Axisymmetric Free Vibrations of Shells of Revolution Having General Meridional Curvature*

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An exact method using power series expansion is presented for solving axisymmetric free vibration problems for shells of revolution having a meridionally varying curvature. The governing equations of motion and the boundary conditions are derived from the stationary conditions of the Lagrangian of the shells of revolution. The method is demonstrated for shells of revolution having cycloidal, parabolical, elliptical, catenary and hyperbolical meridional curvatures. The natural frequencies and the mode shapes are calculated numerically.

Key Words: Vibration, Shells of Revolution, Varying Meridional Curvature, Axisymmetric Vibration, Exact Solutions, Natural Frequency, Mode Shape

1. Introduction

Shells of revolution having a meridionally varying curvature are the most basic structural components in the design of aerospace structures, cooling towers and various storage vessels. Hence, the free vibration characteristics of these shells is of great importance as design criteria. Although there are a large number of references that deal with the vibrations of shells of revolution^{(1)~(3)}, most of them are studied using approximate solutions. Also compared with the cylindrical shells and the conical shells⁽⁴⁾, sufficient engineering data have not been presented. Recently the authors have analyzed the vibrations of shells of revolution having a constant meridional curvature using the classical thin shell theory^{(5)~(7)} and the improved thick shell theory^{(8)~(10)}. The exact natural frequencies have been presented and the vibrational characteristics have been clarified in these studies.

The purpose of the present work is to present an exact and general method for analyzing the free vibration problems for shells of revolution having a meridionally varying curvature by extending the previous theory. But, only the axisymmetric vibrations of thin shell of revolution was treated here to avoid mathematical complexities. The equations of motion and the boundary conditions are derived from the stationary conditions of the Lagrangian of the shell of revolution. The equations of motion are solved exactly by a power series expansion. The method is demonstrated for shells of revolution having elliptical, cycloidal, parabolical, catenary and hyperbolical meridional curvatures. The natural frequencies and the mode shapes are calculated numerically.

2. Theory

Let us consider the axisymmetric free vibrations of thin shells of revolution whose meridian of the cross section is a plane curve. Let $(\bar{x}, \bar{y})$ be the orthogonal coordinates. Figure 1 shows a thin shell of revolution the middle surface of which is made up by rotating a plane curve with respect to $\bar{x}$ axis (or $\bar{y}$ axis for elliptical meridian). Take the arc length s measured along the meridian, the angular coordinates θ (an angle between the normal to the arbitrary point A on the middle surface and $\bar{y}$ axis) and ϕ , z axes in the direction of the normal to the point A with positive inward. Let the principal radius of curvature of the meridian be $\bar{AC}=R_1(\theta)$ ($d\theta/ds=1/R_1$) and that of the parallel circle be $\bar{AB}=R_2(\theta)$, and the thickness of the shell be h . The displacements in the θ , ϕ and z directions $(\bar{u}, \bar{v}, \bar{w})$ are written as follows:

$$(\bar{u}, \bar{v}, \bar{w}) = \{u(\theta), v(\theta), w(\theta)\} \sin \bar{p} \bar{t} \dots (1)$$

where $\bar{p}$ and $\bar{t}$ denote the circular frequency and the time. Assuming a thin-walled shell whose h/R_1 is sufficiently small in comparison with unity, the normal and the shearing strains at any point are derived from Novozhilov⁽¹¹⁾ as

Fig.1 Geometry and coordinate system of a shell of revolution

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$$\left. \begin{aligned} \varepsilon_\theta &= \varepsilon_1 - z\kappa_1, \quad \varepsilon_\varphi = \varepsilon_2 - z\kappa_2 \\ \gamma_{\theta\varphi} &= \omega - 2z\tau \end{aligned} \right\} \dots\dots\dots (2)$$

where

$$\left. \begin{aligned} \varepsilon_1 &= \frac{1}{R_1} \left(\frac{\partial \bar{u}}{\partial \theta} - i\bar{v} \right), \quad \varepsilon_2 = -\frac{1}{R_2} (\bar{u} \tan \theta + i\bar{v}) \\ \kappa_1 &= \frac{1}{R_1} \frac{\partial}{\partial \theta} \left\{ \frac{1}{R_1} \left(\frac{\partial \bar{u}}{\partial \theta} + i\bar{v} \right) \right\}, \quad \kappa_2 = -\frac{\tan \theta}{R_1 R_2} \left(\frac{\partial \bar{u}}{\partial \theta} \right. \\ &\quad \left. + i\bar{v} \right), \quad \omega = \frac{R_2 \cos \theta}{R_1} \frac{\partial}{\partial \theta} \left(\frac{\bar{v}}{R_2 \cos \theta} \right) \\ \tau &= \frac{1}{R_2} \left(\frac{1}{R_1} \frac{\partial \bar{v}}{\partial \theta} + \frac{\tan \theta}{R_2} \bar{v} \right) \end{aligned} \right\} \dots\dots\dots (3)$$

Assuming that normal stresses are in a state of plane stress and the strain-stress relations obey Hooke's law, one obtains

$$\left. \begin{aligned} \sigma_\theta &= E(\varepsilon_\theta + \nu \varepsilon_\varphi) / (1 - \nu^2) \\ \sigma_\varphi &= E(\varepsilon_\varphi + \nu \varepsilon_\theta) / (1 - \nu^2) \\ \tau_{\theta\varphi} &= E\gamma_{\theta\varphi} / 2(1 + \nu) \end{aligned} \right\} \dots\dots\dots (4)$$

where E and ν are Young's modulus and Poisson's ratio, respectively. Let us now define the Lagrangian for a vibratory period (τ) as follows:

$$\begin{aligned} L &= \left(\frac{\tau}{\tau'} \right) \frac{1}{2} \int_{\theta_1}^{\tau'} \int_{-\theta_2}^{\theta_2} \int_0^{2\pi} \int_0^{2\pi} \rho \left\{ \left(\frac{\partial \bar{u}}{\partial t} \right)^2 + \left(\frac{\partial \bar{v}}{\partial t} \right)^2 \right. \\ &\quad \left. + \left(\frac{\partial \bar{w}}{\partial t} \right)^2 \right\} - (\sigma_\theta \varepsilon_\theta + \sigma_\varphi \varepsilon_\varphi + \tau_{\theta\varphi} \gamma_{\theta\varphi}) \\ &\quad \times R_1 R_2 \cos \theta d\theta d\phi dz d\bar{t} \dots\dots\dots (5) \end{aligned}$$

where ρ denotes the mass density and θ_1 and θ_2 are the values of θ at both ends.

Substituting Eqs. (1)~(4) into Eq. (5) and integrating the latter with respect to ϕ , z and $\bar{t}$, the Lagrangian is separated into a term related to flexural-extensional vibration consisting of u and w and a term related to torsional vibration consisting of v which is neglected hereafter. Let us put the curvatures as

$$1/R_1 = G_1 \Phi_1(\theta)/l_0, \quad 1/R_2 = G_2 \Phi_2(\theta)/l_0 \dots\dots (6)$$

where G_1 and G_2 are the constants determined by the shape of the curve and l_0 is a representative length. Substituting Eq. (6) into the Lagrangian related to flexural-extensional vibration, one obtains

$$\begin{aligned} -L / \left(\frac{\pi D}{G_1 G_2 l_0^2} \right) &= \int_{\theta_1}^{\theta_2} \left\{ -\alpha' (u^2 + w^2) \right. \\ &\quad \left. + \beta [(G_1 \Phi_1 U_1 - G_2 \Phi_2 U_2)^2 \right. \\ &\quad \left. + 2\xi G_1 G_2 \Phi_1 \Phi_2 U_1 U_2] + G_1^2 \Phi_1^2 \right. \\ &\quad \times \left\{ G_1 \frac{d}{d\theta} (\Phi_1 U_3) - G_2 \Phi_2 \tan \theta \cdot U_3 \right\}^2 + 2\xi \\ &\quad \times G_1^2 G_2 \Phi_1^2 \Phi_2 \tan \theta \cdot U_3 \frac{d}{d\theta} (\Phi_1 U_3) \left. \right\} \frac{\cos \theta}{\Phi_1 \Phi_2} d\theta \end{aligned} \dots\dots\dots (7)$$

where

$$U_1 = \frac{du}{d\theta} - w, \quad U_2 = u \tan \theta + w, \quad U_3 = \frac{dw}{d\theta} + u \dots\dots\dots (8)$$

and

$$\left. \begin{aligned} \alpha' &= \rho h \bar{p}^2 l_0^2 / D, \quad \beta = 12 l_0^2 / h^2 \\ D &= E h^3 / 12(1 - \nu^2), \quad \xi = 1 - \nu \end{aligned} \right\} \dots\dots\dots (9)$$

Taking the first variation of L to obtain the stationary condition of the Lagrangian, $\delta L = 0$, one obtains

$$\begin{aligned} -\delta L / \left(\frac{2\pi D}{G_1 G_2 l_0^2} \right) &= \int_{\theta_1}^{\theta_2} (E_1 \delta u + E_2 \delta w) d\theta \\ &\quad + \left[T_1 \delta u + T_2 \delta w + M_1 \delta \left(\frac{dw}{d\theta} \right) \right]_{\theta_1}^{\theta_2} \dots\dots\dots (10) \end{aligned}$$

where

$$\begin{aligned} M_1 &= G_1^2 \frac{\cos \theta}{\Phi_1^2} \Phi_1^2 \frac{d}{d\theta} (\Phi_1 U_3) \\ &\quad - (1 - \xi) G_1^2 G_2 \Phi_1^2 \sin \theta \cdot U_3 \\ T_1 &= M_1 + \beta G_1^2 \frac{\cos \theta}{\Phi_1^2} \Phi_1 U_1 \\ &\quad - \beta (1 - \xi) G_1 G_2 \cos \theta \cdot U_2 \\ T_2 &= -\frac{dM_1}{d\theta} + \left\{ G_1^2 \frac{\cos \theta}{\Phi_1^2} \frac{d\Phi_1}{d\theta} - G_1^2 G_2 \sin \theta \right\} \\ &\quad \times \left\{ G_1 \Phi_1 \frac{d}{d\theta} (\Phi_1 U_3) - G_2 \Phi_1 \Phi_2 \tan \theta \cdot U_3 \right\} \\ &\quad + \xi G_1^2 G_2 \Phi_1 \sin \theta \left\{ -\frac{d}{d\theta} (\Phi_1 U_3) + \frac{d\Phi_1}{d\theta} U_3 \right\} \\ T_3 &= -\beta G_2 \frac{\sin \theta}{\Phi_1} \{ (1 - \xi) G_1 \Phi_1 U_1 - G_2 \Phi_2 U_2 \} \\ M_2 &= -\beta G_1 \frac{\cos \theta}{\Phi_1^2} \{ G_1 \Phi_1 + (1 - \xi) G_2 \Phi_2 \} U_1 \\ &\quad + \beta G_2 \frac{\cos \theta}{\Phi_1} \{ (1 - \xi) G_1 \Phi_1 + G_2 \Phi_2 \} U_2 \end{aligned} \dots\dots\dots (11)$$

$$\begin{aligned} E_1 &= -\frac{dT_1}{d\theta} + T_2 + \frac{dM_1}{d\theta} + T_3 - \alpha' \frac{\cos \theta}{\Phi_1 \Phi_2} u \\ E_2 &= -\frac{dT_2}{d\theta} + M_2 - \alpha' \frac{\cos \theta}{\Phi_1 \Phi_2} w \end{aligned} \dots\dots\dots (12)$$

Euler's equations (equations of motion) are

$$E_1 = 0, \quad E_2 = 0 \dots\dots\dots (13)$$

The boundary conditions at $\theta = \theta_1$ and $\theta = \theta_2$ are

$$T_1 u = T_2 w = M_1 \frac{dw}{d\theta} = 0 \dots\dots\dots (14)$$

Employing the method which was developed in Ref. (12), one easily obtains the quantities shown in Table 1. In the table, $\bar{x}$ and $\bar{y}$ are orthogonal coordinates, while a and b are lengths and t is a parameter. The nondimensional variable θ for ellipse indicates an angle between $\bar{x}$ axis and the normal to the arbitrary point A on the middle surface, and for the other curves it does an angle between $\bar{y}$ axis and the normal. The variable coefficients of Eq. (13) may be expanded in infinite power series of θ as

$$\left. \begin{aligned} \Phi_1^2 &= \Psi_1 \sum_{m=0}^{\infty} \zeta_m^{(1)} \theta^{2m}, \quad \frac{\Phi_1}{\Phi_2} = \Psi_2 \sum_{m=0}^{\infty} \zeta_m^{(2)} \theta^{2m} \\ \Phi_3 &= \sum_{m=0}^{\infty} \zeta_m^{(3)} \theta^{2m} \end{aligned} \right\} \dots\dots (15)$$

where $\zeta_0^{(1)} = \zeta_0^{(2)} = 1$ for each curve except $\zeta_0^{(2)}$ $= (2-k)/4$ for cycloid. For convenience of calculation, expressions for Φ_3 are obtained by putting $G_1 = G_2$. Expressions for Ψ_1 , Ψ_2 , $\zeta_m^{(1)}$, $\zeta_m^{(2)}$ and $\zeta_m^{(3)}$ are given in Table 1. Furthermore the following expansion, which are common to each curve, should be made.

$$\begin{aligned} \left(\Phi_1 \Phi_2, \frac{1}{\Phi_1 \Phi_2}, \frac{1}{\Phi_1^2 \Phi_2} \right) &= \sum_{m=0}^{\infty} \left(\frac{G_1}{G_2} \Psi_1 \zeta_m^{(4)}, \zeta_m^{(5)}, \right. \\ &\quad \left. \zeta_m^{(6)} \right) \theta^{2m}, \quad \left(\cos \theta, \cos 2\theta, \frac{\Phi_1}{\Phi_2} \cos \theta, \frac{\cos \theta}{\Phi_1^2 \Phi_2} \right. \\ &\quad \left. \Phi_1 \frac{d\Phi_1}{d\theta} \sin \theta, \frac{\Phi_2}{\Phi_1} \cos \theta, \frac{\cos \theta}{\Phi_1 \Phi_2} \right) = \sum_{m=0}^{\infty} \left(A_m^{(1)}, \right. \\ &\quad \left. A_m^{(2)}, \Psi_2 A_m^{(3)}, A_m^{(4)}, \Psi_1 A_m^{(5)}, \frac{G_1}{G_2} A_m^{(6)}, A_m^{(7)} \right) \theta^{2m} \\ &\quad \left(\sin \theta, \tan \theta, \Phi_1^2 \sin \theta, \Phi_1 \Phi_2 \tan \theta, \right. \\ &\quad \left. \frac{\Phi_2}{\Phi_1} \sin \theta, \frac{\cos \theta}{\Phi_2} \frac{d\Phi_1}{d\theta} \right) = \sum_{m=0}^{\infty} \left(B_m^{(1)}, B_m^{(2)}, \Psi_1 B_m^{(3)}, \right. \\ &\quad \left. \frac{G_1}{G_2} \Psi_1 B_m^{(4)}, \frac{G_1}{G_2} B_m^{(5)}, \frac{G_2}{G_1} \Psi_1 B_m^{(6)} \right) \theta^{2m+1} \\ &\quad \left(\Phi_1 \Phi_2 \tan \theta \right) \sin \theta = \sum_{m=0}^{\infty} \frac{G_1}{G_2} \Psi_1 C_m^{(1)} \theta^{2m+2} \end{aligned} \dots\dots\dots (16)$$

where

$$\begin{aligned}\zeta_m^{(4)} &= \sum_{p=0}^m \zeta_{m-p}^{(1)} \zeta_p^{(3)}, \quad \zeta_0^{(5)} = \frac{G_2}{G_1 \Psi_1 \zeta_0^{(4)}}, \quad \zeta_0^{(6)} = \frac{1}{\Psi_1 \zeta_0^{(4)}} \\ A_0^{(1)} &= A_0^{(2)} = B_0^{(1)} = B_0^{(2)} = 1, \quad \zeta_m^{(6)} = - \sum_{p=1}^m \zeta_{m-p}^{(1)} \zeta_p^{(4)} / \zeta_0^{(4)} \\ \zeta_m^{(6)} &= - \sum_{p=1}^m \zeta_{m-p}^{(6)} \zeta_p^{(4)} / \zeta_0^{(4)}, \quad A_m^{(1)} = \frac{(-1)^m}{(2m)!} \\ A_m^{(2)} &= \frac{(-1)^{m+2m}}{(2m)!}, \quad B_m^{(1)} = \frac{(-1)^m}{(2m+1)!}, \\ B_m^{(2)} &= B_m^{(1)} - \sum_{p=1}^m B_{m-p}^{(2)} A_p^{(1)} \quad (m \geq 1) \\ (A_m^{(3)}, A_m^{(4)}, \dots, A_m^{(7)}) &= \sum_{p=0}^m (\zeta_{m-p}^{(2)} A_p^{(1)}, \zeta_{m-p}^{(6)} A_p^{(1)}, \\ &\quad (m-p) \zeta_{m-p}^{(1)} B_p^{(1)}, \zeta_{m-p}^{(2)} A_p^{(1)}, \zeta_{m-p}^{(6)} A_p^{(1)}) \\ (B_m^{(3)}, B_m^{(4)}, B_m^{(5)}) &= \sum_{p=0}^m (\zeta_{m-p}^{(1)} B_p^{(1)}, \zeta_{m-p}^{(4)} B_p^{(1)}, \\ &\quad \zeta_{m-p}^{(3)} B_p^{(1)}), \quad B_m^{(6)} = \sum_{p=1}^{m+1} A_{m+1-p}^{(4)} \zeta_p^{(1)} \\ C_m^{(1)} &= \sum_{p=0}^m B_{m-p}^{(4)} B_p^{(1)}\end{aligned}$$

.....(17)

Equation(13) has two solutions: one in which u is an odd function of θ and w is an even function of θ , and the other in which u is an even function of θ and w is an odd functions of θ . That is;

(i) In the case that u is an odd function of θ , one takes:

$$u = \sum_{m=0}^{\infty} P_m \theta^{2m+1}, \quad w = \sum_{m=0}^{\infty} R_m \theta^{2m} \quad \text{.....(18)}$$

where P_m and R_m are undetermined coefficients which are determined in turn as follows. Substituting Eq.(18) into Eq.(13) yields

$$\begin{aligned}\sum_{m=2}^{\infty} \{-(F_{1s} + F_{2s})P_{m-1} - (2m)F_{1s}R_m + f_{ms}\} \theta^{2m-3} \\ = 0 \\ \sum_{m=2}^{\infty} \{(2m-3)F_{1s}P_{m-1} + (2m-3)(2m)F_{1s}R_m \\ + f_{ms}\} \theta^{2m-4} = 0\end{aligned}$$

.....(19)

where F_{1s}, F_{2s}, f_{ms} and f_{mb} are series defined in the Appendix. From Eq.(19), the coefficients P_{m-1} , R_m ($m \geq 2$) are successively obtained in terms of P_0 , R_0 and R_1 with the last three left undetermined. Hence, three independent solutions arise from Eq.(18)

(ii) In the case that u is an even function of θ , one takes:

$$u = \sum_{m=0}^{\infty} P_m \theta^{2m}, \quad w = \sum_{m=0}^{\infty} R_m \theta^{2m+1} \quad \text{.....(20)}$$

where P_m and R_m are undetermined coefficients. Substituting Eq.(20) into Eq.(13) yields

$$\begin{aligned}\sum_{m=1}^{\infty} \{-(F_{1s} + F_{2s})P_m - (2m+1)F_{1s}R_m \\ + f_{ms}\} \theta^{2m-2} = 0 \\ \sum_{m=2}^{\infty} \{(2m-2)F_{1s}P_m + (2m-2)(2m+1)F_{1s}R_m \\ + f_{ms}\} \theta^{2m-3} = 0\end{aligned}$$

.....(21)

where F_{1s} , F_{2s} , f_{ms} and f_{mb} are series defined in the Appendix. From Eq.(21), the coefficients P_m , R_m ($m \geq 2$) are successively obtained with P_0 , R_0 and R_1 left undetermined, while P_1 is obtained from the first equation of Eq.(21) with $m=1$. Hence, three independent solutions arise from Eq.(20). In this way, six independent solutions for the complete problem are obtained. The general solutions of Eq.(13) are expressed by combining linearly these six independent solutions. However, when the cross section is symmetric about $\theta=0$, the vibrations are divided into symmetric and antisymmetric vibrations about the axis of symmetry passing through the axis $\theta=0$. Then the displacements given by Eq.(18) such as

$$(u, w) = \sum_{i=1}^3 \lambda_i (u_i, w_i) \quad \text{.....(22)}$$

are the solutions for the symmetric modes of vibration with $\lambda_1 \sim \lambda_3$ are arbitrary constants, and those from Eq.(20) are for the antisymmetric modes. Each of the independent solutions is obtained by setting one of P_0 , R_0 and R_1 equal to unity and the others equal to zero.

3. Numerical Calculations

In the numerical calculations, Poisson's ratio was taken as $\nu=0.3$ and the displacement functions u and w were calculated by retaining

Table 1 Transformation of variable for various curves and representative length, etc.
[$\mu = (a^2 - b^2) / (a^2 + b^2)$ denotes a flatness]

Curve	Expression	Transformation of variable	i_0^2	$\Phi_1(\theta)$	$1/R_1$	$\Phi_3 = R_1/R_2$
(1) Cycloid	$\bar{x} = a(t + \sin t)$ $\bar{y} = a(1 + \cos t - k)$	$\theta = t/2$ $(-\pi < t < \pi)$	a^2	$\frac{1}{\cos \theta}$	$\frac{\Phi_1}{4a}$	$\frac{2(1 + \cos 2\theta)}{1 + \cos 2\theta - k}$
(2) Parabola	$\bar{y} = \frac{1}{ak} \bar{x}^2 - a$	$\tan \theta = \frac{2}{ak} \bar{x}$	$\frac{a^2}{4}$	$(1 + \cos 2\theta)^{1/2}$	$\frac{\Phi_1}{\sqrt{2} ak}$	$\frac{4k}{4 - k + (4 + k) \cos 2\theta}$
(3) Ellipse	$\bar{x} = a \cos t$ $\bar{y} = b \sin t$	$\tan \theta = \frac{a}{b} \tan t$	$\frac{a^2 + b^2}{2}$	$(1 + \mu \cos 2\theta)^{1/2}$	$\frac{\Phi_1}{b_0(1 - \mu^2)}$	$\frac{1 - \mu}{1 + \mu \cos 2\theta}$
(4) Catenary	$\bar{y} = a \cosh \frac{\bar{x}}{b}$	$\tan \theta = \frac{a}{b} \sinh \frac{\bar{x}}{b}$	$\frac{a^2 + b^2}{2}$	$(1 + \cos 2\theta)(1 + \mu \cos 2\theta)^{1/2}$	$\frac{\Phi_1}{2b_0(1 - \mu)}$	$\frac{1 - \mu}{1 + \mu \cos 2\theta}$
(5) Hyperbola	$\bar{x} = b \sinh ht$ $\bar{y} = a \cosh ht$	$\tan \theta = \frac{a}{b} \tan ht$	$\frac{a^2 + b^2}{2}$	$(\mu + \cos 2\theta)^{1/2}$	$\frac{\Phi_1}{b_0(1 - \mu^2)}$	$\frac{1 - \mu}{\mu + \cos 2\theta}$
	Ψ_1	$\zeta_m^{(1)}$	Ψ_2	$\zeta_m^{(2)}$	$\zeta_m^{(3)}$	$\zeta_m^{(4)}$
(1)	1	$-\sum_{p=1}^m \zeta_{m-p}^{(1)} A_p^{(2)}/2$	$\frac{G_2}{G_1}$	$\left[A_m^{(2)} - 2 \sum_{p=1}^m \zeta_{m-p}^{(2)} A_p^{(2)} \right]/4$	$\frac{4}{2-k}$	$\left[2A_m^{(2)} - \sum_{p=1}^m \zeta_{m-p}^{(2)} A_p^{(2)} \right]/(2-k)$
(2)	8	$\frac{3A_m^{(2)}}{8} \left[\frac{5+3^{2m-1}}{4} + 2^{2m-1} \right]$	$\frac{2G_2}{kG_1}$	$\frac{4+k}{8} A_m^{(2)}$	$\frac{k}{2}$	$-\frac{4+k}{8} \sum_{p=1}^m \zeta_{m-p}^{(2)} A_p^{(2)}$
(3)	$(1+\mu)^2$	$\frac{3A_m^{(2)}}{(1+\mu)^2} \left[\mu^3 \frac{1+3^{2m-1}}{4} + \mu^2 2^{2m-1} + \mu \right]$	$G_2(1+\mu)^2$	$\frac{\mu}{1+\mu} A_m^{(2)}$	$\frac{1-\mu}{1+\mu}$	$-\frac{\mu}{1+\mu} \sum_{p=1}^m \zeta_{m-p}^{(2)} A_p^{(2)}$
(4)	$4(1+\mu)$	$\frac{A_m^{(2)}}{4(1+\mu)} \left[\mu \frac{3+3^{2m}}{4} + (1+2\mu) 2^{2m-1} + 2 + \mu \right]$	$2G_2(1+\mu)$	$\frac{\mu}{1+\mu} A_m^{(2)}$	$\frac{1-\mu}{1+\mu}$	$-\frac{\mu}{1+\mu} \sum_{p=1}^m \zeta_{m-p}^{(2)} A_p^{(2)}$
(5)	$(1+\mu)^2$	$\frac{3A_m^{(2)}}{(1+\mu)^2} \left[\frac{1+3^{2m-1}}{4} + \mu 2^{2m-1} + \mu^2 \right]$	$G_2(1+\mu)^2$	$\frac{1}{1+\mu} A_m^{(2)}$	$\frac{1-\mu}{1+\mu}$	$-\frac{1}{1+\mu} \sum_{p=1}^m \zeta_{m-p}^{(2)} A_p^{(2)}$

120 terms for each of the coefficients P_n and R_n in Eqs. (18) and (20). Table 2 shows the convergence of the solutions for the case of representative parameters. The numbers in the table are the number of digits of accuracy of the functions. The rate of convergence of the solutions depend upon the curves and parameters. From Eq. (14), one finds 8 kinds of boundary conditions at each end of the shell. In this report, the following three conditions are numerically calculated.

$$\left. \begin{aligned} u=w=\frac{dw}{d\theta}=0 & \quad (\text{clamped}) \\ u=w=M_1=0 & \quad (\text{supported}) \\ T_1=T_2=M_1=0 & \quad (\text{free}) \end{aligned} \right\} \dots\dots\dots (23)$$

Considering the above boundary conditions, one obtained the frequency equation in the form of a third order determinant for both symmetric and antisymmetric modes. To obtain computational cost saving α, β, k, μ were first chosen, and then a search was made for the

values of θ which satisfied the frequency equations. These values of θ were denoted as θ_0 and the relation between α^h/β and θ_0 was shown for the results hereafter. From the relation between the axis of revolution and

Table 2 Relations between the number of terms and converged digits of the solution (Symmetric, $\beta=5000$, $\alpha^h/\beta=1.0$)

Number of terms	20	40	60	80	100	120
Ellipse	$\mu=0.2$ u, v, w'	6	13	20		
	$\theta=1.0$ T_1, T_2, M_1	6	14	20		
Cycloid	$k=0$ u, v, w'	0	0	2	12	20
	$\theta=1.0$ T_1, T_2, M_1	0	0	2	10	20
Catenary	$\mu=0.2$ u, v, w'	0	4	8	13	20
	$\theta=1.0$ T_1, T_2, M_1	0	4	8	13	17
Parabola	$k=1.0$ u, v, w'	1	6	11	15	18
	$\theta=0.8$ T_1, T_2, M_1	1	6	10	17	20
Hyperbola	$\mu=0.6$ u, v, w'	1	4	8	12	17
	$\theta=0.8$ T_1, T_2, M_1	0	4	8	13	17

Table 3 The values of θ_0 against α^h/β ($\beta=5000$)

Curve	α^h/β	Modes	Symmetric			Antisymmetric		
			C-C	S-S	F-F	C-C	S-S	F-F
Cycloid $k=0$	0.9	1st	0.081199	0.054415		0.12857	0.10289	
		2nd	0.18288	0.15610	0.077682	0.23271	0.20698	0.12921
		3rd	0.28836	0.26122	0.18148	0.33904	0.31351	0.23433
		4th	0.39814	0.37017	0.28799	0.44450	0.42294	0.34269
Parabola $k=1.0$	4.0	1st	0.21026	0.14383		0.31372	0.25592	
		2nd	0.42714	0.37376	0.19719	0.51398	0.47077	0.31672
		3rd	0.59947	0.56040	0.42445	0.64863	0.63096	0.51856
		4th	0.72824	0.69980	0.59970	0.78061	0.75590	0.66934
Ellipse $\mu=0.2$	8.0	1st	0.24842	0.16930		0.37144	0.30082	
		2nd	0.52234	0.45132	0.23150	0.63239	0.57711	0.37809
		3rd	0.77102	0.71102	0.51831	0.78062	0.78015	0.64994
		4th	0.98955	0.93885	0.77281	0.89720	0.84588	0.88742
Catenary $\mu=0.2$	8.0	1st	0.24597	0.16815		0.36230	0.29504	
		2nd	0.49364	0.43071	0.22785	0.59097	0.54153	0.36477
		3rd	0.68955	0.64370	0.48777	0.71544	0.71301	0.59488
		4th	0.82688	0.77940		0.84513	0.83607	
Hyperbola $\mu=0.6$	8.0	1st	0.58084	0.52042	0.29761	0.66538	0.62414	0.45309
		2nd	0.73956	0.70554	0.57436	0.71488	0.71386	0.66660
		3rd						
		4th

Fig.2 Frequency curves (Both ends clamped)

Fig.4 Frequency curves (Both ends clamped)

Fig.3 Frequency curves (Both ends clamped)

Fig.5 Frequency curves (Both ends clamped)

the plane curve, shells of revolution having elliptical, cycloidal and parabolical meridian are positive Gaussian curvature shells, while shells of revolution having catenary and hyperbolical meridian are negative Gaussian curvature shells.

Table 3 presents accurate data of θ_0 against α^4/β in the case where $\beta=5000$ ($l_0/h=20.4$). The symbols C-C, S-S, and F-F denote the boundary conditions for both ends clamped, supported and free respectively. The frequency of the symmetric first mode of vibration becomes zero for F-F condition, which represents the translation of the shell as a rigid body. The S-S condition gives close values to C-C condition.

Figures 2-6 show the relation between α^4/β and θ_0 of shells of revolution having cycloidal, parabolical, catenary, hyperbolical and elliptical meridional curvatures. The curves are the first, second, third, ... modes of vibration in the order from below. It is found that the curves locate higher in order of antisymmetric first, symmetric first, antisymmetric second, symmetric second, ... where α^4/β is small and θ_0 is large, while they locate in order of symmetric first, antisymmetric first, symmetric second, antisymmetric second, ... where α^4/β is larger. In the case of hyperbolical shells of revolution, as seen in Fig.5 the value of α^4/β decreases rapidly to zero as θ_0 comes close to unity. The value of θ_0 which makes α^4/β zero is obtained from $\theta_0 = \tan^{-1} \sqrt{(1+\mu)/(1-\mu)}$ and it gives $\theta_0=1.107$ when $\mu=0.6$ ($a/b=2$). In this case, the shell has infinite length. Setting $\mu=0$ (i.e. $a/b=1$) for elliptical shells of revolution, one obtains $\alpha^4/\beta-\theta_0$ relation for spherical shells. In the case that $\mu=0$, $\beta=10^4$ ($l_0/h \approx 28.9$), $\alpha^4/\beta=3.0$ and both ends clamped, for example one obtains the value of θ_0 for the first through third modes of symmetric vibration as 0.23871, 0.48440, 0.73288 and those of antisymmetric one as 0.31829, 0.57014, 0.82203 which completely coincide with the known results of Ref.(6).

Figures 7 and 8 show the mode shapes. Mode numbering corresponds to the numbered points of the frequency curves which are shown in Fig.6. They are the displacements u and w for $0 \leq \theta \leq \theta_0$ in which λ_1 of Eq.(22) is taken as unity and the scales of u and w are magnified or reduced appropriately to represent them exactly. As seen from the figures, the mode shapes vary with θ_0 even for the same mode of vibration.

4. Conclusions

An exact and general method has been developed for analyzing the free vibration problems for shells of revolution of which the meridian of the cross section is a plane curve. Here, the problem is confined to the axisymmetric vibrations of thin shells of revolution.

For the sake of specificity, the method is demonstrated for shells of revolution having cycloidal, parabolical, elliptical, catenary and hyperbolical meridional curvatures. The natural frequencies and mode shapes are calculated numerically and the vibration characteristics of these shells are clarified.

The numerical calculations presented

here were carried out on an ACOS system 1000 computer of the Tohoku University Computing Center.

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Fig.6 Frequency curves (Both ends clamped, the numbers 1~11, 1'~9' denote the points at which the modes are calculated)

Fig.7 Mode shapes corresponding to numbered points in Fig.6 (symmetric)

Fig.8 Mode shapes corresponding to numbered points in Fig.6 (antisymmetric)

Appendix

(i) Expressions of F_{1S} , F_{2S} , $f_{\pi A}$, $f_{\pi b}$ in Eq. (19).

$$\begin{aligned}
F_{1S} &= G_1^\dagger \Psi_1 \Psi_2 \xi_0^{(1)} A_0^{(3)} (2m-2)(2m-1), \quad F_{2S} = \beta G_1^\dagger \Psi_2 A_0^{(3)} (2m-2)(2m-1) \\
f_{m\alpha} &= \sum_{p=0}^{m-1} \left[G_1^\dagger G_2 \Psi_1 \{ \Psi_1 B_{m-1-p}^{(6)} + (\xi-1) B_{m-1-p}^{(1)} \} - G_1^\dagger \Psi_1 \Psi_2 (2m-2) A_{m-p}^{(3)} \right] \sum_{q=0}^p \{ (p+q-1) \xi_{p-q}^{(1)} (2qR_q + P_{q-1}) \\
&\quad + G_1^\dagger G_2 \Psi_1 (\xi-1) \{ A_{m-1-p}^{(6)} - (2m-2) B_{m-1-p}^{(3)} \} (2pR_p + P_{p-1}) + \beta G_1 (2m-2) \{ G_1 \Psi_2 A_{m-1-p}^{(2)} R_p \\
&\quad + (1-\xi) G_2 A_{m-1-p}^{(1)} \left(\sum_{q=0}^{p-1} P_{p-1-q} B_q^{(2)} + R_p \right) \} \right] - \sum_{q=0}^{m-1} G_1^\dagger \Psi_1 \Psi_2 A_0^{(3)} (2m-2)(m+q-1) \xi_{m-q}^{(1)} (2qR_q + P_{q-1}) \\
&\quad + \sum_{p=0}^{m-2} \left[G_1^\dagger G_2 \Psi_1 C_{m-2-p}^{(1)} (2pR_p + P_{p-1}) - \beta G_1^\dagger \Psi_2 (2m-2) A_{m-1-p}^{(3)} (2p+1) P_p - \alpha^* A_{m-2-p}^{(2)} P_p \right. \\
&\quad \left. + \beta G_1 G_2 \left\{ (\xi-1) B_{m-2-p}^{(1)} ((2p+1)P_p - R_p) + B_{m-2-p}^{(6)} \left(\sum_{q=0}^{p-1} P_{p-1-q} B_q^{(2)} + R_p \right) \right\} \right] \\
f_{mb} &= \sum_{p=0}^{m-1} \left[(2m-3) \{ G_1^\dagger \Psi_1 \Psi_2 (2m-2) A_{m-p}^{(3)} - G_1^\dagger G_2 \Psi_1 \{ \Psi_1 B_{m-1-p}^{(6)} + (\xi-1) B_{m-1-p}^{(1)} \} \} \sum_{q=0}^p (p+q-1) \xi_{p-q}^{(1)} \right. \\
&\quad \left. \times (2qR_q + P_{q-1}) + G_1^\dagger G_2 \Psi_1 (1-\xi) (2m-3) \{ A_{m-1-p}^{(6)} - (2m-2) B_{m-1-p}^{(3)} \} (2pR_p + P_{p-1}) \right] \\
&\quad + \sum_{q=0}^{m-1} G_1^\dagger \Psi_1 \Psi_2 A_0^{(3)} (2m-3)(2m-2)(m+q-1) \xi_{m-q}^{(1)} (2qR_q + P_{q-1}) \\
&\quad + \sum_{p=0}^{m-2} \left[\beta G_1 \{ (\xi-1) G_2 A_{m-2-p}^{(1)} - G_1 \Psi_2 A_{m-2-p}^{(3)} \} ((2p+1)P_p - R_p) + \beta G_1 G_2 \{ (1-\xi) A_{m-2-p}^{(1)} + A_{m-2-p}^{(6)} \} \right. \\
&\quad \left. \times \left(\sum_{q=0}^{p-1} P_{p-1-q} B_q^{(2)} + R_p \right) - G_1^\dagger G_2 \Psi_1 (2m-3) C_{m-2-p}^{(1)} (2pR_p + P_{p-1}) - \alpha^* A_{m-2-p}^{(2)} R_p \right]
\end{aligned}$$

(ii) Expressions of F_{1A} , F_{2A} , $f_{m\alpha}$, $f_{\pi b}$ in Eq. (21).

$$\begin{aligned}
F_{1A} &= G_1^\dagger \Psi_1 \Psi_2 \xi_0^{(1)} A_0^{(3)} (2m-1)(2m), \quad F_{2A} = \beta G_1^\dagger \Psi_2 A_0^{(3)} (2m-1)(2m) \\
f_{m\alpha} &= \beta G_1^\dagger \Psi_2 (2m-1) \sum_{p=1}^m A_{m-p}^{(3)} R_{p-1} + \sum_{p=0}^{m-1} \left[G_1^\dagger G_2 \Psi_1 \{ \Psi_1 B_{m-1-p}^{(6)} + (\xi-1) B_{m-1-p}^{(1)} \} - G_1^\dagger \Psi_1 \Psi_2 (2m-1) A_{m-p}^{(3)} \right] \\
&\quad \times \sum_{q=0}^p (p+q) \xi_{p-q}^{(1)} \{ (2q+1)R_q + P_q \} + G_1^\dagger G_2 \Psi_1 (\xi-1) \{ A_{m-1-p}^{(6)} - (2m-1) B_{m-1-p}^{(3)} \} \{ (2p+1)R_p + P_p \} \\
&\quad + \beta G_1 G_2 (\xi-1) \left\{ B_{m-1-p}^{(1)} (2pP_p - R_{p-1}) - (2m-1) A_{m-1-p}^{(1)} \left(\sum_{q=0}^p P_{p-q} B_q^{(2)} + R_p \right) \right\} - \beta G_1^\dagger \Psi_2 (2m-1) \\
&\quad \times A_{m-p}^{(3)} (2p) P_p - \alpha^* A_{m-1-p}^{(2)} P_p \left. - \sum_{q=0}^{m-1} G_1^\dagger \Psi_1 \Psi_2 A_0^{(3)} (2m-1)(m+q) \xi_{m-q}^{(1)} \{ (2q+1)R_q + P_q \} \right. \\
&\quad \left. + \sum_{p=0}^{m-2} \left[G_1^\dagger G_2 \Psi_1 C_{m-2-p}^{(1)} ((2p+1)R_p + P_p) + \beta G_1 G_2 B_{m-2-p}^{(6)} \left(\sum_{q=0}^p P_{p-q} B_q^{(2)} + R_p \right) \right] \right] \\
f_{mb} &= \sum_{p=0}^{m-1} \left[(2m-2) \{ G_1^\dagger \Psi_1 \Psi_2 (2m-1) A_{m-p}^{(3)} - G_1^\dagger G_2 \Psi_1 \{ \Psi_1 B_{m-1-p}^{(6)} + (\xi-1) B_{m-1-p}^{(1)} \} \} \sum_{q=0}^p (p+q) \xi_{p-q}^{(1)} \{ (2q \right. \\
&\quad \left. + 1)R_q + P_q \} + G_1^\dagger G_2 \Psi_1 (1-\xi) (2m-2) \{ A_{m-1-p}^{(6)} - (2m-1) B_{m-1-p}^{(3)} \} \{ (2p+1)R_p + P_p \} \right. \\
&\quad \left. + \beta G_1 \{ (\xi-1) G_2 A_{m-1-p}^{(1)} - G_1 \Psi_2 A_{m-1-p}^{(3)} \} (2pP_p - R_{p-1}) \right] + \sum_{q=0}^{m-1} G_1^\dagger \Psi_1 \Psi_2 A_0^{(3)} (2m-2)(2m-1) \\
&\quad \times (m+q) \xi_{m-q}^{(1)} \{ (2q+1)R_q + P_q \} + \sum_{p=0}^{m-2} \left[\beta G_1 G_2 \{ (1-\xi) A_{m-2-p}^{(1)} + A_{m-2-p}^{(6)} \} \left(\sum_{q=0}^p P_{p-q} B_q^{(2)} + R_p \right) \right. \\
&\quad \left. - G_1^\dagger G_2 \Psi_1 (2m-2) C_{m-2-p}^{(1)} ((2p+1)R_p + P_p) - \alpha^* A_{m-2-p}^{(2)} R_p \right]
\end{aligned}$$

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