Surface Temperature and Surface Heat Flux

Determination of the Inverse Heat Conduction Problem*

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A method for determining the surface temperature and surface heat flux of a one-dimensional solid, is developed by integration of Duhamel's integral which includes unknown temperature or unknown heat flux in its integrand. Specific forms of surface condition determination are developed for sample inverse problems: slab, hollow cylinder, cylinder, hollow sphere, sphere.

Discussion on the effect of available information at an interior point due to damped system and the effect of variation of surface conditions on their formulations shows that these formulations are capable of representing the unknown surface conditions except for small time interval followed by discontinuous change of surface conditions.

A small time interval which can not describe any surface condition is demonstrated by a numerical example.

Key Words: Thermal Engineering, Thermal Conduction, Transient Heat Conduction, Inverse Problem, Duhamel's Integral

I. Introduction

In a one-dimensional transient heat conduction problem, the relation between surface and interior temperature variation is described by Duhamel's integral. To find the surface temperature in its integrand giving the interior temperature, the author has presented a method based on the concept of continuous surface temperature (1). The method shows that Duhamel's integral can be transformed in to an equation which contains unknown surface temperature and its derivatives and that the transformed equation can describe the surface temperature and the surface heat flux expressed by an interior temperature and its derivatives when the interior point is adiabatic end. And those descriptions represent the surface conditions for any time except for a small time interval followed by discontinuous change of surface temperature.

Generally, an interior point can be selected arbitrarily in a one-dimensional solid and both the temperature and the heat flux at the interior point are functions of time, respectively.

As well known in the direct problem, these variations are represented by the equation of heat conduction with the initial and boundary conditions given by the temperature and/or heat flux. It can be said that both the interior temperature and interior heat flux variations at the interior point are indicative of the whole temperature field in the solid.

Therefore, it can be considered that the temperature and its flux at an interior point of a solid will be able to determine the whole temperature field in the solid including its surface.

From this stand point, a method for determining a temperature and its flux at any point in a one-dimensional solid with given temperature and its flux at an interior point will be developed by integration of Duhamel’s integral in chapter II. It will be shown that the result is equivalent to that obtained by other methods (2)-(5) in chapter III, it will be shown the specific forms of surface conditions for a one-dimensional geometry.

And their characteristics will be discussed in chapter IV.

II. Temperature and its Flux Variation at Any Point in One-dimensional Solid Expresssed by Known Temperature and its Flux Variations at a Point in the Solid

Let \( \theta(\xi_n, \tau') \) and \( \frac{\partial \theta(\xi_n, \tau')}{\partial \xi_n} \) be known temperature and known flux at \( \xi_n \) and let \( \theta(\xi', \tau') \) and \( \frac{\partial \theta(\xi', \tau')}{\partial \xi_n} \) be unknown temperature and unknown flux at \( \xi_n \) where \( \xi' \) and \( \tau' \) are dimensionless coordinate and time normalized by the points \( \xi_n \) and \( \xi_e \).

Assume that the temperature field \( \theta(\xi', \tau') \) in the region between \( \xi_n \) and \( \xi_e \) consists of two unknown temperature distributions, i.e., \( \theta_l(\xi', \tau') \) and \( \theta_r(\xi', \tau') \), which satisfy the following conditions

\[
\theta_l(\xi_n, \tau') = \theta(\xi_n, \tau'), \frac{\partial \theta_l(\xi_n, \tau')}{\partial \xi_n} = 0, \quad \ldots \ldots \ldots \ldots (1)
\]

\[
\theta_r(\xi_n, \tau') = 0, \frac{\partial \theta_r(\xi_n, \tau')}{\partial \xi_n} = -\frac{\partial \theta(\xi_n, \tau')}{\partial \xi_n}, \quad \ldots \ldots \ldots \ldots (2)
\]

Therefore, unknown temperature \( \theta(\xi', \tau') \) and unknown flux \( \frac{\partial \theta(\xi', \tau')}{\partial \xi_n} \) at \( \xi_n \) can be

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given as the sum of \( \theta_i(x, r') \) and \( \theta_j(x, r') \) and its flux at \( x \).

The solutions between unknown \( \theta_i(x, r') \) and \( \theta_j(x, r') \) and given \( \theta_i(x, r') \) and \( \theta_j(x, r') \) and \( \partial \theta / \partial x |_{x=0} \) are expressed by Duhamel's integral as follows:

\[
\theta(x_n, r') = \int_0^{x_n} \theta(x_n, \lambda) \frac{\partial U_i(x_n, r'-\lambda)}{\partial x} \, d\lambda ,
\]

\[
\frac{\partial \theta}{\partial x} |_{x_n} = \frac{1}{u_n} \int_0^{x_n} \frac{\partial \theta}{\partial x} \left[ \frac{\partial U_i(x_n, r'-\lambda)}{\partial x} \right] \, d\lambda ,
\]

\[
U_i(x_i, r') \quad \text{and} \quad U_j(x_i, r') \quad \text{are temperature response with the following initial and boundary conditions}
\]

\[
U_i(x_i, r') = 0 , \quad r' = 0 ,
\]

\[
U_i(x_i, r') = 1 , \quad r' > 0 ,
\]

\[
\frac{\partial U_i}{\partial x} \bigg|_{x_i} = 0 ,
\]

\[
\frac{\partial U_j}{\partial x} \bigg|_{x_i} = -1 = u_i , \quad r' > 0 ,
\]

\[
U_j(x_i, r') = 0 .
\]

These \( U_i(x_i, r') \) and \( U_j(x_i, r') \), generally, can be represented by

\[
U_i(x_i, r') = 1 - \sum_{x_i} a_{i,x}e^{-\alpha x_i} ,
\]

\[
U_j(x_i, r') = \sum_{x_i} b_{i,x} e^{-\alpha x_i} ,
\]

A method for determining the unknown \( \theta_i(x_i, r') \) in the integrand of Eq. (3) has been discussed in the preceding paper. The result shows that, when \( \theta_i(x_i, r') \) is known from \( L \)-th derivatives, \( \theta_i(x_i, r') \) can be treated as an unknown continuous function of time having \( L \)-th derivatives even if \( \theta_i(x_i, r') \) is discontinuous at the surface \( x_i \).

Based on these results it is possible to handle Eq. (3) by a simpler procedure than the preceding one. Integration of Eq. (3) and its derivatives of order \( L \) are given by

\[
\theta(x_n, r') = \sum_{x_i} (-1)^L \phi(x_i, 0) \frac{\partial^L \theta_i(x_i, r')}{\partial x^L} ,
\]

\[
\frac{\partial \theta}{\partial x} |_{x_n} = \sum_{x_i} (-1)^L \phi(x_i, 0) \frac{\partial^L \theta_i(x_i, r')}{\partial x^L} + R' ,
\]

\[
\frac{\partial^2 \theta}{\partial x^2} |_{x_n} = \sum_{x_i} (-1)^L \phi(x_i, 0) \frac{\partial^L \theta_i(x_i, r')}{\partial x^L} + R'' ,
\]

where

\[
\phi(x_i, r') = \sum_{x_i} a_{i,x} e^{-\alpha x_i} ,
\]

\[
R' = \frac{L}{L-1} \sum_{x_i} (-1)^L \phi(x_i, 0) \frac{\partial^L \theta_i(x_i, r')}{\partial x^L} + R'' ,
\]

\[
\frac{\partial \theta}{\partial x} |_{x_n} = \sum_{x_i} (-1)^L \phi(x_i, 0) \frac{\partial^L \theta_i(x_i, r')}{\partial x^L} + R'' ,
\]

\[
\frac{\partial^2 \theta}{\partial x^2} |_{x_n} = \sum_{x_i} (-1)^L \phi(x_i, 0) \frac{\partial^L \theta_i(x_i, r')}{\partial x^L} + R'' ,
\]

For the similar reason (A), the term \( R' \) in Eq. (14) can be ignored and combining Eq. (14) with Eq. (12) yields

\[
\frac{\partial \theta}{\partial x} |_{x_n} = \sum_{x_i} (-1)^L \phi(x_i, 0) \frac{\partial^L \theta_i(x_i, r')}{\partial x^L} + R'' ,
\]

\[
\frac{\partial^2 \theta}{\partial x^2} |_{x_n} = \sum_{x_i} (-1)^L \phi(x_i, 0) \frac{\partial^L \theta_i(x_i, r')}{\partial x^L} + R'' ,
\]

where

\[
V_i = \sum_{x_i} (-1)^L \phi(x_i, 0) C_{i,-k} .
\]

By the same procedure, \( \frac{\partial \theta}{\partial x} |_{x_n} \) in the integrand of Eq. (4) and \( \theta_i(x_i, r') \) can be found as follows

\[
\frac{\theta}{\partial x} |_{x_n} = \sum_{x_i} (-1)^L \phi(x_i, 0) \frac{\partial^L \theta_i(x_i, r')}{\partial x^L} + R'' ,
\]

\[
\frac{\partial^2 \theta}{\partial x^2} |_{x_n} = \sum_{x_i} (-1)^L \phi(x_i, 0) \frac{\partial^L \theta_i(x_i, r')}{\partial x^L} + R'' .
\]
\[ \theta_e (\xi_e, \tau_e) = \frac{s}{\lambda_e D_e} \frac{\partial^2 \theta}{\partial \xi_e^2} \mid_{\xi_e} \]  

(20)

where 

\[ W_e = 1 / \sqrt{\tau_e} (\xi_e, 0) \]  

(21)

\[ W_e = - \sum \left( -1 \right)^i \phi (\xi_e, 0) W_{i, \tau} \]  

(22)

\[ \phi (\xi_e, \tau_e) \]  

(23)

Utilizing Eqs. (12), (20), (17) and (19), unknown temperature \( \theta (\xi_e, \tau_e) \) and unknown flux \( \frac{\partial \theta}{\partial \xi_e} \) at \( \xi_e \) \( \xi_e \) \( \xi_e \) \( \xi_e \) as follows

\[ \theta (\xi_e, \tau_e) = \theta (\xi_e, \tau_e) + \theta (\xi_e, \tau_e) \]  

(24)

\[ \frac{\partial \theta}{\partial \xi_e} \mid_{\xi_e} = \frac{\partial \theta}{\partial \xi_e} + \frac{\partial \theta}{\partial \xi_e} \mid_{\xi_e} \]  

(25)

\[ \frac{\partial \theta}{\partial \xi_e} \mid_{\xi_e} = \sum \frac{1}{\xi_e} V_e \frac{\partial \theta}{\partial \xi_e} \mid_{\xi_e} \]  

(26)

where \( L_e \) \( L_e \) \( L_e \) \( L_e \) because derivatives for known temperature and flux are given by increasing \( e \) successively.

Eqs. (26) and (26) are introduced in disregard of \( R_e \) \( R_e \) \( R_e \) \( R_e \) and corresponding terms in Eq. (8). However, it will not always be expected that \( R_e \) \( R_e \) \( R_e \) \( R_e \) and corresponding terms can be ignored. It will be shown in the following chapter that there is limitation to applicability of Eqs. (25) and (26).

The form of Eqs. (25) and (26) for \( q \) slab was presented by Carslaw and Jaeger, but no discussion on it was made by them. Their results for a slab are equal to the present ones.

Burgraff has assumed the form of the unknown temperature \( \theta (\xi_e, \tau_e) \) as follows

\[ \theta (\xi_e, \tau_e) = \frac{s}{\lambda_e D_e} \frac{\partial^2 \theta}{\partial \xi_e^2} \mid_{\xi_e} \]  

(27)

\[ \frac{\partial \theta}{\partial \xi_e} \mid_{\xi_e} = \sum \frac{1}{\xi_e} p_n (\xi_e) \frac{\partial \theta}{\partial \xi_e} \mid_{\xi_e} \]  

(28)

Supposing that this Eq. (27) must satisfy the heat conduction equation, Burgraff has shown \( f_n (\xi_e) \) and \( g_n (\xi_e) \) for slab, sphere, cylinder, hollow sphere and hollow cylinder.

Comparing Eq. (27) with Eq. (25), Burgraff's assumption was the same as for the present results except that the upper limit of summation is infinite or finite.

And it will be proved in Appendix A that Burgraff's coefficients in Eq. (27) are equivalent to the present coefficient in Eqs. (25) and (26) for any one-dimensional geometry.

Makhin et al. have used the Laplace transform for a slab and their result was the same as Burgraff's case.

Shoji has shown by the Laplace transform that the temperature and the heat flux at a point in a slab are described by the known temperatures at two points in the slab. Those descriptions are the same as those which can be deduced from Eqs. (25) and (26) utilizing the known temperature and the heat flux given by the direct problem with the known temperatures at two points.

III. Surface Temperature and Surface Heat Flux Determination for One-dimensional Geometry

Let \( \xi \) and \( \tau \) be dimensionless coordinate and time normalized by one-dimensional geometrical form. Let \( \theta (\xi_e, \tau_e) \) and \( \frac{\partial \theta}{\partial \xi_e} \mid_{\xi_e} \) be known temperature and known flux at a point \( \xi_e \) and \( \theta (\xi_e, \tau_e) \) and \( \frac{\partial \theta}{\partial \xi_e} \mid_{\xi_e} \) be unknown temperature and unknown flux at a point \( \xi_e \).

These unknown temperature and unknown flux are given by Eqs. (25) and (26) as follows

\[ \theta (\xi_e, \tau_e) = \sum \frac{1}{\xi_e} f_n (\xi_e) \]  

(29)

\[ \frac{\partial \theta}{\partial \xi_e} \mid_{\xi_e} = \sum \frac{1}{\xi_e} g_n (\xi_e) \]  

(30)

where \( \xi_e \) is a conversion factor for transformation of the coordinate system \( (\xi_e, \tau_e) \) to \( (\xi_e, \tau_e) \).

In the following part of this chapter, specific forms of the coefficient \( \xi_e \), \( \xi_e \), \( \xi_e \), \( \xi_e \) and \( \xi_e \) for one-dimensional geometry will be shown. When a value of \( \xi_e \) corresponds to the case of surface in the following expression of each coefficient, Eqs. (28) and (29) will give the description for the surface conditions.

III.1 Slab

Using a dimensionless coordinate system normalized by the thickness of a slab, \( \xi_e \) and \( \xi_e \) are given as

\[ 0 \leq \xi_e \leq 1, \quad 0 \leq \xi_e \leq 1 \]

and the surfaces are at \( \xi_e = 0 \) or at \( \xi_e = 1 \).

The coefficients are as follows

\[ \xi_e = \frac{\xi_e - \xi_e}{\xi_e - \xi_e} \]  

(30)

\[ \xi_e = \frac{1}{2(1 + \xi_e)}, \quad \xi_e = \frac{1}{2(1 + \xi_e)} \]  

(31)

\[ \xi_e = \frac{1}{2(1 - \xi_e)}, \quad \xi_e = \frac{1}{2(1 - \xi_e)} \]  

(32)

where \( \xi_e \) and \( \xi_e \) are those shown in the preceding paper and \( \xi_e \) and \( \xi_e \) can be obtained similarly for \( \xi_e \) and \( \xi_e \).

III.2 Hollow cylinder

\( \xi_e \) and \( \xi_e \) are given as

\[ \xi_e = \frac{\xi_e - \xi_e}{\xi_e - \xi_e} \]  

(33)

\[ \xi_e = \frac{1}{2(1 + \xi_e)}, \quad \xi_e = \frac{1}{2(1 + \xi_e)} \]  

(34)

\[ \xi_e = \frac{1}{2(1 - \xi_e)}, \quad \xi_e = \frac{1}{2(1 - \xi_e)} \]  

(35)

where \( \xi_e \) and \( \xi_e \) are those shown in the preceding paper and \( \xi_e \) and \( \xi_e \) can be obtained similarly for \( \xi_e \) and \( \xi_e \).
\( \delta \leq \xi_a \leq 1, \delta \leq \xi_t \leq 1 \) for \( \delta = \text{inner dia./outer dia.} \).

\( 1 \leq \xi_a \leq \delta, 1 \leq \xi_t \leq \delta \) for \( \delta = \text{outer dia./inner dia.} \).

and the surfaces are at \( \xi_a = 1 \) or at \( \xi_t = 1 \).

Either \( \delta \) is larger or smaller than unity, \( \xi \) is

\[ \xi = \xi_t \]  

(32)

and \( C_i, D_i, V_i \), and \( W_i \) can be obtained by a similar procedure shown in the preceding paper. Their recurrence formulas are given as follows

\[ C_0 = 1, \quad V_0 = 0, \]

\[ C_i = \left[ \frac{1}{\xi_t} \left( \frac{1}{\xi_t} \right)^{-1} \right] C_{i-1} \]

\[ - \left( \frac{1}{\xi_t^2} \right) V_{i-1} \quad \text{for } 2 \left( \frac{1}{\xi_t} \right)^{-1} C_{i-1} \]

\[ \times \left( \frac{1}{\xi_t^2} \right) V_{i-1} - p \ln \left( \frac{\xi_a}{\xi_t} \right) \]

\[ = \frac{1}{\xi_t} \left( \frac{1}{\xi_t} \right)^{-1} (2k) C_{i-1} - V_{i-1} \]

\[ - p \cdot \frac{1}{(2l/n)!} \left( \frac{\xi_a}{\xi_t} \right)^{2l} \]

\[ D_2 = 2k C_{i-1} - V_{i-1} \]

\[ D_i = \frac{1}{\xi_t} \left( \frac{1}{\xi_t} \right)^{-1} \]

\[ \times \left( \frac{1}{\xi_t^2} \right) V_{i-1} + \left( \frac{\xi_a}{\xi_t} \right)^{2l+1} \]

\[ \times \left( \frac{1}{\xi_t^2} \right) + p \ln \left( \frac{\xi_a}{\xi_t} \right) \]

\[ W_i = \frac{1}{\xi_t} \left( \frac{1}{\xi_t} \right)^{-1} (2k) C_{i-1} - V_{i-1} \]

\[ + p \cdot \frac{1}{(2l/n)!} \left( \frac{\xi_a}{\xi_t} \right)^{2l+1}, \]

where \( (2k)! \) is \( 2k! \) and

\[ p = 1. \]  

(35)

III.3 Cylinder

Let the dimensionless coordinate system be normalized by the radius of a cylinder.

III.3.1 The case of \( 0 < \xi_a = \xi_t < 1 \)

The coefficients are given by Eqs. (32) - (35) in the same way as in the case of hollow cylinder. However, the surface is at \( \xi_t = 1 \) only.

III.3.2 The case of \( \xi_a = 0 \) and \( 0 < \xi_t < 1 \)

The surface is at \( \xi_t = 1 \), and the coefficients are as follows

\[ \xi = \xi_t, \]  

(36)

\[ C_t = 1/(2l/n)^t, \quad V_t = 2l/(2l/n)^t \]  

(37)

\[ D_t = W_t = 0, \]

which can be deduced from Eqs. (33) and (34) with

\[ \xi_n = 0 \]  

followed by \( p = 0. \)

(38)

III.3.3 The case of \( 0 < \xi_a \leq 1 \) and \( 0 < \xi_t < \xi_a \)

This is the case of the direct problem. However, the temperature and the flux at a point \( \xi_t \) in the present form can be given as follows,

i) When \( 0 < \xi_t < \xi_a \), the coefficients are given by Eqs. (32) - (35).

ii) When \( \xi_a = 0 \), the coefficients are as follows

\[ \xi = \xi_a, \]  

(39)

\[ C_t = 1, \quad D_t = 0, \]

\[ C_i = \frac{1}{(2l/n)!} \left( \frac{1}{\xi_a} \right)^{2l} \]  

(40)

\[ D_i = \frac{1}{(2l/n)!} \left( \frac{1}{\xi_a} \right)^{2l}, \]

\[ V_i = W_i = 0, \]

which are deduced from Eqs. (33) and (34) with \( p = 0. \)

III.4 Hollow sphere

Using a dimensionless coordinate system normalized by an outer radius, \( \xi_a \) and \( \xi_t \) are given as

\[ \delta \leq \xi_a \leq 1 \quad \text{and} \quad \delta \leq \xi_t \leq 1 \]

where \( \delta \) is (inner dia./outer dia.). The surfaces are at \( \xi_t = \delta \) or at \( \xi_t = 1 \).

From the fact that the transient heat conduction equation expressed in one-dimensional spherical coordinate can be transformed into the one expressed in one-dimensional Cartesian coordinate and from the result of III.1 in slab, the coefficients for hollow cylinder are given as follows

\[ \xi = \xi_t - \xi_a, \]  

(41)

\[ C_t = (1+2\xi_a)(\xi_t)/(2l+1), \]

\[ D_t = (\xi_a)/(2l+1), \]

\[ V_t = 2l(1+2l-\xi_t)^{2l+1}/(2l+1), \]

(42)

\[ W_t = \xi_t(2l+\xi_t)(\xi_t)/(2l+1). \]

III.5 Sphere

Let the dimensionless coordinate system be normalized by the radius of a sphere.

III.5.1 The case of \( 0 \leq \xi_a \leq 1 \) and \( \xi_a \leq \xi_t \leq 1 \)

The surface is at \( \xi_t = 1 \) and the coefficients are given by Eqs. (41) and (42) in the same way as in the case of hollow sphere.
This is the case of the direct problem and the solution in the present form can be given as follows

\[\begin{align*}
\theta(x, t) &= 0, \quad \tau \leq \tau_n \\
\theta(x, t) &= G(t, \tau) + \frac{\partial \theta}{\partial \tau}(x, \tau_n)^n, \quad \tau > \tau_n
\end{align*}\]

(45) 

where \(a = b_x\) or \(b_x\), and \(b_y\) is Biot number, the temperature variation \(\theta(x, t)\) and \(\frac{\partial \theta}{\partial t}\) and their flux are given in the direct problem as follows

\[\begin{align*}
\theta(x, t) &= 0, \quad \tau \leq \tau_n \\
\frac{\partial \theta}{\partial \tau}(x, \tau_n) &= G(t, \tau) + \frac{\partial \theta}{\partial \tau}(x, \tau_n), \quad \tau > \tau_n
\end{align*}\]

(46) 

And

\[\begin{align*}
\phi(x, t) &= \sum \frac{h_x H_x(x)}{\gamma x} e^{-\gamma x t}, \quad \phi(x, t) = \frac{\partial \phi}{\partial x}
\end{align*}\]

(47) 

The terms of \(U^*, z, H_x, b_x\) and \(b_y\) are temperature response at a finite time, eigen value, eigen function and a coefficient, respectively, which describe the interior temperature response with unit step change of surface conditions corresponding to \(n = 0\) in Eq. (45).

When the interior temperature and its flux at the interior point \(x_n\) are given by Eqs. (46), Eqs. (28) and (29) are reduced to

\[\begin{align*}
\theta(x, t) &= u_x \sum \frac{h_x H_x(x)}{\gamma x} e^{-\gamma x t}, \quad \phi(x, t) = \frac{\partial \phi}{\partial x}
\end{align*}\]

(48) 

where \(L \geq n\) and

\[\begin{align*}
A_i(x, t) &= \sum \frac{(-1)^n z^{n+1}}{(x, t)}\frac{\partial^2 \theta}{\partial \tau^i}(x, t) \\
B_i(x, t) &= \sum \frac{(-1)^n z^{n+1}}{(x, t)}\frac{\partial^2 \phi}{\partial \tau^i}(x, t)
\end{align*}\]

(49) 

As shown in Appendix B, \(A_i\) and \(B_i\) are given by

\[\begin{align*}
A_i(x, t) &= \phi(x, t), \quad B_i(x, t) = \psi(x, t)
\end{align*}\]

then, Eq. (48) differs from Eq. (46) in its second term. However, when \(L\) is infinite in the second term of Eq. (49), \(M_i^r\) and \(N_i^r\) are given by

\[\begin{align*}
M_i^r &= (-1)^n z i^\psi(x, t) \quad \text{and} \quad N_i^r = (-1)^n z i^\phi(x, t)
\end{align*}\]

as shown in Appendix C, then, Eq. (48) is equal to Eq. (46).

Thus, when full information about the interior temperature and its flux at the interior point \(x_n\) in the solid is available, Eqs. (28) and (29) can describe the exact temperature and its flux at any point \(x\) in the solid including the surfaces.

However, available information at an interior point is incomplete due to the damping of the surface conditions. Therefore, upper limit \(L\) of the sum in Eqs. (28) and (29) or in Eq. (48) is inevitably finite.

With finite \(L\), the second term of Eq. (48) becomes smaller for larger \(r > \tau_n\) and it becomes smaller than the first term of Eq. (48). Then, Eq. (48) is equal to Eq. (46) approximately.

With finite \(L\) and smaller \(r > \tau_n\), whether Eq. (48) gives true values or not depends on \(L\), the position of \(x_n\) and \(x_n\), and Biot number as can be seen qualitatively by Eqs. (61) and (62) in Appendix C for a slab. This means that Eqs. (28) and (29) can not describe the true temperature and its flux in small time interval, called an
unresolved time interval, following a discontinuous change of surface conditions.

In the discrete problem the degree 1 of available information and the positions, \( \xi_s \) and \( \xi_e \) are given, but variational form of surface condition \( u(0, r) \) and \( B_t \) are unknown.

Effect of \( G_1 \) on the unresolved time interval is maximum at \( n = 0 \), which corresponds to a step change of surface conditions, as comparing \( |M_n - M_n'| \) with \( |M_n - M_n'|/r_0 \) for a slab as shown in the preceding paper for a hollow cylinder.

As well known by the direct problem, \( B_t \) directly specifies the eigenvalue \( \gamma_n \) of Eq. (47). For the same \( s \) the value of \( \gamma_n \) is minimum at \( B_t = 0 \) for surface temperature change or at \( B_t = \infty \) for surface flux change. An \( \eta \) number affects mainly \( M_0 \) or \( M_0' \) through \( \gamma_n \) of the exponential terms, when \( M_0' \) or \( M_0 \) with small \( \gamma_n \) is nearly a true value at a specific time, \( M_0' \) or \( M_0 \) with large \( \gamma_n \) should have nearly a true value at the time. Then, when \( M_0' \) or \( M_0 \) at specific time is nearly a true value at \( B_t = 0 \) for surface temperature change or at \( B_t = \infty \) for surface flux change, \( M_0' \) or \( M_0 \) at the time should have nearly a true value for any \( B_t \) number.

Thus, it can be concluded that there exists a relatively large unresolved time interval evaluated by Eqs. (28) and (29) or (48) with \( u(0, r) \) for \( V(0, r) \) of Eqs. (5), (6) as the interior temperature and interior flux.

These can be confirmed by numerical examples. The results show that a slab and a hollow cylinder show that the unresolved time interval is maximum in the case of \( u(0, r) \) with \( \xi_s \) specified at surface.

Figure 1 shows the unresolved time interval \( \Delta t_{s} \) for surface temperature and for surface flux with \( u(0, \xi_s, r) \) as the interior temperature in a slab. The unresolved time interval is judged by the maximum \( r = r_0 \) satisfying the condition

\[
\text{true value - calc. value} \quad \text{in Eqs. (26) or (29)} \quad |r_0| > 0.001
\]

for each \( s \) and \( \xi_s \).

In Fig. 1 solid lines and broken lines show unresolved time intervals for the step changed surface (\( \xi_s = 0 \)) and for the adiabatic end (\( \xi_s = 1 \)), respectively.

As shown in Fig. 1, the closer is the interior point to the surface and/or the greater is the degree of available interior information, the smaller is the unresolved time interval \( \Delta t_{s} \). And Fig. 1 shows that the unresolved time interval \( \Delta t_{s} \) for surface flux is greater than that for surface temperature for the same \( \xi_s \) and \( s \) values.

Figure 2 shows the unresolved time interval \( \Delta t_{f} \) with \( L = 4 \) and 8 for hollow cylinder (inner dia./outer dia. = 6 = 0.2, 0.4, 0.6, 0.8) and for a cylinder (\( s = 0 \)). The unresolved time interval \( \Delta t_{f} \) is a dimensionless time with (outer radius - inner radius) as length unit.

Broken lines (chained lines) show the unresolved time intervals calculated by Eqs. (20), (29), (32) - (38) with \( u(0, r) \) and \( \partial u/\partial \xi_s \) as interior temperature and interior flux when the step temperature changes at inner surface (at outer surface)
and another surface is adiabatic.
In Fig. 2 the solid lines show the unresolved time interval \( \Delta t \) for a slab as shown in Fig. 1.
From Fig. 2 it can be seen that the unresolved time interval for a hollow cylinder is the same as for a slab with dimensionless time and coordinate normalized by thickness of the solid.
It can be said that Eqs. (28) and (29) can describe the unknown surface temperature and unknown surface heat flux except for a small unresolved time interval, as shown in Figs. 1 and 2, following a discontinuous change of surface conditions.

V. Conclusion

As well known by the direct problem, interior temperature and interior heat flux variations in a one-dimensional solid are determined by the equation of heat conduction with the initial and boundary conditions. Inversely, speaking, it can be considered that the temperature and heat flux all over the one-dimensional solid can be defined by interior temperature and interior heat flux variations at an interior point in the solid.

Based on this concept, the inverse problem has been discussed and the major conclusions are drawn as follows:

i) A method, for determining the temperature and heat flux at any point in a one-dimensional solid when the temperature and heat flux at an interior point in the solid are prescribed as functions of time, has been developed by integration of Duhemel's integral which includes unknown temperature and unknown heat flux in its integrand.

ii) The equations to describe the temperature and the heat flux are the same as those which have been introduced by different methods other than the present procedure.

iii) The present equations have the coefficients which are related as functions of both temperature response with surface of unit step temperature change and adiabatic end and one with surface of unit step flux change and zero temperature end. Specific forms of the equations have been developed for sample inverse problems: slab, hollow cylinder, cylinder, hollow sphere and sphere.

iv) Utilizing the known temperature response to investigate the characteristics of the present equations, the following results have been obtained:

a) The equations give exact temperature and exact heat flux at any point in the solid if available information is complete.
b) Even when the available information is incomplete, the equations give nearly exact temperature and heat flux after the elapse of a time following a discontinuous change of surface conditions.
c) During small time interval following a discontinuous change of surface conditions, the equations can not give exact temperature and heat flux depending on the degree of available information and on the position of the interior point.
d) The maximum value of the small time interval which can not give the surface conditions has been determined by the numerical calculations for a slab and a hollow cylinder.

In this paper the interior temperature and the interior heat flux have been given as error-less data in numerical examples. It will be necessary to discuss limitation and accuracy of surface conditions obtained by the interior conditions with some sort of errors which might appear in practical applications.

References


Appendix A. Burggraf's Equation and Present Formulation

Burggraf (3) has shown that \( f_s(\xi') \) in Eq. (27) should satisfy the following differential equation and the following boundary conditions

\[
\begin{align*}
\phi^1 f_0(\xi') &= 0, \quad \phi^1 f_0(\xi') = f_{s-1}(\xi'), \quad n \geq 1 \\
\phi^2 f_0(\xi) &= 1, \quad f_0(\xi) = 0, \quad n \geq 1 \\
\frac{\partial f_0}{\partial \xi} &= 0, \quad n \geq 0. \\
\end{align*}
\]

(51)

It can be confirmed that the following \( f_s(\xi') \) satisfies the condition Eq. (51)

\[
f_s(\xi') = (-1)^n \phi(\xi', 0)
- \sum_{n=0}^{\infty} (-1)^n \phi(\xi, 0) f_{s-1}(\xi'), \quad (52)
\]

where \( \phi(\xi, 0) \) is given by Eq. (9). Considering that Eq. (7) with infinite \( L_e \)
instead of finite \( L \) and with \( \xi' \) instead of \( \xi_n \) satisfy the equation of heat conduction and utilizing the condition of \( U_i(\xi', 0) \) in Eq. (5), \( \phi_i(\xi', 0) \) must satisfy the following conditions:

\[
\begin{align*}
\phi_i(\xi', 0) &= 0, \\
\frac{d^2 \phi_i(\xi', 0)}{d \xi'^2} &= -\phi_i^{-1}(\xi', 0), \\
\frac{d \phi_i(\xi', 0)}{d \xi'} &= 1, \quad \phi_i(\xi', 0) = 0, \\
\frac{d^2 \phi_i(\xi', 0)}{d \xi'^2} &= 0, \quad I \geq 1.
\end{align*}
\]

(53)

At \( \xi = \xi_n \), Eq. (52) gives

\[
\phi_i(\xi_n) = \sum_{I} (-1)^I \phi_i(\xi_n, 0) f_{-I}(\xi_n).
\]

(54)

with Eq. (53). Differentiation of Eq. (52) with respect to \( \xi' \) gives

\[
\frac{d \phi_i}{d \xi'} |_{\xi_n} = 0,
\]

\[
\frac{d^2 \phi_i}{d \xi'^2} |_{\xi_n} = (-1)^I \phi_i(\xi_n, 0),
\]

\[
-\frac{d \phi_i}{d \xi'} |_{\xi_n} = (-1)^I \phi_i(\xi_n, 0) \frac{d \phi_i}{d \xi'} |_{\xi_n}.
\]

(55)

Comparison of Eqs. (54) and (55) with Eqs. (13) and (18) gives

\[
f_n(\xi_n) = C_n, \quad \frac{d f_n}{d \xi_n} |_{\xi_n} = V_n.
\]

(56)

A similar procedure to that applied for \( g_n(\xi') \) in Eq. (27) yields

\[
g_n(\xi_n) = D_n, \quad \frac{d g_n}{d \xi_n} |_{\xi_n} = W_n.
\]

(57)

Appendix B. Evaluation of \( A_i(\xi_n, \xi_m) \) and \( B_i(\xi_n, \xi_m) \)

Just like \( \phi_i'(\xi', 0) \) described in Appendix A, \( \phi_i(\xi', 0) \) in Eq. (47) has the following relations:

\[
\frac{d \phi_i'(\xi', 0)}{d \xi'} = 0, \quad \frac{d^2 \phi_i'(\xi', 0)}{d \xi'^2} = -\phi_i^{-1}(\xi', 0).
\]

(58)

It can be confirmed easily for a slab to give

\[
\frac{d A_i}{d \xi_n} = 0, \quad \frac{d B_i}{d \xi_n} = 0.
\]

(59)

utilizing Eq. (58) and \( \psi(\xi' \xi_n) \).

In the case of a hollow cylinder, differentiation of \( A_i(\xi_n, \xi_m) \) with respect to \( \xi_n \) gives

\[
\frac{d A_i}{d \xi_n} = \sum_{I} (-1)^I \psi_{-I}(\xi_n, 0) \left( \frac{d C_n}{d \xi_n} + D_n \right).
\]

(60)

\[
\frac{d \psi(\xi' \xi_n)}{d \xi_n} = \sum_{I} (-1)^I \psi_{-I}(\xi_n, 0) \left( \frac{d C_n}{d \xi_n} + D_n \right),
\]

\[
\times \psi(\xi' \xi_n).
\]

(61)

Evaluating bracket term in this equation with Eqs. (33) – (35), it is shown that the term is equal to zero.

Thus, in the case of either slab or hollow cylinder, \( A_i(\xi_n, \xi_m) \) and \( B_i(\xi_n, \xi_m) \) are independent of \( \xi_m \), and they give

\[
A_i(\xi_n, \xi_m) = A_i(\xi_n, \xi_n) = \phi_i(\xi_n, 0),
\]

(62)

\[
B_i(\xi_n, \xi_m) = B_i(\xi_n, \xi_n) = \psi(\xi_n, 0).
\]

Appendix C. Evaluation of \( M_i^e \) and \( N_i^e \)

Utilizing \( \phi_i \) and \( \psi_i \) defined by Eq. (47), \( N_i^e \) and \( N_i^e \) give

\[
M_i^e(x, y) = \sum_{I} \left\{ H_i(\xi_n) \sum_{I} (-1)^I \right\} \left( C_i(\gamma_i z) + \frac{1}{\gamma_i} \frac{\partial H_i}{\partial z} \right) \left( \frac{\gamma_i}{\gamma_i^2} \right)^{I+1} \times D_i(\gamma_i z)^{I+1} \left( \frac{\gamma_i}{\gamma_i^2} \right)^{I+1} \times V_i(\gamma_i z)^{I+1} \left( \frac{\gamma_i}{\gamma_i^2} \right)^{I+1}.
\]

(63)

\[
N_i^e(x, y) = \sum_{I} \left\{ H_i(\xi_n) \sum_{I} (-1)^I \right\} \left( C_i(\gamma_i z) + \frac{1}{\gamma_i} \frac{\partial H_i}{\partial z} \right) \left( \frac{\gamma_i}{\gamma_i^2} \right)^{I+1} \times D_i(\gamma_i z)^{I+1} \left( \frac{\gamma_i}{\gamma_i^2} \right)^{I+1} \times V_i(\gamma_i z)^{I+1} \left( \frac{\gamma_i}{\gamma_i^2} \right)^{I+1}.
\]

(64)

For a slab, \( z \) is equal to \( \xi_n - \xi_m \) and each sum in bracket with infinite \( L \) is equal to

\[
s_{\gamma_n} \gamma_n z - \xi_n \text{ or } \gamma_n \text{ \gamma_n z} - \xi_n.
\]

Then, expressions in bracket with infinite \( L \) in Eqs. (61) and (62) are \( H_i(\xi_n) \) and \( \partial H_i/\partial z \), respectively, because \( H_i(\xi_n) \) is \( s_{\gamma_n} \gamma_n z - \xi_n \text{ or } \gamma_n \text{ \gamma_n z} - \xi_n \).

For a hollow cylinder, \( z \) is equal to \( \xi_n \). Multiply each equation in Eq. (33) by \( (-1)^{I} \gamma(\gamma_i z)^{I+1} \) and make the infinite sum regarded with suffix \( \xi \) in those equations. These equations give

\[
\sum_{I} (-1)^I C_i(\gamma_i \xi_n)^{I+1} = \frac{\pi}{2} \gamma_i \xi_n B_i,
\]

(65)

\[
\sum_{I} (-1)^I V_i(\gamma_i \xi_n)^{I+1} = \frac{\pi}{2} \gamma_i \xi_n B_i \gamma_i \xi_n B_i,
\]

(66)

\[
\gamma_i \xi_n B_i \gamma_i \xi_n B_i \times \left( \frac{J_i(\gamma_i \xi_n)}{\gamma_i \xi_n B_i} \right).
\]

The two equations in Eq. (34) after a similar procedure give

\[
\sum_{I} (-1)^I D_i(\gamma_i \xi_n)^{I+1} = \frac{\pi}{2} \gamma_i \xi_n B_i,
\]

(67)

\[
\sum_{I} (-1)^I W_i(\gamma_i \xi_n)^{I+1} = \frac{\pi}{2} \gamma_i \xi_n B_i \gamma_i \xi_n B_i \times \left( \frac{J_i(\gamma_i \xi_n)}{\gamma_i \xi_n B_i} \right).
\]

(68)

\[
\gamma_i \xi_n B_i \gamma_i \xi_n B_i \times \left( \frac{J_i(\gamma_i \xi_n)}{\gamma_i \xi_n B_i} \right).
\]

On the other hand, \( H_i(\xi) \) can be written as

\[
H_i(\xi) = J_i(\gamma_i \xi_n) A_i(\gamma_i \xi_n).
\]

Utilizing this \( H_i(\xi) \) with any representation of \( A_i(\xi) \) and Eqs. (63) and (64),
expression in bracket with infinite \( L \) in Eqs. (61) and (62) gives \( H_s(\alpha) \) and \( \partial H_s/\partial \xi|_{\text{in}} \), respectively.

Thus, in the case of either slab or hollow cylinder, \( M_s^n \) and \( N_s^n \) give

\[
M_s^n = (-1)^n n! d^n(\xi, \tau) \\
N_s^n = (-1)^n n! d^n(\xi, \tau)
\]

(65)

Equations (50) and (65) with \( n = 0 \) give

\[
U_i(\xi, \tau) = \sum_{i=1}^{\infty} \left( C_i \alpha^2 \frac{\partial^i U_i(\xi, \tau)}{\partial \tau^i} \right) + D_i \alpha^2 \frac{\partial^i U_i|_{\text{in}}}{\partial \tau^i},
\]

\[
\frac{\partial U_i}{\partial \xi} \bigg|_{\text{in}} = \sum_{i=1}^{\infty} V_i \alpha^2 \frac{\partial^i U_i(\xi, \tau)}{\partial \tau^i} + \sum_{i=1}^{\infty} W_i \alpha^2 \frac{\partial^i U_i|_{\text{in}}}{\partial \tau^i}.
\]

(66)

When \( \xi_s \) is selected as the surface generated by the step unit change of surface condition, each equation in Eq. (66) gives

\[
1 = \sum_{i=1}^{\infty} \left( C_i \alpha^2 \frac{\partial^i U_i(\xi, \tau)}{\partial \tau^i} \right) + D_i \alpha^2 \frac{\partial^i U_i|_{\text{in}}}{\partial \tau^i},
\]

\[
-1 = \sum_{i=1}^{\infty} V_i \alpha^2 \frac{\partial^i U_i(\xi, \tau)}{\partial \tau^i} + \sum_{i=1}^{\infty} W_i \alpha^2 \frac{\partial^i U_i|_{\text{in}}}{\partial \tau^i}.
\]

(67)

(68)

These equations are defined for \( \tau > 0 \).

Each \( U_i(\xi, \tau) \), \( \partial U_i/\partial \xi|_{\text{in}} \) and its time derivatives in those equations are all zero at \( \tau = 0 \). Thus, Eqs. (67) or (68) can be used as a representation with continuous function for positive and negative infinite regions of \( \tau \) for a step unit discontinuous change at \( \tau = 0 \).