Free Vibrations of Thick Toroidal Shells

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In this paper, the free vibrations of thick toroidal shells with circular cross section are analyzed by using an improved thick shell theory. The equations of motion and the boundary conditions are derived from the stationary conditions of the Lagrangian of the toroidal shell. The equations of motion are solved exactly by a power series expansion and then natural frequencies and mode shapes are obtained. Effects of various parameters upon natural frequencies are clarified through an analysis of numerical results. The results by the present improved thick shell theory are compared with those by the classical thin shell theory and the effects of the rotatory inertia and shear deformation upon natural frequencies and mode shapes are clarified.

Key Words: Vibration, Thick Torus, Thick Shell Theory, Exact Solution, Natural Frequency, Mode Shapes

1. Introduction

The toroidal shell is utilized in many areas of engineering. For example, the toroidal geometrical shape is being considered as a basic structural element in Tokamak-type fusion reactor[1], rotating space station[2] and in space vehicle liquid storage containers[3]. Therefore, knowledge of the detailed vibrational characteristics of this shell is of great importance as a design criteria. As studies of the free vibrations of toroidal shells based on the classical thin shell theory (we call it classical theory hereafter), there are papers by Bauders and Amsenäke[4] using matrix iteration technique, by Liepins[5] using finite difference method, by Sobel and Flügge[6] using Fourier series expansions and by the authors[7] using power series solution.

On the other hand, as studies of the free vibrations of toroidal shells based on the improved thick shell theory (we call it improved theory hereafter), there are only two papers by McGill and Lenzen[8] who treated axisymmetric vibration using finite difference methods. However, comparisons with the classical theory have not been made in their studies.

The purpose of the present work is to present a method for analyzing the free vibrations of thick toroidal shells with circular cross section by using the improved theory. The equations of motion and the boundary conditions are derived from the stationary conditions of the Lagrangian of the toroidal shell. The equations of motion are solved exactly by a power series expansion and then natural frequencies and mode shapes are obtained numerically. The results obtained by the present improved theory are compared with those by the classical theory and the effects of rotatory inertia and shear deformation upon natural frequencies and mode shapes are clarified.

2. Theory

2.1 Lagrangian, equations of motion and boundary conditions

The geometry and the coordinate system for a toroidal shell with circular cross section are shown in Fig.1. The radius of curvature of the meridian is denoted by \( R \) and the distance between the center of the circular cross section \( D \) and the axis of revolution \( n \) by \( a \) and the thickness of the shell is denoted by \( h \). Take the angular coordinates \( \phi, \omega, \zeta \) axes as shown in the figure. Let \( P \) denote a point located on the normal at a distance \( z \) from the point \( O \) on the middle surface. Let us consider the transformation of variable as \( \psi + \pi / 2 = \omega \) and employ a nondimensional coordinate \( \theta = \psi / \pi \) (-1 ≤ 1).

We assume displacements at a point \( P \) in the \( \phi, \omega, \zeta \) directions as follows:

\[
(q, \epsilon, w) = \left(\begin{array}{c}
u + \frac{z}{R}w, z + \frac{\zeta}{R}w, w
\end{array}\right) \times \sin \phi
\] (1)

where, \( q, \epsilon, w \) and \( \omega \) are the displacements in the \( \phi, \omega, \zeta \) directions at the middle surface \( (z=0) \) and \( \nu, \epsilon, w, \phi, \omega, \zeta \) are the angular

![Fig.1 The geometry and coordinate system of a toroidal shell](image_url)
displacements of the cross sections normal to the φ and ω axes while \( \frac{\partial}{\partial t} \) and \( t \) denote the circular frequency and the time.

The normal and shearing strains at any point are expressed by Love\(^{10} \) as follows.

\[
\begin{align*}
\epsilon_r &= \frac{1}{R + z} \left( \frac{\partial^2}{\partial \theta^2} + \phi \sin \theta \frac{\partial}{\partial \theta} + \phi \cos \theta \frac{\partial}{\partial \phi} \right) \\
\epsilon_\phi &= \frac{1}{R + z} \left( \frac{\partial}{\partial \theta} + \phi \frac{\partial}{\partial \phi} \right) \\
\gamma_{\theta\phi} &= -\frac{1}{R + z} \left( \frac{\partial}{\partial \phi} - \phi \frac{\partial}{\partial \theta} \right)
\end{align*}
\]

where \( \epsilon_r, \epsilon_\phi, \) and \( \gamma_{\theta\phi} \) are Young's modulus, Poisson's ratio, and the shear coefficient, respectively. Components \( \tau_{r\theta} \) and \( \tau_{\phi\theta} \) are assumed to be expressed with use of the same shear coefficients. The Lagrangian for a vibratory period can be found as follows:

\[
L = \frac{1}{2} \left( \frac{\partial \epsilon_r}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \epsilon_\phi}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \gamma_{\theta\phi}}{\partial t} \right)^2 - \left( \epsilon_r \frac{\partial \epsilon_r}{\partial t} + \epsilon_\phi \frac{\partial \epsilon_\phi}{\partial t} \right)
\]

where \( r, \gamma \) and \( g \) are the period, the specific weight, and the acceleration due to gravity. Substituting Eq. (1), (2), and (4) into Eq. (5), we expand \( 1/r \) and \( 1/(R + z) \) into power series of \( z \) as follows.

\[
\frac{1}{r} = \frac{1}{S} \left( 1 + \frac{z}{S} \cos \theta + \frac{z^2}{S^2} \cos^2 \theta + \cdots \right)
\]

\[
\frac{1}{R + z} = \frac{1}{R} \left( 1 + \frac{z}{R} \cos \theta + \frac{z^2}{R^2} \cos^2 \theta + \cdots \right)
\]

Assuming that normal stresses are in a state of plane stress and strain-stress relations obey Hooke's law, one obtains

\[
\begin{align*}
\sigma_r &= \frac{E}{1-\nu^2}(\epsilon_r + \nu \epsilon_\phi) \\
\sigma_\phi &= \frac{E}{1-\nu^2}(\epsilon_\phi + \nu \epsilon_r) \\
\tau_{\theta\phi} &= \frac{E}{2(1+\nu)} \gamma_{\theta\phi}
\end{align*}
\]

Integrating Eq. (5) with respect to \( z \) and \( t \) and omitting the terms of order larger than \( z^2 \), one obtains

\[
\begin{align*}
\frac{1}{S} \int_{\theta} \int_{\phi} \left[ \frac{E}{1-\nu^2} \left( \epsilon_r \frac{\partial \epsilon_r}{\partial t} + \epsilon_\phi \frac{\partial \epsilon_\phi}{\partial t} \right) + \frac{E}{2(1+\nu)} \gamma_{\theta\phi} \frac{\partial \gamma_{\theta\phi}}{\partial t} \right] d\theta d\phi
\end{align*}
\]

The first variation of \( L \), which gives the stationary condition of the Lagrangian \( \delta L = 0 \), takes a form as follows;

\[
\begin{align*}
\delta L &= \frac{1}{S} \int_{\theta} \int_{\phi} \left[ \frac{E}{1-\nu^2} \left( \epsilon_r \frac{\partial \epsilon_r}{\partial t} + \epsilon_\phi \frac{\partial \epsilon_\phi}{\partial t} \right) + \frac{E}{2(1+\nu)} \gamma_{\theta\phi} \frac{\partial \gamma_{\theta\phi}}{\partial t} \right] d\theta d\phi \\
&- \int_{\theta} \int_{\phi} \left[ T_1 \delta \epsilon_r + T_2 \delta \epsilon_\phi + T_3 \delta \gamma_{\theta\phi} \right] d\theta d\phi
\end{align*}
\]

where

\[
\begin{align*}
T_1 &= \frac{E}{1-\nu^2} \left( \frac{\partial \epsilon_r}{\partial t} \right) \\
T_2 &= \frac{E}{1-\nu^2} \left( \frac{\partial \epsilon_\phi}{\partial t} \right) \\
T_3 &= \frac{E}{2(1+\nu)} \left( \frac{\partial \gamma_{\theta\phi}}{\partial t} \right)
\end{align*}
\]
Equations $E_{q}$ = 0 ($q = 1, 2, \ldots, 5$) are Euler's ones (equations of motion). Equations $T_{1} = T_{2} = T_{3} = T_{4} = T_{5} = T_{6}$ = 0 and $T_{1} = T_{2} = T_{3} = T_{4} = T_{5} = T_{6}$ = 0 are boundary conditions at $\theta = 0$, $\theta$ and $\phi = \pi$, $\pi$, respectively. Boundary conditions with respect to $\phi$ are of no use since we consider complete shells with $\phi = 0$ and $\phi = 2\pi$, hereafter.

2.2 Solutions of equations of motion

Let us put the displacements as follows;

\[
\begin{align*}
\eta(\theta, \phi) &= \sum_{n} \psi_{n}(\theta) \psi_{n}(\phi) \sin n\theta \\
\xi(\theta, \phi) &= \sum_{n} \psi_{n}(\theta) \psi_{n}(\phi) \sin n\theta \\
\end{align*}
\]

where $n$ denotes the circumferential (direction of $\phi$) wave number. Also let the symbols $T_{n}$, $T_{2n}$, $\ldots$, $N_{n}$ denote the functions obtained by substituting Eq.(12) into Eq.(10) and by omitting $\sin n\theta$ or $\cos n\theta$. Substituting Eq.(12) into Eq.$E_{q}$ = 0 ($q = 1, 2, \ldots, 5$) yields the following simultaneous ordinary differential equations with variable coefficients.

\[
\begin{align*}
\frac{d^{2}T_{n}}{d\theta^{2}} + nT_{n} - T_{n} \sin n\theta + T_{n} \cos n\theta + T_{n} \sin \theta \psi_{n}(\theta) \phi_{n}(\theta) + T_{n} \cos \theta \psi_{n}(\phi) = 0 \\
\frac{d^{2}T_{n}}{d\phi^{2}} + nT_{n} + nT_{n} \sin n\theta + nT_{n} \cos n\theta + nT_{n} \sin \theta \psi_{n}(\theta) \phi_{n}(\theta) + nT_{n} \cos \theta \psi_{n}(\phi) = 0 \\
\frac{d^{2}M_{n}}{d\theta^{2}} - nM_{n} - T_{n} \sin \theta \phi_{n}(\theta) \phi_{n}(\theta) + T_{n} \sin \theta \psi_{n}(\phi) \psi_{n}(\phi) = 0 \\
\frac{d^{2}M_{n}}{d\phi^{2}} - nM_{n} + nM_{n} \sin \theta \phi_{n}(\theta) \phi_{n}(\theta) + nM_{n} \sin \theta \psi_{n}(\phi) \psi_{n}(\phi) = 0
\end{align*}
\]

To obtain the solutions of Eq.(13), variable coefficients such as $\cos n\theta$, $\sin n\theta$, $\phi_{n}$, etc., can be expanded in an infinite power series of $\theta$ as

\[
\begin{align*}
\cos n\theta &= A_{1}^{(n)} \cos n\theta + A_{2}^{(n)} \cos 2n\theta + \cdots + A_{m}^{(n)} \cos mn\theta \\
\sin n\theta &= B_{1}^{(n)} \sin n\theta + B_{2}^{(n)} \sin 2n\theta + \cdots + B_{m}^{(n)} \sin mn\theta \\
\end{align*}
\]

where

\[
\begin{align*}
A_{1}^{(n)} &= 1, \quad A_{2}^{(n)} = (-1)^{n} x^{n+1}/(2m+1), \quad B_{1}^{(n)} = x, \quad B_{2}^{(n)} = (-1)^{n} x^{n}/(2m+1) \\
A_{p}^{(n)} &= A_{p}^{(1)}(1/n)^{p/(1+1/k)} A_{2p}^{(n)} = A_{2p}^{(1)}(1/n)^{p/(1+1/k)} \quad (k = 1, 2, \ldots, n) \\
B_{p}^{(n)} &= A_{p}^{(1)}(1/n)^{p/(1+1/k)} \quad (k = 1, 2, \ldots, n) \\
A_{2n}^{(n)} &= A_{2n}^{(1)} A_{2n}^{(1)} + A_{2n}^{(2)} A_{2n}^{(2)} + \cdots + B_{n}^{(n)} A_{n}^{(n)} + A_{2n}^{(n)} A_{2n}^{(n)} \\
B_{2n}^{(n)} &= B_{2n}^{(1)} A_{2n}^{(1)} + B_{2n}^{(2)} A_{2n}^{(2)} + \cdots + B_{2n}^{(n)} A_{2n}^{(n)} + B_{2n}^{(n)} A_{2n}^{(n)} \\
C_{1}^{(n)} &= C_{1}^{(n)} A_{1}^{(n)} + C_{1}^{(n)} A_{2}^{(n)} + \cdots + C_{n}^{(n)} A_{n}^{(n)}
\end{align*}
\]

Equation(13) has two solutions; one in which $u_{n}$, $v_{n}$ are even functions of $\theta$, and $u_{m}$, $v_{m}$ are odd functions of $\theta$, and the other in which $u_{n}$, $v_{n}$, $u_{m}$, $v_{m}$ are even functions of $\theta$. That is;

1. In the case in which $u_{n}$ is an even function of $\theta$, one takes

\[
\begin{align*}
u_{n} &= \sum_{m} \psi_{m}(\theta) \phi_{m}(\phi) \psi_{m}(\phi) \cos m\theta \\
\omega_{n} &= \sum_{m} \psi_{m}(\theta) \phi_{m}(\phi) \psi_{m}(\phi) \sin m\theta \\
\end{align*}
\]

where $\psi_{m}$, $\phi_{m}$, $\omega_{m}$, $\phi_{m}$ are the coefficients to be determined. Substituting Eq.(16) into Eq.(13), one obtains

\[
\begin{align*}
\frac{d^{2}T_{n}}{d\theta^{2}} + nT_{n} - T_{n} \sin n\theta + T_{n} \cos n\theta + T_{n} \sin \theta \psi_{n}(\theta) \phi_{n}(\phi) + T_{n} \cos \theta \psi_{n}(\phi) = 0 \\
\frac{d^{2}M_{n}}{d\theta^{2}} - nM_{n} - T_{n} \sin \theta \phi_{n}(\theta) \phi_{n}(\phi) + T_{n} \sin \theta \psi_{n}(\phi) \psi_{n}(\phi) = 0
\end{align*}
\]
\[
\sum_{i=1}^{n} \left[ \left( \frac{1}{i} \right) (2m-k) \right] \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} = 0
\]

\[
\sum_{i=1}^{n} \left[ \left( \frac{1}{i} \right) (2m-k) \right] \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} = 0
\]

\[
\sum_{i=1}^{n} \left[ \left( \frac{1}{i} \right) (2m-k) \right] \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} = 0
\]

\[
\sum_{i=1}^{n} \left[ \left( \frac{1}{i} \right) (2m-k) \right] \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} = 0
\]

where

\[ f_{m} = \alpha_{m} \delta_{m} - \alpha_{m} \delta_{m+1} + \alpha_{m} \delta_{m+2} + \ldots + \alpha_{m} \delta_{m+n} + \sum_{i=1}^{n} \left( \frac{1}{i} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} \]

\[ + \beta \left( \frac{1}{\beta} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} - \beta \left( \frac{1}{\beta} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} \]

\[ + \beta \left( \frac{1}{\beta} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} - \beta \left( \frac{1}{\beta} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} \]

\[ f_{m} = \alpha_{m} \delta_{m} - \alpha_{m} \delta_{m+1} + \alpha_{m} \delta_{m+2} + \ldots + \alpha_{m} \delta_{m+n} + \sum_{i=1}^{n} \left( \frac{1}{i} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} \]

\[ + \beta \left( \frac{1}{\beta} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} - \beta \left( \frac{1}{\beta} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} \]

\[ f_{m} = \alpha_{m} \delta_{m} - \alpha_{m} \delta_{m+1} + \alpha_{m} \delta_{m+2} + \ldots + \alpha_{m} \delta_{m+n} + \sum_{i=1}^{n} \left( \frac{1}{i} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} \]

\[ + \beta \left( \frac{1}{\beta} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} - \beta \left( \frac{1}{\beta} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} \]

\[ f_{m} = \alpha_{m} \delta_{m} - \alpha_{m} \delta_{m+1} + \alpha_{m} \delta_{m+2} + \ldots + \alpha_{m} \delta_{m+n} + \sum_{i=1}^{n} \left( \frac{1}{i} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} \]

\[ + \beta \left( \frac{1}{\beta} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} - \beta \left( \frac{1}{\beta} \right) (2m-k) \frac{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)}{s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n)} \]

provided that \( \beta = 0 (\beta < 0) \) and \( c_{1} = c_{2} = c_{3} = c_{4} = 0 (\beta < 0) \). These conditions and Eq. (19) hold also for Eq. (22). Considering Eq. (17), the coefficients \( \alpha_{m} \), \( \beta_{m} \), \( \alpha_{m} \), \( \alpha_{m} \) and \( \alpha_{m} (m \geq 2) \) are obtained successively with \( \alpha_{0} - \beta_{0} \), \( \alpha_{0} - \beta_{0} \), \( \alpha_{0} - \beta_{0} \) and \( \alpha_{0} - \beta_{0} \) left undetermined. Hence, five independent solutions arise from Eq. (16).

(ii) In the case that \( \omega_{s} \) is an odd function of \( \theta \), one takes

\[ u_{m} = \sum_{n=1}^{\infty} \pi_{m}^{\omega_{n}} \pi_{m} = \sum_{n=1}^{\infty} \pi_{m}^{\omega_{n}} \pi_{m} = \sum_{n=1}^{\infty} \pi_{m}^{\omega_{n}} \pi_{m} \]

where \( \pi_{m} \), \( \pi_{m} \), \( \pi_{m} \) and \( \pi_{m} \) are coefficients to be determined. Substituting Eq. (20) into Eq. (13), one obtains

\[ \sum_{i=1}^{n} \left( \frac{1}{i} \right) (2m-k) \left( s/(s+k+1) + s/(s+k+2) + \ldots + s/(s+k+n) \right) = 0 \]
where

\[ f_{in} = \gamma\omega_n\beta \frac{x}{k - x} = \frac{x}{x - \gamma\omega_n\beta} + 4\pi \left( \frac{1}{2}x(2x + 1) + \frac{1}{2}x^2 + \frac{1}{2}x^3 + x \right) \]

\[ + \frac{1}{2}x \left( \frac{1}{2}x^2 + \frac{1}{2}x^3 + x \right) \]

Considering Eq. (21), the coefficients \( a_m, b_n, \phi_m, \psi_m, c_n, d_n \) and \( e_m \) are obtained successively with \( e_0, b_0, a_0, d_0 \) and \( e_0 \) left undetermined. Hence, five independent solutions arise from Eq. (20). In this way, ten independent solutions are obtained. The general solutions to Eq. (13) are expressed by combining linearly these ten independent solutions as follows:

\[ \psi(x, t) = \sum_{m} (A_m \cos(\lambda_m x) + B_m \sin(\lambda_m x)) e^{-\gamma \omega_m t} \]

where \( \lambda_1, \lambda_2 \) are arbitrary constants. However, when the profile and boundary conditions are symmetric about the \( x \)-plane, the vibrations are divided into a symmetric vibration and an antisymmetric one about this plane. The expressions obtained by combining linearly five independent solutions from Eq. (16) are general solutions for symmetric vibration and those from Eq. (20) are ones for antisymmetric vibration, respectively.

These five independent solutions can be obtained by setting one of \( a_0, b_0, c_0, d_0 \) and \( e_0 \) equal to unity and the others equal to zero.

3. Numerical Calculations

In the calculations, Poisson's ratio and the shear coefficient are taken as \( \nu = 0.3 \) and \( \kappa = \pi \nu / 12 \) (the value obtained by Miersky \((22) \)), respectively and \( \gamma, \kappa, \) etc., of Eqs. (16) and (20) are calculated by retaining about 150 terms of the coefficients \( a_m, b_m \), etc. Table 1 shows the relations between the number of terms and converged digits of the solution for the case of representative parameters. The parameters \( \beta \) and \( k \) have large effect on the convergence of the solutions and generally the convergence becomes better as \( \beta \) or \( k \) becomes smaller. Incidentally, calculations were carried out on quadruple precision format by using ACOS system 1000 computer of Tohoku University to avoid dropping digits which often occurred on double precision.
format. From Eq. (9), there are 32 kinds of boundary conditions at each end of the shell. In this report, the clamped conditions at the both ends are numerically calculated. On the other hand, for the case of a completely closed torus in θ-direction, the boundary conditions at \( \theta = \pi \) are automatically determined from the properties of the solutions at \( \theta = 0 \). Then

\[
T_{\theta r} = T_{\theta z} = T_{r r} = T_{z z} = M_{r r} = M_{z z} = 0 \quad \text{(Symmetric vibration)}
\]

\[
M_{\theta r} = M_{\theta z} = M_{r r} = M_{z z} = 0 \quad \text{(Antisymmetric vibration)}
\]

Considering these boundary conditions, one obtains a frequency equation in the fifth order determinant for each case.

In Table 2 and Figs. 2-5, the nondimensional frequencies obtained by the present method are compared with those obtained by the classical theory which developed in Ref. (8). Table 2 shows effects of the thickness parameter \( \beta \) of the shell upon the frequency in the case where \( \beta = 1000, 500, 200 \) (Ref. 9, 13, 6.45, 4.08). With an increase in \( \beta \), the frequency as well as the differences between the improved theory and the classical theory increase. This tendency becomes prominent as the vibration becomes higher modes. Thus one can not ignore the influence of the rotatory inertia and the shear deformation. For example, the differences between the two theories are up to 6.1% for the antisymmetric third mode of vibration in \( \beta = 200 \).

Figure 2 shows the frequency curves in which the torus does not completely close in \( \theta \)-direction. With an increase in \( \alpha \), as shown schematically, the torus is gradually closed, and at \( \alpha = 1.0 \) is given the frequency of a torus whose innermost periphery is clamped along the circle with radius \( (a - \delta) \). The differences between the two theories become larger as \( \alpha \) becomes smaller or as the vibration becomes higher modes.

Figure 3 shows the frequency curves obtained from Eq. (24) in which the torus completely closes in \( \theta \)-direction. For this boundary condition, the frequency curves of the first symmetric and antisymmetric vibrations almost superpose on each other. The second and third modes of vibration show similar tendencies. Thus, to avoid complexity, only the antisymmetric vibrations are shown in Fig. 3. In this boundary condition, the influence of the rotatory inertia and the shear deformation seems to be comparatively small since the differences between the two theories are small until the antisymmetric third mode.

Figure 4 shows effects of torus radii ratio \( k \) upon the frequency.

![Fig. 2 Frequency curves (Both ends clamped, S denotes symmetric vibration and A does antisymmetric one and suffixes indicate the modes of vibration. These figure captions are same in Figs. 3-5)](image)

![Fig. 3 Frequency curves (Completely closed torus in \( \theta \)-direction, Antisymmetric vibration)](image)

![Fig. 4 Effects of torus radii ratio: \( k \) upon the frequency (Both ends clamped)](image)

![Fig. 5 Relation between the frequency and the circumperential wave number; \( n \) (Both ends clamped)](image)
Though the frequency increases with $k$, the differences between the two theories are almost unchangeable.

Figure 5 shows the relation between the frequency and the circumferential wave number $n$. Though the frequency gradually increases with $n$, the differences between the two theories are almost unchangeable.

Figure 6 shows the mode shapes of $\omega$ corresponding to the eigenvalues in the case where $\beta=500$ which are found in Table 2. In calculations of mode shapes, $\lambda_1$ of Eq.(23) is taken as unity and the scales of displacements are magnified or reduced appropriately. The values of the symmetric second mode in the classical theory are those of $\omega_{2n}$. It is found that the mode shapes of the two theories are quite similar.

4. Conclusions

In this paper, the free vibrations of thick toroidal shells with circular cross section are analyzed by using an improved theory. The equations of motion are solved exactly by a power series expansion. The results obtained by the present improved theory are compared with those by the classical theory and the effects of the rotatory inertia and the shear deformation upon natural frequencies and mode shapes are clarified.

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