Axisymmetric Free Vibrations of Shells of Revolution Having
General Meridional Curvature*

By Tadashi KOSAWADA**, Katsuoshi SUZUKI**, and Shin TAKAHASHI**

An exact method using power series expansion is presented for solving axisymmetric free vibration problems for shells of revolution having a meridionally varying curvature. The governing equations of motion and the boundary conditions are derived from the stationary conditions of the Lagrangian of the shells of revolution. The method is demonstrated for shells of revolution having cycloidal, parabolical, elliptical, catenary and hyperbolical meridional curvatures. The natural frequencies and the mode shapes are calculated numerically.

Key Words: Vibration, Shells of Revolution, Varying Meridional Curve, Axisymmetric Vibration, Exact Solutions, Natural Frequency, Mode Shape

1. Introduction

Shells of revolution having a meridionally varying curvature are the most basic structural components in the design of aerospace structures, cooling towers and various storage vessels. Hence, the free vibration characteristics of these shells is of great importance as design criteria. Although there are a large number of references that deal with the vibrations of shells of revolution \( (1) \sim (3) \), most of them are studied using approximate solutions. Also compared with the cylindrical shells and the conical shells \( (4) \), sufficient engineering data have not been presented. Recently the authors have analyzed the vibrations of shells of revolution having a constant meridional curvature using the classical thin shell theory \( (4) \) and the improved thick shell theory \( (5) \). The exact natural frequencies have been presented and the vibrational characteristics have been clarified in these studies.

The purpose of the present work is to present an exact and general method for analyzing the free vibration problems for shells of revolution having a meridionally varying curvature by extending the previous theory. But, only the axisymmetric vibrations of thin shell of revolution was treated here to avoid mathematical complexities. The equations of motion and the boundary conditions are derived from the stationary conditions of the Lagrangian of the shell of revolution. The equations of motion are solved exactly by a power series expansion. The method is demonstrated for shells of revolution having elliptical, cycloidal, parabolical, catenary and hyperbolical meridional curvatures. The natural frequencies and the mode shapes are calculated numerically.

2. Theory

Let us consider the axisymmetric free vibrations of thin shells of revolution whose meridian of the cross section is a plane curve. Let \((x, y)\) be the orthogonal coordinates. Figure 1 shows a thin shell of revolution the middle surface of which is made up by rotating a plane curve with respect to \(x\) axis (or \(y\) axis for elliptical meridian). Take the arc length \(s\) measured along the meridian, the angular coordinates \(\theta\) (an angle between the normal to the arbitrary point \(A\) on the middle surface and \(y\) axis) and \(\phi, z\) axes in the direction of the normal to the point \(A\) with the positive inward. Let the principal radius of curvature of the meridian be \(\alpha = R_1(\theta)\) and that of the parallel circle be \(\alpha = R_2(\theta)\), and the thickness of the shell be \(h\). The displacements of the \(x, y\) and \(z\) directions \((\bar{u}, \bar{v}, \bar{w})\) are written as follows:

\[
(\bar{u}, \bar{v}, \bar{w}) = (u(\theta), v(\theta), w(\theta)) \sin \bar{s} \quad \cdots \quad (1)
\]

where \(\bar{u}\) and \(\bar{v}\) denote the circular frequency and the time. Assuming a thin-walled shell whose \(\alpha = R_1\) is sufficiently small in comparison with unity, the normal and the shearing strains at any point are derived from Novozhilov \( (1) \) as

![Fig.1 Geometry and coordinate system of a shell of revolution](image-url)
\[ \epsilon_x = \varepsilon_1 - 2\varepsilon_2, \ \epsilon_y = \varepsilon_1 - 2\varepsilon_2 \]
\[ \gamma_{xy} = \omega \varepsilon \]

where
\[ \varepsilon_x = \frac{1}{K} \left( \frac{\partial u}{\partial x} \right), \quad \varepsilon_y = \frac{1}{K} \left( \frac{\partial v}{\partial y} \right), \quad \gamma_{xy} = -\frac{1}{K} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]
\[ u_x = \frac{1}{K} \left( \frac{1}{\partial \theta} \right), \quad u_y = -\frac{1}{K} \left( \frac{\partial \theta}{\partial \phi} \right), \quad \gamma_{xy} = \tan \theta \left( \frac{\partial \phi}{\partial \theta} \right) + \theta \left( \frac{\partial \theta}{\partial \phi} \right) \]
\[ r = \frac{1}{K} \left( \frac{1}{\partial \theta} \right), \quad R = \frac{1}{K} \left( \frac{\partial \theta}{\partial \phi} \right) \]

Assuming that normal stresses are in a state of plane stress and the strain-stress relations obey Hooke's law, one obtains
\[ \alpha = \frac{A(\alpha + \omega)}{(1 - \nu)} \]
\[ \eta = \frac{A(\eta + \omega)}{(1 - \nu)} \]
\[ \tau = \frac{A(\tau + \omega)}{(1 - \nu)} \]

where \( E \) and \( \nu \) are Young's modulus and Poisson's ratio, respectively. Let us now define the Lagrangian for a vibratory period (\( T \)) as follows:
\[ L = \frac{1}{2} \mu \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \mu \left( \frac{\partial \theta}{\partial t} \right)^2 \]
\[ \mu \left( \frac{\partial u}{\partial x} \right)^2 + \mu \left( \frac{\partial v}{\partial y} \right)^2 \]
\[ R \cos \theta \left( \frac{\partial \theta}{\partial \phi} \right)^2 \]

where \( \mu \) denotes the mass density and \( \phi_1 \) and \( \phi_2 \) are the values of \( \theta \) at both ends.

Substituting Eqs. (1)–(4) into Eq. (5) and integrating the latter with respect to \( \phi \), \( x \), and \( y \), the Lagrangian is separated into a term related to flexural-extentional vibration consisting of \( u \) and \( v \) and a term related to torsional vibration consisting of \( \phi \) which is neglected hereafter. Let us put the curvatures as
\[ 1/R = G_1 \phi_1, \quad 1/R = G_2 \phi_2 \]

where \( G_1 \) and \( G_2 \) are the constants of proportionality to the shape of the curve and \( l_0 \) is a representative length. Substituting Eq. (6) into the Lagrangian related to flexural-extensional vibration, one obtains
\[ -L = \int \left[ \frac{1}{2} \mu \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \mu \left( \frac{\partial \theta}{\partial t} \right)^2 - \mu \left( \frac{\partial u}{\partial x} \right)^2 - \mu \left( \frac{\partial v}{\partial y} \right)^2 + 2G_1 \phi_1 \frac{\partial u}{\partial x} + 2G_2 \phi_2 \frac{\partial v}{\partial y} \right] \]
\[ + \mu \left( \frac{\partial u}{\partial x} \right)^2 + \mu \left( \frac{\partial v}{\partial y} \right)^2 \]

where
\[ \frac{\partial u}{\partial x} = u_x, \quad \frac{\partial v}{\partial y} = v_y, \quad \frac{\partial u}{\partial x} = u_y, \quad \frac{\partial v}{\partial y} = v_x \]

and
\[ \sigma = -\mu \phi_1 \left( \frac{\partial u}{\partial x} \right)^2, \quad \sigma_0 = -\mu \phi_2 \left( \frac{\partial v}{\partial y} \right)^2 \]
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Euler's equations (equations of motion) are
\[ \frac{d}{dt} \left( \frac{\partial \phi}{\partial t} \right) + \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) = 0 \]

Employing the method which was developed in Ref. (12), one easily obtains the quantities shown in Table 1. In the table, \( x \) and \( y \) are orthogonal coordinates, while \( c \) and \( b \) are lengths and \( t \) is a parameter. The nondimensional variable \( \theta \) for ellipse indicates an angle between \( x \) axis and the normal to the movable point \( A \) on the middle surface, and for the other curves it does an angle between \( y \) axis and the normal. The variable coefficients of Eq. (13) may be expanded in infinite power series of \( \theta \) as
\[ \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x} \left( 1 + \sum \theta^2 \phi_2 \frac{\partial \phi}{\partial x} \right) \]
\[ \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial y} \left( 1 + \sum \theta^2 \phi_3 \frac{\partial \phi}{\partial y} \right) \]

where \( \phi_1(1) = \phi_2(1) = 1 \) for each curve except \( \phi_1(2) = 2 \left( k-l \right) / 3 \) for cycloid. For convenience of calculation, expressions for \( \phi_1 \) are obtained by putting \( G_1 = G_2 \). Expressions for \( \phi_2, \phi_3, \phi^{(2)}, \phi^{(3)} \) and \( \phi^{(3)} \) are given in Table 1. Furthermore the following expansion, which are common to each curve, should be made.
\[ \phi \left( \frac{\partial \phi}{\partial x} \right)^2 = \sum \left( \frac{\partial \phi}{\partial x} \right)^2 \]
\[ \phi \left( \frac{\partial \phi}{\partial y} \right)^2 = \sum \left( \frac{\partial \phi}{\partial y} \right)^2 \]

Taking the first variation of \( L \) to obtain the stationary condition of the Lagrangian, \( L = 0 \), one obtains
\[ \frac{\partial}{\partial \theta} \left( \frac{1}{G_1 \phi_1} \right) = \int \epsilon_1 \left( E_1 \delta u + E_2 \delta v \right) \]

\[ + T \delta u + T \delta v + M \left( \frac{\partial \phi}{\partial x} \right) \]

where
\[ M = G_1 \cos \theta \left( \frac{\partial \sigma}{\partial t} \right) \]

\[ - (1 - \xi) G_1 \sigma_1 \sin \theta \]

\[ T = M_1 + 1 \xi G_1 \sigma_1 \cos \theta \]

\[ T = \frac{2M_1}{(1 - \xi) G_1 \sigma_1 \sin \theta} \]

\[ T = -G_1 \phi_1 \sin \theta \left( 1 - \xi - G_1 \phi_1 \sin \theta \right) \]

\[ M_1 = -G_1 \phi_1 \sin \theta \left( 1 - \xi - G_1 \phi_1 \sin \theta \right) \]

\[ -G_1 \phi_1 \sin \theta \left( 1 - \xi - G_1 \phi_1 \sin \theta \right) \]

\[ + G_1 \phi_1 \sin \theta \left( 1 - \xi - G_1 \phi_1 \sin \theta \right) \]

\[ E_1 = \frac{dT}{dt} + T \delta u + M_1 \delta \sin \theta \]

\[ E_1 = \frac{dT}{dt} + M_1 \delta \sin \theta \]

\[ E_1 = \frac{dT}{dt} + M_1 \delta \sin \theta \]
Equation (13) has two solutions: one in which \( \theta \) is an odd function of \( \theta \) and \( \omega \) is an even function of \( \theta \), and the other in which \( \omega \) is an even function of \( \theta \) and \( \omega \) is an odd function of \( \theta \). That is:

1. In the case that \( \theta \) is a odd function of \( \theta \), one takes:
   \[
   u = \frac{1}{F_0} A_0 d\theta, \quad \omega = \frac{1}{F_0} C_0 d\theta
   \]  

   where \( F_0 \) and \( R_0 \) are undetermined coefficients which are determined in turn as follows. Substituting Eq. (18) into Eq. (13) yields
   \[
   \begin{align*}
   & \sum \left( -2f_m F_{m+1} P_{m+1} + (2m+1)F_m P_m R_m \omega \right) d\theta = 0 \\
   & \sum \left( (2m-3)F_m P_m R_m + (2m-3)(2m)F_m R_m \omega \right) d\theta = 0
   \end{align*}
   \]

   where \( F_{1A}, F_{2A}, f_{mA}, f_{mB} \) are series defined in the Appendix. From Eq. (19), the coefficients \( F_{1A}, R_{A}(m+2) \) are successively obtained with \( R_0, R_1, \) and \( R_2 \) left undetermined. Hence, three independent solutions arise from Eq. (18).

   3. Numerical Calculations

   In the numerical calculations, Poisson's ratio was taken as \( \nu = 0.3 \) and the displacement functions \( u \) and \( \omega \) were calculated by retaining...
120 terms for each of the coefficients \( P_m \) and \( \bar{R}_n \) in Eqs. (18) and (20). Table 2 shows the convergence of the solutions for the case of representative parameters. The numbers in the table are the number of digits of accuracy of the functions. The rate of convergence of the solutions depend upon the curves and parameters. From Eq. (14), one finds 8 kinds of boundary conditions at each end of the shell. In this report, the following three conditions are numerically calculated.

\[
\begin{align*}
\frac{\partial w}{\partial x} &= 0 \quad \text{(clamped)} \\
\frac{\partial w}{\partial x} &= \bar{M}_I = 0 \quad \text{(supported)} \\
\bar{T}_I &= \bar{T}_O = M_{10} = 0 \quad \text{(free)}
\end{align*}
\]

Considering the above boundary conditions, one obtained the frequency equation in the form of a third order determinant for both symmetric and antisymmetric modes. To obtain computational cost saving, \( \alpha, \beta, \lambda, \mu, \omega \) were first chosen, and then a search was made for the values of \( \theta \) which satisfied the frequency equations. These values of \( \theta \) were denoted as \( \theta_0 \) and the relation between \( \alpha/\beta \) and \( \theta_0 \) was shown for the results hereafter. From the relation between the axis of revolution and

<table>
<thead>
<tr>
<th>Mode of curve</th>
<th>Symmetric</th>
<th>Antisymmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipse, ( \mu=0.2 )</td>
<td>1: 0.12057, 0.20289</td>
<td>1: 0.32073, 0.55360</td>
</tr>
<tr>
<td>Ellipse, ( \mu=0.8 )</td>
<td>1: 0.32027, 0.55360</td>
<td>1: 0.52198, 0.85759</td>
</tr>
<tr>
<td>Parabola, ( k=1.0 )</td>
<td>1: 0.20225, 0.40450</td>
<td>1: 0.20225, 0.40450</td>
</tr>
<tr>
<td>Parabola, ( k=2.0 )</td>
<td>1: 0.20225, 0.40450</td>
<td>1: 0.20225, 0.40450</td>
</tr>
<tr>
<td>Parabola, ( k=3.0 )</td>
<td>1: 0.20225, 0.40450</td>
<td>1: 0.20225, 0.40450</td>
</tr>
<tr>
<td>Parabola, ( k=4.0 )</td>
<td>1: 0.20225, 0.40450</td>
<td>1: 0.20225, 0.40450</td>
</tr>
<tr>
<td>Parabola, ( k=5.0 )</td>
<td>1: 0.20225, 0.40450</td>
<td>1: 0.20225, 0.40450</td>
</tr>
<tr>
<td>Parabola, ( k=6.0 )</td>
<td>1: 0.20225, 0.40450</td>
<td>1: 0.20225, 0.40450</td>
</tr>
<tr>
<td>Parabola, ( k=7.0 )</td>
<td>1: 0.20225, 0.40450</td>
<td>1: 0.20225, 0.40450</td>
</tr>
</tbody>
</table>

**Table 2** Relations between the number of terms and converged digits of the solution (Symmetric, \( \beta=5000 \), \( \alpha/\beta=1.0 \))

<table>
<thead>
<tr>
<th>Curve</th>
<th>( \alpha/\beta )</th>
<th>( \alpha/\beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipse</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Ellipse</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>Ellipse</td>
<td>3.0</td>
<td>3.0</td>
</tr>
<tr>
<td>Ellipse</td>
<td>4.0</td>
<td>4.0</td>
</tr>
<tr>
<td>Ellipse</td>
<td>5.0</td>
<td>5.0</td>
</tr>
<tr>
<td>Ellipse</td>
<td>6.0</td>
<td>6.0</td>
</tr>
<tr>
<td>Ellipse</td>
<td>7.0</td>
<td>7.0</td>
</tr>
</tbody>
</table>

**Fig. 2** Frequency curves (Both ends clamped)

**Fig. 3** Frequency curves (Both ends clamped)

**Fig. 4** Frequency curves (Both ends clamped)

**Fig. 5** Frequency curves (Both ends clamped)
the plane curve, shells of revolution having elliptical, cycloidal and parabolical meridian are positive Gaussian curvature shells, while shells of revolution having catenary and hyperbolical meridian are negative Gaussian curvature shells.

Table 3 presents accurate data of $\theta_0$ against $a'/b'$ in the case where $b'=5000$ ($L/h=30.4$). The symbols C-C, S-S, and P-F denote the boundary conditions for both ends clamped, supported and free respectively. The frequency of the symmetric first mode of vibration becomes zero for P-F condition, which represents the translation of the shell as a rigid body. The S-S condition gives close values to C-C condition.

Figures 2-6 show the relation between $a'/b'$ and $\theta_0$ of shells of revolution having cycloidal, parabolical, catenary, hyperbolical and elliptical meridional curvatures. The curves are the first, second, third... modes of vibration in the order from below. It is found that the curves locate higher in order of antisymmetric first, symmetric first, antisymmetric second, symmetric second, where $a'/b'$ is small and $\theta_0$ is large, while they locate in order of symmetric first, antisymmetric first, symmetric second, antisymmetric second... where $a'/b'$ is larger. In the case of hyperbolical shells of revolution, as seen in Fig.5 the value of $a'/b'$ decreases rapidly to zero as $\theta_0$ comes close to unity. The value of $\theta_0$ which makes $a'/b'$ zero is obtained from $\theta_0=\tan(\pi+\theta)/\pi$ and it gives $\theta_0=1.107$ when $\nu=0.5(b/a=2)$. In this case, the shell has infinite length. Setting $\nu=0.5$ (i.e. $a/b=1$) for elliptical shells of revolution, one obtains $a'/b' - \theta_0$ relation for spherical shells. In the case that $\nu=0.5$, $b'=10^3(L/h=28.9)$, $a'/b'=3.0$ and both ends clamped, for example one obtains the value of $\theta_0$ for the first through third modes of symmetric vibration as 0.23871, 0.48840, 0.73288 and those of antisymmetric one as 0.31829, 0.57014, 0.82203 which completely coincide with the known results of Ref. (6).

Figures 7 and 8 show the mode shapes. Mode numbering corresponds to the numbered points of the frequency curves which are shown in Fig.6. They are the displacements $u$ and $w$ for $0\leq\theta_0$ in which $\lambda_1$ of Eq.(22) is taken as unity and the scales of $u$ and $w$ are magnified or reduced appropriately to represent them exactly. As seen from the figures, the mode shapes vary with $\theta_0$ even for the same mode of vibration.

4. Conclusions

An exact and general method has been developed for analyzing the free vibration problems for shells of revolution of which the meridian of the cross section is a plane curve. Here, the problem is confined to the axisymmetric vibrations of thin shells of revolution.

For the sake of specificity, the method is demonstrated for shells of revolution having cycloidal, parabolical, elliptical, catenary and hyperbolical meridional curvatures. The natural frequencies and mode shapes are calculated numerically and the vibration characteristics of these shells are clarified.

The numerical calculations presented here were carried out on an ACOS system 1000 computer of the Tohoku University Computing Center.

Finally, the authors wish to express their thanks to Mr. YAMAGUCHI, Technical Official in Faculty of Engineering, Yamagata University for her help in the numerical calculations.
Appendix

(i) Expressions of $F_1$, $F_2$, $f_1$, $f_2$, $f_3$ in Eq. (19).

\[
\]

\[
f_1 = \sum_{j=1}^{m} \left[ (G(G, W, W, W, W, + (\xi - 1)W, +), \xi - (2m - 1)W, +) \frac{1}{2} (p + q - 1) \sum_{e=1}^{3} (2p_e + R_e) \right] - G(W, W, A, W, W, +) \frac{1}{2} (2p + R) \]

\[
+ G(G, W, (\xi - 1)W, +, e, - (2m - 1)W, +) (2p + R) + R_s + R_t + R_e + R_f
\]

\[
+ \sum_{j=1}^{m} \left[ (G(G, W, W, W, W, + (\xi - 1)W, +), \xi - (2m - 1)W, +) \frac{1}{2} (2p + R) \right] - G(W, W, A, W, W, +) \frac{1}{2} (2p + R)
\]

\[
+ G(G, W, (\xi - 1)W, +, e, - (2m - 1)W, +) (2p + R) + R_s + R_t + R_e + R_f
\]

\[
f_2 = \sum_{j=1}^{m} \left[ (G(G, W, W, W, W, + (\xi - 1)W, +), \xi - (2m - 1)W, +) \frac{1}{2} (2p + R) \right] - G(W, W, A, W, W, +) \frac{1}{2} (2p + R)
\]

\[
+ G(G, W, (\xi - 1)W, +, e, - (2m - 1)W, +) (2p + R) + R_s + R_t + R_e + R_f
\]

(ii) Expressions of $F_3$, $F_4$, $f_1$, $f_2$, $f_3$, $f_4$ in Eq. (21).

\[
\]

\[
f_3 = \sum_{j=1}^{m} \left[ (G(G, W, W, W, W, + (\xi - 1)W, +), \xi - (2m - 1)W, +) \frac{1}{2} (2p + R) \right] - G(W, W, A, W, W, +) \frac{1}{2} (2p + R) + R_s + R_t + R_e + R_f
\]

\[
+ G(G, W, (\xi - 1)W, +, e, - (2m - 1)W, +) (2p + R) + R_s + R_t + R_e + R_f
\]

\[
+ \sum_{j=1}^{m} \left[ (G(G, W, W, W, W, + (\xi - 1)W, +), \xi - (2m - 1)W, +) \frac{1}{2} (2p + R) \right] - G(W, W, A, W, W, +) \frac{1}{2} (2p + R)
\]

\[
+ G(G, W, (\xi - 1)W, +, e, - (2m - 1)W, +) (2p + R) + R_s + R_t + R_e + R_f
\]

\[
f_4 = \sum_{j=1}^{m} \left[ (G(G, W, W, W, W, + (\xi - 1)W, +), \xi - (2m - 1)W, +) \frac{1}{2} (2p + R) \right] - G(W, W, A, W, W, +) \frac{1}{2} (2p + R)
\]

\[
+ G(G, W, (\xi - 1)W, +, e, - (2m - 1)W, +) (2p + R) + R_s + R_t + R_e + R_f
\]

\[
+ \sum_{j=1}^{m} \left[ (G(G, W, W, W, W, + (\xi - 1)W, +), \xi - (2m - 1)W, +) \frac{1}{2} (2p + R) \right] - G(W, W, A, W, W, +) \frac{1}{2} (2p + R)
\]

\[
+ G(G, W, (\xi - 1)W, +, e, - (2m - 1)W, +) (2p + R) + R_s + R_t + R_e + R_f
\]

\[
- G(W, W, W, W, W, + (\xi - 1)W, +), \xi - (2m - 1)W, +) \frac{1}{2} (2p + R) \]

\[
+ G(G, W, (\xi - 1)W, +, e, - (2m - 1)W, +) (2p + R) + R_s + R_t + R_e + R_f
\]

\[
+ \sum_{j=1}^{m} \left[ (G(G, W, W, W, W, + (\xi - 1)W, +), \xi - (2m - 1)W, +) \frac{1}{2} (2p + R) \right] - G(W, W, A, W, W, +) \frac{1}{2} (2p + R)
\]

\[
+ G(G, W, (\xi - 1)W, +, e, - (2m - 1)W, +) (2p + R) + R_s + R_t + R_e + R_f
\]

\[
- G(W, W, W, W, W, + (\xi - 1)W, +), \xi - (2m - 1)W, +) \frac{1}{2} (2p + R) \]

\[
+ G(G, W, (\xi - 1)W, +, e, - (2m - 1)W, +) (2p + R) + R_s + R_t + R_e + R_f
\]

References
