An Efficient Time-marching Scheme for Solving Compressible Euler Equations

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An implicit time-marching finite-difference scheme is proposed for analysing steady two-dimensional inviscid transonic flows. The scheme is based on the well-known Beam-Warming delta-form approximate factorization scheme, but this is improved in the following two points: (i) in order to treat the fixed wall boundary condition without difficulty, momentum equations of contravariant velocity components as fundamental equations in curvilinear coordinates are used. (ii) To calculate stably with a sufficiently large Courant number, the central-difference of the Crank-Nicholson method is replaced by the upstream-difference of the Robert-Weiss method. The upstreaming is performed on the basis of the theory of characteristics and does not influence the accuracy of the solution. The flows through a converging-diverging nozzle and a symmetric wing are calculated. The calculated results agree well with the existing theories.

Key Words: Compressible Flow, Numerical Analysis, Transonic Flow, Time-marching Method, Finite-difference Method, Shock Capturing Method

1. Introduction

An implicit time-marching method was proposed in the previous paper(1) for analysing steady two-dimensional inviscid transonic flows. The time-marching methods are divided into explicit and implicit methods, and the implicit method seems to be more suitable for obtaining steady state solutions, because the method is capable of taking a sufficiently large time interval at, Courant number. However, in the widely used existing implicit time-marching methods, such as the Beam-Warming approximate-factorization(AF) scheme(2), the delta-form AF scheme(3), and their extensions to the nonorthogonal curvilinear coordinate grid by Steger et al.(4)(5), it is difficult to obtain a stable solution unless the Courant number is smaller than 2. The reason is that for the Courant number beyond this limit the diagonally dominant condition of the set of linear equations to be solved is lost. And, for a very large Courant number, the neighboring grid points in the finite-difference equations become almost independent, and the every other points connect closely. In the previous paper, the convection term was made 'upstreaming' by using the Robert-Weiss one-sided difference scheme(6) instead of the above-mentioned central-difference schemes, in order to take a sufficiently large Courant number. Although state of flow at a point depends more strongly on the upstream region than on the downstream region, the influence from the downstream can not be ignored for the subsonic flow. In the old and the new Denton scheme(7)(8), the upstreaming for stabilization is made with a partial downstreaming for pressure or density. And, in the Steger and Warming scheme(9), the upstreaming is performed according to sign of the phase velocities of waves propagating along each characteristics. This upstreaming was adopted in the previous paper as a reasonable procedure.

The present paper follows the basic idea of the previous paper, but the theory is reconstructed from the fundamental equations in order to treat the solid wall boundary easily, in a word, the momentum equations of contravariant velocities \( U \) and \( V \) on the transformed plane \( \xi \) are solved. The treatments of the solid wall boundary\((n = \text{const.})\) for the viscous flow are rather easy, though the flow near the wall which varies abruptly has to be computed precisely using a fine grid. On the other hand, for the inviscid flow the fundamental equations must be solved as well on this boundary, because only the condition \( V = 0 \) is prescribed here. This matter becomes more clear from the theory of characteristics, that is, according to this theory the same treatments as for the interior points are required at the boundary points, except that the equation of the wave towards the boundary from the outside is replaced by the only one boundary condition \( V = 0 \). Such treatments can be successful, provided that a velocity component along the boundary is introduced(10). Therefore, we will use here the momentum equations of \( U \) and \( V \). Then, as mentioned later, the \( \xi \)-sweep of the AF scheme along the wall becomes the same as that in the interior region, the \( n \)-sweeps can be performed including the boundary points as well, and the delay of the whole computation related with the solid wall treatments is avoided completely.

In the following, we shall first describe the details of the present method, then show validity of this method as a shock capturing method in some numerical examples.

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Nomenclature

\( \lambda, \beta : \text{see Eq. (17)} \)
\( \text{CFL} : \text{Courant number} \)
\( c : \text{velocity of sound} \)
\( E, F : \text{see Eq. (2)} \)
\( E, F : \text{see Eq. (9)} \)
\( e : \text{stagnation internal energy per unit volume} = \rho c_e (u^2 + v^2)/2 \)
\( G_0 = (g_{zz}/g_{11})^{1/2} \)
\( G_1 = g_{12}/g_{11} \)
\( G_2 = g_{12}/g_{22} \)
\( g_{11}, g_{12}, g_{22} : \text{metrics, see Eq. (11)} \)
\( I : \text{identity matrix} \)
\( J : \text{Jacobi, see Eq. (3)} \)
\( k : \text{ratio of specific heats} \)
\( k = k - 1 \)
\( M : \text{Mach number} \)
\( p : \text{static pressure} \)
\( q : \text{see Eq. (7)} \)
\( \text{RHS} : \text{see Eq. (16)} \)
\( S_{\xi}, S_{\eta} : \text{matrices composed of eigenvectors of} \lambda \text{ and} \beta, \text{respectively, see Eq. (22)} \)
\( t : \text{time} \)
\( U, V : \text{contravariant velocities, see Eq. (4)} \)
\( u, v : \text{x- and y-components of velocity, respectively} \)
\( x, y : \text{Cartesian coordinates} \)
\( \phi : \text{see Eq. (18)} \)
\( \rho \phi : \text{see Eq. (15)} \)
\( \Delta t : \text{time interval} \)
\( \xi, \eta : \text{grid spaces on computational plane} \)
\( \hat{\xi}, \hat{\eta} : \text{upstream-difference operators with respect to} \xi \text{ and} \eta, \text{respectively} \)
\( \lambda_A, \lambda_B : \text{diagonal matrices composed of eigenvalues of} \lambda \text{ and} \beta, \text{respectively} \)
\( \xi, \eta : \text{coordinates on computational plane} \)
\( \rho : \text{density} \)
\( \phi^e : \text{see Eq. (13)} \)

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\( j, j^N : \text{value of grid point} \xi_j \eta_j \text{ at time} t^N \)

2. Fundamental Equations

The governing equations for the unsteady two-dimensional inviscid compressible flows of a perfect gas are conservation of mass, momentum, and energy, and can be expressed in general curvilinear coordinates \( \xi, \eta \) as

\[
\dot{\xi} + \dot{\eta} = 0 \quad \quad \text{------------------------- (1)}
\]

where

\[
\begin{bmatrix}
\rho \\
p \\
\rho u \\
\rho v \\
\rho (u + \xi) \\
\rho (v + \eta)
\end{bmatrix}
= \begin{bmatrix}
\rho \dot{U} \\
\rho \dot{V} \\
\rho \dot{u} \\
\rho \dot{v} \\
\rho \dot{x} + \xi \\
\rho \dot{y} + \eta
\end{bmatrix} \quad \text{------------------------- (2)}
\]

The metrics of transformation \( \xi, \eta \ldots \) are expressed as

\[
\dot{\xi} = \eta, \dot{\eta} = -\xi \quad \text{------------------------- (3)}
\]

and the contravariant velocities \( U \) and \( V \) are

\[
U = \dot{\xi} \lambda + \xi \dot{\lambda} \\
V = \dot{\eta} \lambda + \eta \dot{\lambda}
\]

which are the velocities on \( \xi \eta \)-plane.

Eqs. (1) and (2) have been used in most of the existing studies as the fundamental equations in curvilinear coordinates. But these equations involve some troubles in the treatments of solid wall boundary conditions, that is, estimation of the starting values in the time-marching method, instability in an early stage of the computation, and slow convergence of the solution. In the present paper, the velocities \( u \) and \( v \) on \( xy \)-plane in \( \xi \) are replaced by the velocities \( U \) and \( V \) on \( \xi \eta \)-plane, in order to treat more directly in the calculation the solid wall boundary condition.

\[
U = 0 \quad \text{or} \quad V = 0 \quad \text{------------------------- (5)}
\]

Then, the first and the fourth equations in Eq. (2) are not altered, but the second and third equations are replaced by their linear combination. That is,

\[
\begin{bmatrix}
\rho \\
\rho U \\
\rho V \\
\rho (u + \xi)
\end{bmatrix}
= \begin{bmatrix}
\dot{\xi} \\
\dot{\eta} \\
\dot{\lambda} \\
\dot{\lambda}
\end{bmatrix} \quad \text{------------------------- (6)}
\]

where

\[
\dot{\xi} = J (\dot{\xi}) = 0
\]

\[
\dot{\eta} + \dot{\dot{\xi}} + \vec{F}_e + \vec{R}_e = 0
\]

Then, the values of \( \vec{R}_e \) are generally small for smoothly varying grids.

\[
\begin{bmatrix}
\rho \dot{U} \\
\rho \dot{V} \\
\rho \dot{u} + \xi \dot{\lambda} \\
\rho \dot{v} + \eta \dot{\lambda}
\end{bmatrix}
= \begin{bmatrix}
\rho \dot{U} + g_{11} p \rho \\
\rho \dot{V} + g_{12} p \\
\rho \dot{u} + \xi \dot{\lambda} + g_{11} \dot{\lambda} \rho \\
\rho \dot{v} + \eta \dot{\lambda} + g_{12} \dot{\lambda} \rho
\end{bmatrix}
\quad \text{------------------------- (7)}
\]

and

\[
\vec{R}_e = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\quad \text{------------------------- (8)}
\]

The values of \( \vec{R}_e \) are generally small for smoothly varying grids.

\[
\vec{R}_e = \vec{R}_e = 0
\]

\[
\xi, \eta \quad \text{in Eqs. (8) and (10) are the i-th components of} \vec{E} \text{ and} \vec{F}, \text{respectively. The values of} \vec{E} \text{ and} \vec{F} \text{ can be evaluated from the values of} \vec{q}, \text{the equation of state, and the following relations.}
\]

\[
p = k\rho - \rho \phi^2
\]

where

\[
\phi^2 = \left( \frac{k}{2} \right) \left( \frac{u^2 + v^2}{2} \right)
\]

\[
\dot{\xi} = \frac{\epsilon (u^2 + v^2)}{2} - 2g_{11} \rho \dot{U} + g_{11} \dot{\lambda} \rho
\]

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The fundamental equations in the present method are Eqs. (6) to (10), and here the initial value problem of Eq. (6) is solved using an implicit time-marching method to analyse steady flows.

3. Delta-form Approximate-factorization Scheme

The basic ideas in the proposed method are the same as in the previous paper[1]. So, we here give only an outline of the method. The central-difference schemes like the Crank-Nicholson scheme have been used in the existing methods by Beam-Warming[3] and Steger[4], but in the previous paper the Robert-Weiss one-sided difference scheme was applied for the purpose of ‘upstreaming’. And the linearization of the fundamental equations by the Taylor expansion and the rewriting to a delta-form approximate-factorization scheme similar to the existing methods were done in order to resolve them into one-dimensional problems and to make the computation easy. The fundamental equation to which these techniques are applied become

\( (I + \Delta t \hat{A} \Delta t \hat{B}) Q = \text{RHS} \) \hspace{1cm} (14)

where

\( I + \Delta t \hat{A} \Delta t \hat{B} \) \hspace{1cm} (15)

\( \text{RHS} = \partial U / \partial q \) \hspace{1cm} (16)

\( \hat{A} = 1 / 2, \hat{B} = \Delta t / \Delta t \) are the upstream-difference operators precisely mentioned later, and \( \Delta t \) is an approximate operator of \( \hat{L} \) with the second-order accurate central-difference. Matrices \( \hat{A} = \partial E / \partial q \) and \( \hat{B} = \partial F / \partial q \) in the case of Eq. (9) are

\[
\hat{A} = \\
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-U + g_1 \Phi & 2U - g_1 a & -g_1 b & \Phi \\
-U + g_2 \Phi & V - g_2 a & U - g_2 \Phi & \Phi \\
-k_a U + 2g_1 \Phi & k_a \Phi & -\Phi & a U - g_1 \Phi & \Phi \\
-k_b U + 2g_2 \Phi & k_b \Phi & -\Phi & b U - g_2 \Phi & \Phi \\
0 & 1 & 0 & 0 \\
-U + g_1 \Phi & 2U - g_1 a & -g_1 b & \Phi \\
-U + g_2 \Phi & V - g_2 a & U - g_2 \Phi & \Phi \\
-k_a U + 2g_1 \Phi & k_a \Phi & -\Phi & a U - g_1 \Phi & \Phi \\
-k_b U + 2g_2 \Phi & k_b \Phi & -\Phi & b U - g_2 \Phi & \Phi \\
\end{bmatrix}
\]

where

\( a = \hat{K} (x + y) = \hat{K} (g_{11} U - g_{11} V) \)
\( b = \hat{K} (x + y) = \hat{K} (g_{11} V - g_{11} U) \)
\( 2 \Phi = a U + b V \)

Since the right hand side of Eq. (14) is a finite-difference equation of Eq. (6) for the steady flow, if this residual (RHS) approaches zero, then the correction \( \Delta \hat{Y} \) also approaches zero, and the solution \( \hat{Y} \) of Eq. (6) comes to converge. The accuracy of the solution in this delta-form scheme depends only on the right hand side of Eq. (14). On the other hand, the stability, convergence and CPU-time during the computing process mainly depend on the left hand side operators. In this left hand side which is derived from Eq. (6) originally, the techniques of the upstreaming and the factorization are applied in the stage till Eq. (14), and further some modifications are performed in the coming stage.

4. Application of Characteristic Theory

As mentioned in the previous paper, the one-sided differences of Eq. (14) must be taken on the upstream side to make the coefficient matrix of the set of linear equations diagonally dominant. But the interpretation of this upstreaming is not necessarily clarified. The finite volume method by Denton[7] (7) is a delta-form explicit time-marching scheme, and during the computing process the upstreaming for the general variables together with the downsteaming for the pressure is performed according to sign of the velocity. And in the new method improved by Denton[8] (8), the corrections obtained for each cell are generally added to the downstream side cell points, but the correction of density is made also to the upstream side cell points according to the Mach number. Such corrections to the downstream side mean taking the upstream-difference from a different point of view. In the Steger-Warming implicit scheme[9] (9), the upstreaming is performed according to sign of the phase velocities of waves propagating along each characteristics but to the flow velocity. Chakravarthy[11] proposed a similar upstreaming for the solid wall boundary. Such a treatment of upstreaming seems to be reasonable in conformity with the physical phenomenon.

The eigenvalues (phase velocities of waves) \( \lambda_A \) and \( \lambda_B \) of the matrices \( \hat{A} \) and \( \hat{B} \) are

\[
\lambda_A = \begin{cases} 
U, & U \pm c \sqrt{g_{11}} \\
V, & V \pm c \sqrt{g_{11}} 
\end{cases} \hspace{1cm} (19)
\]

which are the same as those in the existing theories\( (5)(9) \). Now introducing the eigenvectors \( \xi_F \) and \( \xi_N \) corresponding to the eigenvalues \( \lambda_A \) and \( \lambda_B \), respectively, the matrices \( \hat{A} \) and \( \hat{B} \) are expressed in the products of matrices

\[
\hat{A} = S_F \hat{A} S_N, \hspace{1cm} \hat{B} = S_F \hat{B} S_N \hspace{1cm} (20)
\]

where \( \lambda_A \) and \( \lambda_B \) are the diagonal matrices of eigenvalues

\[
\begin{bmatrix}
U & 0 \\
0 & U - c \sqrt{g_{11}} \\
V & 0 \\
0 & V - c \sqrt{g_{11}} 
\end{bmatrix}
\]

and \( S_F \) and \( S_N \) are the matrices composed of the eigenvectors \( \xi_F \) and \( \xi_N \), respectively, and given as

\[
\begin{bmatrix}
C_F \n_1 \\
\bar{C}_F \n_1 \\
C_N \n_1 \\
\bar{C}_N \n_1 
\end{bmatrix} \hspace{1cm} (21)
\]

where \( \n_1 \), \( \bar{C}_F \) and \( \bar{C}_N \) are selected in a simple form of
Eq. (23) is a matrix which transforms the fundamental equations (6) in the conservation form into the nonconservation form, and Eqs. (24) and (25) are matrices which further transform this into a system of ordinary differential equations by the theory of characteristics. This theory can be applied under the premises of the linearization and the factorization of the fundamental equations. The order of eigenvalues in Eq. (21) is taken by considering the nonzero diagonal elements in Eqs. (24) and (25), and is the same as the new MacCormack scheme.

Substituting Eq. (20) into Eq. (14), and taking account of the relation \( I = S e^{-1} S e^{-1} \), we get

\[
S e^{-1}(I + \Delta t A \Delta e I) M \\
\times (I + \Delta t A \Delta e I) S d q^m = RHS 
\]

This equation can be easily solved by splitting into the following steps.

1. \( \Delta e q^m = S e^{-1} RHS \)  
2. \( \xi - \text{sweep} \)
3. \( \Delta \eta \)  
4. \( \eta - \text{sweep} \)
5. \( \Delta \eta q^m = S e^{-1} \Delta \eta q^m \)

Eqs. (27), (29) and (31) are the product of vector and matrix at each grid point, where

\[
\mathbf{M}^{-1} = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}^{-1} 
\]

The matrix in Eq. (32) depends only on the location of the grid points. On the other hand, Eqs. (28) and (30) are solved implicitly. That is, Eqs. (28) and (30) are systems of linear equations of \( \Delta q \) lying in \( \xi \)-direction and \( \Delta q \) lying in \( \eta \)-direction, respectively, and can be solved as divided into sets of linear equations with tridiagonal (actually almost bidiagonal except for boundaries) matrix. Where, \( |A| \) means to take an absolute value of each element in \( A \). Since the up-streaming in Eqs. (28) and (30) is performed according to sign of the phase velocities of waves propagating along each characteristic, all the coefficient matrices of the above four sets of linear equations are diagonally dominant. Therefore, these equations can be easily solved by Gaussian elimination.

The computation for obtaining a steady state solution by the present time-marching method is executed by the following procedures. First, values of \( q^0 \) are estimated appropriately. Next, the residual values of RHS are computed from Eq. (16), and the corrections \( \Delta q^m \) are made in the steps (1) to (V) above mentioned. The advanced values \( q^{n+1} \) are determined from Eq. (15). Then, these computations are repeated until all the residuals approach zero and the solution converges. For the purpose of capturing the shock wave, it is necessary to take into account of the fourth-order artificial dissipation term like reference (2) in the RHS. If the values of this term are adjusted suitably, the Courant numbers \( |A| \Delta t/\Delta \xi \) and \( |A| \Delta t/\Delta \eta \) can take a sufficient large value.

5. Treatments of Solid Wall Boundary

The condition on this boundary is fundamentally only one condition that the contravariant velocity component across the boundary be zero as shown in Eq. (5). For the inviscid flow there are two cases, one in which the interior region and the solid wall boundary are computed alternately, and one in which the whole region including this boundary is computed by a similar algorithm. The former case contains a method in which the pressure on the boundary is first computed from the momentum equation, and then the remaining quantities are obtained from the condition of the constant stagnation enthalpy on the adiabatic flow (4). In this kind of method the convective term of the whole computation often deteriorates as a result of this boundary treatment. On the other hand, the method using the reflection condition in the latter case is not applicable for the curvilinear coordinates or the curved boundary. And the method applying the theory of characteristics can be used generally, in which the computations of sweep along the boundary are applicable also to the boundary straightaway, and the computations of sweep across the boundary can be executed by using Eqs. (5) at the boundary points instead of a wave equation propagating towards the boundary from the outside (11). The treatments of the solid wall boundary in the present method are closely related to the last method mentioned above, but become more similar to the interior region. We now consider the boundary \( \eta = \text{const} \) whose boundary condition is \( \nu = 0 \). The \( \eta \)-derivatives of RHS in Eq. (26) on this boundary are approximated by the second-order one-sided differences, and the computations of Eqs. (27) to (31) are the very same as the interior region except for Eq. (30). The
computation of Eq. (30) also is similar to the interior region, but in the third equation of Eq. (30) at the bottom boundary (y = 1) the downstream difference is taken to avoid the employment of the exterior grid points. Consequently, the third set of linear equations become

\[
\begin{bmatrix}
1 - c_1 & c_1 & & & & \\
- c_1 & 1 + c_1 & & & & \\
& & & \ddots & & \\
& & & & 1 - c_1 & c_1 \\
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4 \\
q_5 \\
\end{bmatrix}
= \begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4 \\
q_5 \\
\end{bmatrix}
\]

where

\[
c_1 = (\Delta t c(s_2^2/2) \Delta n) \gamma_1, \quad c_2 = (\Delta t (V + c(s_2^2/2) \Delta n) \gamma_2, \quad \ldots
\]

(33)

In the case that \(1 - c_1 \neq 0\) can not be computed conventionally by Gaussian elimination. However, the solution is definite and the relation

\[
(1 - c_1)q_1 = (1 + c_1)q_2 - c_2q_3
\]

holds. Therefore, it is possible to avoid any trouble, if only the first equation of Eq. (33) is replaced by this relation beforehand. The advanced values of \(\vec{q}\) on this boundary are determined by setting that \(\Delta (\vec{q} U) = 0\). It is not possible to carry out such treatments unless the fundamental equations, Eq. (6) are employed.

6. Numerical Examples

We first computed the transonic flows with a shock wave in a converging-diverging nozzle as shown in Fig.1. The computational grid had \(37 \times 11\) grid points, and was generated analytically by using the Laplace equations. The initial conditions were prescribed such that the stagnation pressure \(p_0 = 7.0\), the stagnation density and the velocity \(u_0 = u_1 = 1.0\) at the upstream boundary and the static pressure \(p_2 = 6.0\) at the downstream boundary. And the boundary conditions were prescribed such that the values of \(p_2\) and \(u_2\) and \(u_3 = 0\) at the upstream boundary, the value of \(p_2\) and \(u_1 = 0\) at the downstream boundary, and

\(V = 0\) on the solid walls. An explicit artificial dissipation like reference(2) was added throughout the computational domain to suppress the oscillation near the shock wave. The Mach number contours and pressure contours computed under these conditions are shown in Fig.1. And Fig.2 shows the pressure distributions on the solid wall and along the center line as compared with the results of the one-dimensional flow theory. From these figures, it can be said that the computed results are at least reasonable. Fig.3 shows a comparison of the convergence histories for the maximum values of \(\Delta (\vec{q} U)\) for the Courant numbers 0.4 and 7, 5.

We next computed the flows around a symmetric parabolic wing as shown in Fig.4. The flow conditions of this computation are the same as in the case of uniform flow Mach number \(M = 0.675\) in reference(12). The computational grid shown in Fig.4 had \(65 \times 17\) grid points and was generated using Poisson's equation. Fig.4 shows the computed Mach number contours and Fig.5 the Mach number distributions along the lower and upper boundaries. These results agree very well with reference(12), but small oscillations appear near the shock. Further investigation about the addition of artificial viscosity is necessary to remove these oscillations.

7. Conclusions

An implicit time-marching method for analysing steady two-dimensional inviscid transonic flow problems has been proposed. This method is based on the well-known Beam-Warming delta-form approximate-factorization scheme, and improved in the following points.

(1) By using the momentum equations of contravariant velocities \(U\) and \(V\) as the fundamental equations in curvilinear coordinates, the difficulties concerning the treatments of solid wall boundary, such as

![Fig.1 Computational grid, Mach number contours and pressure contours (p/p0) for converging-diverging nozzle flow](image1)

![Fig.2 Pressure distributions for converging-diverging nozzle flow](image2)

![Fig.3 Converging histories for corrections](image3)
the divergence of solution, instability, and delay of convergence, have been overcome. By the present method, the boundary points can be computed at the same time almost similarly to the interior points.

(2) By using the Robert-Weiss upstream-difference scheme instead of the Crank-Nicholson central-difference scheme, the Courant number was taken sufficiently large and the convergence of solution was accelerated. This upstreaming is performed on the basis of the theory of characteristics, and does not affect the accuracy of the solution.

In order to show the validity of the present method as a shock capturing method, the method was applied to two flow problems. Comparisons of the calculated converging-diverging nozzle flows with the results of the one-dimensional flow theory and of the symmetric parabolic wing flows with the existing numerical method were in satisfactory agreement.

References