varepsilon_\|} = \frac{\varepsilon_{1\text{m}}}{\varepsilon_m},
\]

(8)

where \( \varepsilon_\| \) is the equivalent strain calculated from the displacements given at the boundaries. In the present uniaxial loading case, \( \varepsilon_\| = \varepsilon_x \).

Furthermore, \( x \geq 1 \), in general.

When two kinds of displacements corresponding to the average strains \( \varepsilon_x \) of 0.01 and 0.02 are given, the calculated patterns of strain distribution are the same, though the absolute value of the latter strain is twice as large as the former strain. This is because the material is assumed to be rigid-plastic and the yield stresses \( \sigma_1 \) and \( \sigma_m \) are the same even if the mean strain \( \varepsilon_x \) is different. Hence, the following calculations are carried out only for a representative value of tensile strain of \( \varepsilon_x = 0.01 \).

4. Results of Calculation

4.1 Effect of aspect ratio

Figure 3 shows the deformation patterns for different values of the aspect ratio \( a \) given by Eq.(4)(Fig.1) and the volume fraction \( V_f \). The broken lines in Fig.3 are the results of rigid-plastic calculation and the solid lines are those of the rigid-slip-plastic calculation. The yield stress ratio is taken as \( a = 1.5 \). The scale of the displacement in Fig.3 is magnified by 50. The mark \( \Delta \) in Fig.3 represents the position of the

---

Fig. 4 Strain distribution along M1 circle 
\( (R=1, V_f = 0.1, a = 1.50) \)

---

Fig. 5 Distribution of strain \( \varepsilon_x \) 
\( (\text{unit: } \varepsilon_x, R=1, V_f = 0.1, \ v=1.50) \)
boundary between the inclusion and the matrix.

Figure 4 shows the distribution of the strain components along the circular boundary M1 in the matrix shown in Fig.2. The strain components in Fig.4 are expressed as their ratios to $\varepsilon_r$. The strain near $\Psi = 90^\circ$ in Fig.4 is very small, which is due to the mutual restriction of deformation between the inclusion and the neighbouring matrix.

Figure 5 shows the distribution of the thickness strain $\varepsilon_t$, where the unit is chosen as $\varepsilon_t$. It is seen from Fig.5 that the inclusion hardly deforms in the case of $R=1$.

Figure 6 shows the relation between the coefficients $A_t$, $A_w$ defined by Eq.(7) and the aspect ratio $R$ (Eq.(4)) (the solid line: R-P FEM). It is seen from Fig.6 that $A_t\approx 0$ for the range of $R=0.5-5$. That is, only the matrix deforms while the inclusion remains undeformed.

When $R$ is smaller than 0.5, the mutual restriction of deformation in the loading direction ($x$-direction) between the inclusion and the matrix is relatively tight, and as a result the inclusion deforms and $A_t$ increases. Meanwhile, $A_w$ also increases for the aspect ratio $R\geq 5$, which is due to the mutual restriction of deformation in $y$-direction between the inclusion and the matrix. It follows that the deformation behaviour of materials with inclusion is markedly affected by the shape factor $R$.

4.2 Effect of volume fraction of inclusion

Figures 7 and 8 show the distribution of strain components along M1 circle shown in Fig.2 and the distribution of $\varepsilon_t$ in the material, respectively. The volume fraction is taken as $V_f=0.2$ and other parameters are the same as those for Figs.4 and 5. The calculated strain distributions shown in Figs.7 and 8 are similar to those shown in Figs.4 and 5, though some differences are seen locally.

The mean equivalent strain for the whole material is obtained from the equivalent strain for the respective regions as follows.

![Fig.6 Relation between strain concentration factors and aspect ratio R (a=1.50)](image)

![Fig.7 Strain distribution along M1 circle (R=1, V_f=0.2, a=1.50)](image)

![Fig.8 Distribution of strain $\varepsilon_t$ (unit: $\varepsilon_t$, R=1, V_f=0.2, a=1.50)](image)

![Fig.9 Relation between strain concentration factor $A_w$ and volume fraction $V_f$ of inclusions (a=1.50), the solid line represents Eq.(11)](image)
Dividing both sides of Eq. (9) by $\varepsilon_e$, and substituting Eq. (7), we have

$$1 = \frac{1}{V_i^2} + (1 - V_i) A_M$$  \hspace{1cm} (10)

When the inclusion hardly деforms and the deformation occurs mostly in the matrix, we may assume that $A_i = 0$, and then

$$A_M = 1 / (1 - V_i)$$  \hspace{1cm} (11)

Figure 9 shows the relation between the volume fraction $V_i$ of the inclusion and the strain concentration coefficient $A_M$ of the matrix. The calculated value of $A_M$ for $R = 1$ is in good agreement with the value obtained from Eq. (11), which corresponds to the upper bound of $A_M$.

It was reported experimentally that the deformation of inclusions comes close to that of the matrix when the volume fraction of inclusions is large (32)(33). Such trend, however, is not found in the present results for the range of $V_i = 0.1 - 0.3$. It is seen from Fig. 9 that the value of $A_i$ is almost the same irrespective of the volume fraction $V_i$ of the inclusion. Hence, the matrix deforms much more as the volume fraction of the inclusion increases.

### 4.3 Effect of the ratio of yield stresses of inclusion and matrix

All the results shown above are for the case of $\alpha = 1.5$, where $\alpha$ is the ratio of yield stresses of the inclusion and the matrix. When the yield stress ratio $\alpha$ is the same, no difference is observed in the displacement or the strain distribution, even if their absolute values are different. Hence, only the effect of the yield stress ratio $\alpha$ will be considered in the following.

The deformation of hard inclusion decreases as the value of $\alpha$ increases. Fig. 10 shows the influence of $\alpha$ on $A_i$ and $A_M$. It is seen from Fig. 10 that $A_i$ is almost equal to zero for a wide range of $R$ when $\alpha$ is large. This implies that the deformation of the matrix becomes more inhomogeneous when $\alpha$ increases. The trend is also shown in Table 1, where the difference of strains in the matrix increases with an increase in $\alpha$.

Figure 11 shows the relation between the strain concentration coefficient $A_i$ and the yield stress ratio $\alpha$. It follows that the value of $A_i$ decreases almost linearly with an increase in $\alpha$.

### Table 1 Maximum and minimum values of equivalent strain in inclusion and matrix (unit: $\%$)

<table>
<thead>
<tr>
<th>$V_i$=0.1</th>
<th>$V_i$=0.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$=1.22</td>
<td>$\alpha$=1.5</td>
</tr>
<tr>
<td>Inclusion</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.94</td>
</tr>
<tr>
<td>0.3</td>
<td>0.85</td>
</tr>
<tr>
<td>0.5</td>
<td>0.62</td>
</tr>
<tr>
<td>1.0</td>
<td>0.02</td>
</tr>
<tr>
<td>5.0</td>
<td>0.63</td>
</tr>
<tr>
<td>10.0</td>
<td>0.73</td>
</tr>
<tr>
<td>Matrix</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>1.12</td>
</tr>
<tr>
<td>0.3</td>
<td>1.98</td>
</tr>
<tr>
<td>0.5</td>
<td>3.77</td>
</tr>
<tr>
<td>1.0</td>
<td>4.45</td>
</tr>
<tr>
<td>5.0</td>
<td>2.30</td>
</tr>
<tr>
<td>10.0</td>
<td>1.42</td>
</tr>
</tbody>
</table>

### Fig. 10 Relation between strain concentration factor $A_i$ and yield stress ratio $\alpha$ ($V_i=0.1$)

### Fig. 11 Relation between strain concentration factor $A_i$ and yield stress ratio $\alpha$ ($V_i=0.1$)

### Fig. 12 Relation between strain multiaxiality factor $\alpha$ and aspect ratio $R$ ($\alpha=1.50$)
5. Discussion

5.1 Flow stress

It is important to clarify the relation between the deformation behaviour of the inclusion or the matrix and the mean flow stress of the whole material. Considering the work of plastic deformation, the mean flow stress may be estimated as follows.

The externally applied plastic work is equal to the internal plastic work, which is given as the sum of plastic works of inclusion and matrix (13) (14). When the applied strain and the mean stress are denoted by \( \varepsilon \) and \( \sigma \), respectively, the external work is expressed as \( \sigma \varepsilon \). In the present case, \( \varepsilon = \varepsilon \), then we may write as follows.

\[
\sigma \varepsilon = V/A_0 (\varepsilon A_0) + (1 - V/A_0) \varepsilon \frac{\varepsilon A_0}{\varepsilon A_0} \quad \ldots \quad (12)
\]

The mean flow stress is given from Eq.(12) by

\[
\sigma = V/A_0 (\varepsilon A_0) + (1 - V/A_0) \varepsilon \frac{\varepsilon A_0}{\varepsilon A_0} \quad \ldots \quad (13)
\]

Substituting Eq.(10) into Eq.(13), we have

\[
\sigma = V/A_0 (\varepsilon A_0) + (1 - V/A_0) \varepsilon \frac{\varepsilon A_0}{\varepsilon A_0} \quad \ldots \quad (14)
\]

Then, substituting \( \varepsilon = \sigma A_0 \), the following relation is finally obtained.

\[
\sigma / A_0 = (1 + V/A_0 (\sigma - 1)) \varepsilon \quad \ldots \quad (15)
\]

For the special case of \( A_0 = 0 \), that is, where no deformation occurs in the inclusion, we have

\[
\sigma = \varepsilon \delta \quad \ldots \quad (16)
\]

Meanwhile, when the deformation occurs under the constant strain condition (or the uniform strain condition) we have, substituting \( \varepsilon A_0 = \varepsilon A_0 = \varepsilon \), into Eq.(13),

\[
\sigma = V/A_0 (1 - V/A_0) \varepsilon \quad \ldots \quad (17)
\]

Eq.(17) is the law of mixture well-known for the composite materials. The law of mixture is a simplified formula deduced from the general equation (13) or (15).

Figure 12 shows the relation between the parameter \( x \) of the strain multiaxiality and the aspect ratio \( R \). The value of \( x \) changes nearly periodically with \( R \), though its absolute value is slightly greater than unity. Hence, Eq.(16) may be approximated as follows.

\[
\sigma = A_0 \quad \ldots \quad (18)
\]

Figure 13 shows the equivalent stresses calculated from the average nodal forces in \( x \) and \( y \)-directions with the mark \( \bullet \) or \( \triangle \). The both results are in good agreement, which indicates that the estimation of the mean flow stress from Eq.(15) is adequate. The values calculated from the law of mixture [Eq.(17)] are \( \sigma / A_0 = 1.05 \) and 1.10 for \( V = 0.1 \) and 0.2, respectively, which are shown with the dotted lines in Fig.13.

It follows from the comparison of Fig.6 and Fig.13 that the mean flow stress is influenced by the strain concentration coefficient \( A_0 \), that is, the magnitude of strain in the inclusion. The flow stress of composite materials is also affected by the manner of contribution of inclusions.
to the deformation. The deviation from the law of mixture results from the fact that the inclusion hardly deforms and deformation occurs mostly in the matrix. Thus, the flow stress is lower than that expected from the law of mixture. It is also seen from Figs. 10 and 11 that the value of $A_i$ is smaller for a larger value of yield stress ratio $\sigma_i$. Accordingly, the flow stress decreases.

M. F. Ashby (9)(10) proposed a theory of precipitation hardening of alloys based on the dislocation theory. He assumes that no deformation occurs in inclusions, and hence a misfit strain is produced. The misfit strain is relaxed by introducing dislocations around inclusions, which result in hardening of the alloy. His assumption corresponds to the case of $A_i = 0$ in the above analysis. The assumption is adequate for the aspect ratios close to $R = 1$ as shown in Figs. 6 and 10. It is, however, inadequate for the case of small values or large values of $R$ compared with unity.

5.2 Scatter of strain distribution

The above discussion was mainly on the averaged strain. The scatter of strain distribution is another important factor for the deformation of inhomogeneous materials (34). Therefore, the degree of the scatter of strain distribution in the inclusion as well as in the matrix will be examined in the following.

Table 1 shows the maximum and the minimum values of the equivalent strains in the inclusion and the matrix for respective values of the aspect ratio.

Table 2 shows some examples of the coefficient $V$ of variation which expresses the scatter of the distribution of the equivalent strains quantitatively. It is defined as follows using the ratio of the standard deviation $S$ of strain distribution from the mean strain,

$$V = \frac{S}{\bar{\varepsilon}_i} = 1 - \frac{\sum (\varepsilon_{im} - \bar{\varepsilon}_i)^2}{\sum \varepsilon_{im}}$$

where $\bar{\varepsilon}_i$ is replaced with $\bar{\varepsilon}_{im}$ for the inclusion, with $\bar{\varepsilon}_m$ for the matrix, and with $\bar{\varepsilon}_m$ for the whole material. $dS_i$ is the area of each finite element.

Eshelby (1)(3) pointed out that the strain in the inclusion is constant when uniform macroscopic strain is applied to infinite matrix with an ellipsoidal inclusion, so far as the matrix is elastic. This fact is valid for an elliptic inclusion in a two-dimensional infinite elastic body (plane strain) as well as in the three-dimensional case. Based on Eshelby's analysis, it is often assumed in the so-called self-consistent model used in the analysis of crystal plasticity that the strain is uniform in the ellipsoidal inclusion imbedded in infinite body (4).

It is noteworthy, however, that the strain distribution in the elliptic inclusion is not uniform when the matrix deforms rigid-plastically and it depends on the aspect ratio, the volume fraction and the yield stress ratio, as discussed above (Table 1). Similar trend is seen also in the calculated results under the plane strain condition (35), though the results shown in the present paper are obtained for the plane stress. Nevertheless, the strain distribution in the inclusion is nearly uniform for the range of aspect ratios $R \leq 0.1$ or $R \geq 0.5$, as shown with the sign * in Table 2.

5.3 Change in slip systems

It is possible to take the change in the slip systems after deformation into analysis, by combining the Bishop-Hill theory (23)(24) and the rigid-plastic finite element method, as reported in the previous papers (25)(26). In order to apply this method to a material with an inclusion, the orientation of tensile axis ($x$-axis) of the inclusion and that of the matrix are assumed, for example, to be [110] and [100], respectively. Then, the yield stresses of the inclusion and the
matrix are $\sigma = 3.67k$ and $\sigma_k = 2.45k$, (23)(24), where $k$ is the critical shear stress. These values correspond to the case of the yield stress ratio $\sigma = 1.5$. Fig. 14 shows the elements in which slip systems have changed during deformation. Thus a change in slip systems occurs in the region near the inclusion, where the strain state deviates from the applied uniaxial tension, that is, compressive strain in $x$-direction is comparable with tensile strain in $x$-direction.

The solid lines in Fig. 3 show the deformation patterns after the slip systems have changed. Figs. 15 and 16 show the calculated results of the distribution of strain components along the circle $M$ and the distribution of $\sigma_r$, respectively (K=1). Some examples of the calculated results of the strain concentration coefficients after the change in the slip systems are also shown in Fig. 6. It is seen that the value of $\sigma_r$ (the broken line) is lower than that of the rigid-plastic case, especially for large values of $R$.

The plane stress state is assumed in the above analysis. The deformation behaviour of inclusions in real alloys or composite materials is considered to be in between the plane stress and the plane strain condition. Similar results are obtained from the analysis assuming a plane strain, which will be reported separately(35).

6. Conclusions

The plastic deformation of a material with elliptic inclusions is investigated, where the inclusions are supposed to be placed regularly in two-dimensional arrays and the yield stress of the inclusions is higher than that of the matrix. The plane stress condition is assumed in the rigid-plastic finite element method is used. The change in crystal slip systems is also taken into account in some cases, incorporating Bishop-Hill theory. The main results obtained may be summarized as follows.

(1) The plastic deformation behaviour of the inclusion and the matrix is affected by the shape of the inclusion, the volume fraction and the yield stress ratio. This is because the mutual constraint between the inclusion and the matrix changes mainly according to the shape of the inclusion.

(2) The mean flow stress of the whole material depends on the contribution of deformations to deformation. The mutual restriction of deformation between inclusion and matrix. It is also affected by the multiaxiality of local strain due to the presence of inclusions.

(3) The distribution of plastic strains in an elliptic inclusion is not uniform, though it is uniform when the matrix is elastic.

The authors are indebted to Okayama University Computer Center and Miss M. Wakamatsu for their help in the present study.

References