Self-excited Vibration of a Circular Plate Subjected to Frictional Forces Exerted in Two Regions on Its Outer Circumference

(Part 1, The Case of a Circular Plate without Effect of Internal Resonances)

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Using the averaging method and a direct numerical integration, is analyzed the self-excited vibration of a Mindlin circular plate subjected to comparatively weak frictional forces as a function of relative slip velocity at two points on its outer circumference. It is found from the averaging method that there exist two types of stable steady-state solutions of a single mode with different spatial locations with respect to the two frictional points. This paper also presents a relationship between the generating vibration mode and the angle between the frictional points, and a relationship between the magnitude of each frictional force and generating mode. The results from two different approaches agreed well with each other. Furthermore, there was a good qualitative agreement between experimental and analytical results of the self-excited vibrations generated in a single mode.

Key Words: Vibration, Self-excited Vibration, Frictional Vibration, Vibration of Circular Plate, Averaging Method, Direct Numerical Integration

1. Introduction

There have been a number of investigations relating to frictional vibrations (11-13). Most of them treated those of a single-degree-of-freedom system. Kataoka et al. (14), Yokoi and Nakai (15-16) investigated the vibrations of a beam due to friction. The self-excited oscillations in a multi-degree-of-freedom oscillating circuit similar to the above system were examined by Hayashi et al. (17), taking into account the effect of the internal resonances. In relation to the problem of noise, there have been a number of works on the squeal noise generated in a wheel of a railway vehicle and a brake of a car due to the frictional vibrations. Rudd (18) regarded a wheel as a single-degree-of-freedom system and analysed the wheel squeal. Nakai et al. (19,20) dealt with a wheel as an autonomous system having many degrees of freedom, and investigated the mechanism of the wheel squeal theoretically and experimentally. The present paper describes the self-excited vibrations of a wheel which is rubbed axially with a frictional force at a point on its outer circumference. The frictional force has a characteristic of falling gradient with respect to relative slip velocity. In this paper, the term "frictional force" means a force with such a frictional characteristic.

In practice, the wheel tread brake in a railway vehicle and the drum brake in a car are actually subjected to frictional forces acting at two parts on their outer circumferences. The self-excited vibration in such a system may be different from that for the case that only one frictional force acts. In this report, therefore, the authors analyse the self-excited vibrations of a circular plate accompanied with no internal resonances which can be generated when the friction rods are pressed in the normal direction at two points on its outer circumference and are swept in the axial direction of the circular plate. Mindlin's circular plate theory (21) is applied to the out-of-plane motion of the circular plate. The steady-state vibrations for the nonlinear simultaneous ordinary differential equations obtained from the expansion of the solution in terms of the characteristic modes, are computed by applying the averaging method and a direct numerical integral method (cf. section 2.4) which is a kind of shooting method (22). The application of the averaging method shows that there exist two types of stable self-excited vibrations with different spatial locations with respect to the two frictional points, and makes clear the relationship between the generating vibration mode and the angle between the two frictional points, and the dependence of the generating modes upon the magnitude of the frictional forces. The results derived from both approaches agree very well with each other. Furthermore, a fine qualitative agreement between the experimental and the theoretical results is confirmed.

2. Theoretical analysis

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The authors examine a Mindlin circular plate with steps. The number of steps is assumed to be \( N \). Boundary conditions for inner circumference, \( r = r_s \), which is fixed, and the outer circumference, \( r = R \), which is free, are assumed. Two axial frictional forces \( P_1 \) and \( P_2 \) act respectively at Point 1 \((R, \theta)\) and Point 2 \((R, \theta)\) on the outer circumference of the circular plate as shown in Fig.1. The equations of motion and the stress-displacement relations of a homogeneous ring are represented by the following equations, in which super- and subscripts \( j \) refer to the \( j \)-th ring from the inside \((j = 1, \ldots, N)\):

\[
\begin{align*}
\frac{\partial^2 M_j}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial M_j}{\partial \theta} \right) + \frac{M_j - M_{j-1}}{r} &= Q_i^j \quad \text{(1a)} \\
\frac{\partial^2 Q_i^j}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial Q_i^j}{\partial \theta} \right) - Q_i^j &= \frac{\rho h_i}{12} \frac{\partial^2 \psi_i^j}{\partial t^2} \quad \text{(1b)}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 \psi_i^j}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi_i^j}{\partial \theta} \right) &= \frac{\rho h_i}{12} \frac{\partial^2 w_i^j}{\partial t^2} - \frac{P_1}{r} \eta(R, \theta) - \frac{P_2}{r} \delta(R, \theta) \quad \text{(1c)}
\end{align*}
\]

\[
\begin{align*}
\frac{M_j}{D_i} &= \frac{\partial^2 \psi_i^j}{\partial \theta^2} + \frac{1}{D_i} \left( \psi_i^j + \frac{\partial^2 \psi_i^j}{\partial t^2} \right) \quad \text{(2a)} \\
\frac{M_j}{D_i} &= \frac{1}{D_i} \left( \psi_i^j + \frac{\partial^2 \psi_i^j}{\partial t^2} \right) + \frac{\partial^2 \psi_i^j}{\partial t^2} \quad \text{(2b)} \\
2M_{i+1} - 1 &= \frac{\partial^2 \psi_i^j}{\partial \theta^2} + \frac{\partial^2 \psi_i^j}{\partial \theta^2} \quad \text{(2c)} \\
Q_i^j &= \frac{\pi^2 G}{12} \frac{\partial^2 \psi_i^j}{\partial \theta^2} \quad \text{(2d)} \\
Q_i^j &= \frac{\pi^2 G}{12} \frac{\partial^2 \psi_i^j}{\partial \theta^2} \quad \text{(2e)}
\end{align*}
\]

where \((r, \theta)\) is the polar coordinate of the circular plate, \( M_j, M_{i+1}, M_i, Q_i^j \) and \( Q_i^j \) are moments and shearing forces in the corresponding directions respectively, \( w_i^j, \psi_i^j \) and \( \phi_i^j \) are axial displacement and angular displacements respectively [cf. Fig.1(a)], \( h_i \) is thickness of the \( j \)-th

Fig.1 Coordinates and parameters of circular plate and frictional points

2.1 Natural frequencies and characteristic functions

We obtain the solutions of the equations of free vibration derived from Eq.(1) according to Mindlin\(^{(10)}\). Defining natural circular frequencies by \( \omega_i^m \), where \( k \) is the number of nodal diameters and \( l \) is the number of nodal circles, \( k_i = 0, \ldots, \infty \), and the characteristic functions of \( \psi_i^j, \phi_i^j \) and \( w_i^j \) corresponding to \( \omega_i^m \) by \( \phi_i^m, \phi_i^m, \) and \( w_i^m \) respectively, one set of characteristic modes, that is, the symmetric modes are written as

\[
\begin{align*}
\phi_i^m(r, \theta) &= \phi_i^m(r) \cos k \theta \quad \text{(3a)} \\
\phi_i^m(r, \theta) &= \phi_i^m(r) \sin k \theta \quad \text{(3b)} \\
\omega_i^m(r, \theta) &= \omega_i^m(r) \cos k \theta \quad \text{(3c)} \\
&= \omega_i^m(r) \sin k \theta \quad \text{(3d)} \\
&= \omega_i^m(r) \cos k \theta \quad \text{(3e)} \\
&= \omega_i^m(r) \sin k \theta \quad \text{(3f)}
\end{align*}
\]

If \( \cos \) and \( \sin \) in Eq.(3) are replaced by \( \sin \) and \( -\cos \) respectively, then we obtain the anti-symmetric modes

\[
\begin{align*}
\phi_{i+1}^m(r, \theta) &= \phi_{i+1}^m(r) \sin k \theta \quad \text{(4a)} \\
\phi_{i+1}^m(r, \theta) &= -\phi_{i+1}^m(r) \cos k \theta \quad \text{(4b)} \\
\omega_{i+1}^m(r, \theta) &= \omega_{i+1}^m(r) \sin k \theta \quad \text{(4c)} \\
&= \omega_{i+1}^m(r) \cos k \theta \quad \text{(4d)} \\
&= \omega_{i+1}^m(r) \sin k \theta \quad \text{(4e)} \\
&= \omega_{i+1}^m(r) \cos k \theta \quad \text{(4f)}
\end{align*}
\]

where \( \phi_{i+1}^m(r) \) etc. are the shape functions of the characteristic modes and involve the Bessel functions and 6 or 4 integral constants \(^{(10)}\).

2.2 Equations of self-excited vibrations

By using Eqs.(3) and (4), in order to consider the change of phase with respect to \( \theta \), we assume the solutions of Eq.(1) as follows:

\[
\begin{align*}
\psi_i^j &= \sum_{m=1}^{\infty} \phi_i^m(C_m \cos m\theta - D_m \sin m\theta) \quad \text{(5a)} \\
\psi_i^j &= \sum_{m=1}^{\infty} \phi_i^m(C_m \sin m\theta - D_m \cos m\theta) \quad \text{(5b)} \\
\omega_i^j &= \sum_{m=1}^{\infty} \omega_i^m(C_m \cos m\theta + D_m \sin m\theta) \quad \text{(5c)} \\
&= \sum_{m=1}^{\infty} \omega_i^m(C_m \sin m\theta + D_m \cos m\theta) \quad \text{(5d)}
\end{align*}
\]

where \( C_m \) and \( D_m \) are both unknown functions of time, and the symbol \( \Sigma_m \) means a double summation from zero to \( \infty \) with respect to the subscripts \( m \) and \( z \). When the shearing forces \( P_1 \) and \( P_2 \) act simultaneously at two points of the circular plate, the solutions must be assumed to be linear combinations of the symmetric and anti-symmetric modes in order to represent exactly the generating vibrations.

We substitute Eq.(5) into Eq.(1), and multiply both sides of Eqs.(1a), (1b) and (1c) by Eqs.(3a), (3b) and (3c) respectively and add the resulting equations on both sides. We then integrate the resultant relations over the total region of the circular plate with the help of Eq.(2), the relationship \(^{(10)}\) between natural circular frequencies and characteristic functions and the orthogonal relationship

\[
\sum_{m=1}^{\infty} \int_{r_1}^{r_2} \sum_{j=1}^{N} \left( \left[ \phi_i^m \phi_{i+1}^m + \phi_i^m \phi_{i+1}^m \right] \frac{\partial^2 \psi_i^j}{\partial \theta^2} \right) \frac{h_i^2}{12} \\
+ \sum_{m=1}^{\infty} \int_{r_1}^{r_2} \sum_{j=1}^{N} \left( \left[ \phi_i^m \phi_{i+1}^m + \phi_i^m \phi_{i+1}^m \right] \frac{\partial^2 \psi_i^j}{\partial \theta^2} \right) \omega_i^m \omega_{i+1}^m \right) \frac{\partial \omega_i^m}{\partial \theta} dt d\theta = 0 
\]

for \((m, s) = (k, l)\). We repeat similarly the above process by using Eqs.(4a), (4b) and
\( (4.6) \) in place of Eqs.(3.a), (3.b) and (3.c) respectively. After tedious calculations, the following equation with subscripts \( m \) and \( s \) \((m, s=0, \ldots, \infty) \) is obtained:

\[
\left\{ \begin{align*}
\frac{d^2\omega^a_m(R)}{dt^2} + 2\omega^s_m\omega^a_m \frac{d\omega^a_m}{dt} + \omega^a_m^2 \frac{D^a_m}{D_m^s} \\
= \frac{\rho R}{\rho R^2} \left[ P_i \left( \cos m\theta + P_i \left( \sin m\theta \right) \right) \right]
\end{align*} \right.
\]

The integral constants involved in the characteristic functions of \((m, s)\)-mode are conveniently determined such that they satisfy the following normalized relation:

\[
\frac{1}{\pi} \int_0^{\pi} \left[ \left( \phi^a_m \right)^2 + \left( \phi^s_m \right)^2 \right] \frac{d\theta}{1 + \rho m^2} = \frac{\rho R^2}{12}
\]

where \( \rho \) is Kronecker's delta and \( \phi^i_{m,s} = 0 \) for the axisymmetric modes \((m=0)\).

In the second term on the left side of Eq.(7), the viscous dampsings coupled with respective modes \((\delta_{m,s} \text{ is damping ratio})\) are considered to induce a weak damping of the circular plate.

The forces \( P_i \) and \( P_i \) in Eq.(7) are replaced by

\[
P_i = P_i - C_i \cdot \omega_m \]

where \( f(\omega) \) is the coefficient of friction, \( P_i \) is the normal load, \( V_i \) is sweep velocity of friction rod, \( \nu_i \) is relative slip velocity \( \cdot \nu_i = \omega^s - \omega^a, \nu_i = \omega^s(R, \theta, t) \) and \( \omega_m = \omega(R, \theta, t) \), \( \frac{\partial \omega^a_m}{\partial \omega^s_m} \) and \( \frac{\partial \omega^a_m}{\partial \omega^s_m} \) meaning the vibrational centres of Eq.(7) are decided from

\[
\left\{ \begin{align*}
C_m &= \alpha_{m,s} (P_i f(\nu_i) + P_i f(\nu_i))/\omega_m \\
D_m &= \beta_{m,s} (P_i f(\nu_i) - P_i f(\nu_i))/\omega_m
\end{align*} \right.
\]

where \( \alpha_{m,s} = \rho m \rho R^4, \beta_{m,s} = \rho m \rho R^4 \nu_i \omega_m \).

\[
\nu_i = \omega^s - \omega^a, \omega_m \cos m\theta, \nu_i = \omega^s - \omega^a, \omega_m \sin m\theta
\]

Applying the coordinate transformations

\[
C_m = C_{\infty} + C_m, D_m = D_{\infty} + D_m
\]

we obtain the fundamental equations of the self-excited vibrations with respect to the fluctuating components \( C_m \) and \( D_m \) as follows:

\[
\left\{ \begin{align*}
\frac{d^2\omega^a_m(R)}{dt^2} + 2\omega^s_m\omega^a_m \frac{d\omega^a_m}{dt} + \omega^a_m^2 \frac{D^a_m}{D^\infty} \\
= \alpha_{m,s} (P_i f(\nu_i) + P_i f(\nu_i))/\omega_m
\end{align*} \right.
\]

\[
\frac{d^2\omega^a_m(R)}{dt^2} + 2\omega^s_m\omega^a_m \frac{d\omega^a_m}{dt} + \omega^a_m^2 \frac{D^a_m}{D^\infty} \\
= \beta_{m,s} (P_i f(\nu_i) - P_i f(\nu_i))/\omega_m
\]

where \( f(\nu_i) = c_i |\nu_i| - e_i |\nu_i| + e_i \)

where coefficients \( c_i, e_i, e_i \) are suitable constants representing the characteristic of friction at Point \( i = 1, 2 \).
by the characteristic values of the coefficient matrix in Eq.(18). Namely, if and only if the real parts of the characteristic values are all negative, the corresponding solutions are stable.

The fluctuating displacement and velocity at an arbitrary point on the circular plate corresponding to the steady-state solution of.by and \( b_{ix} \) can be computed from Eqs.(15), (13), (10) and (5c).

2.4 Application of a direct numerical integral method

There have been a number of trials to obtain the periodic solutions of an autonomous system by a direct numerical integration. Krogdahl solved van der Pol's equation numerically by the Runge-Kutta-Gill method, and Urabe presented an algorithm based on the concept of the phase space. In this report, the following algorithm is used in order to ensure the accuracy of the averaging method and to make it applicable to the problem in which there is a certain discontinuity in the characteristic of friction in the future. The present method is much more general than the averaging method treated in section 2.3.

(1) We transform Eq.(11) into a system of first order ordinary differential equations

\[
d\mathbf{y}/dt = f(\mathbf{y})
\]

(19)

where \( \mathbf{y} = (y_1, \ldots, y_n) \) and \( f = (f_1, \ldots, f_n) \).

(2) If the period of the solution of Eq.(19) is \( T \), which is unknown, then setting \( \tau = (2\pi)/T \), the following differential equations have a periodic solution of period \( 2\pi \):

\[
d\mathbf{y}/d\tau = (T/2\pi)f(\mathbf{y})
\]

(20)

(3) Defining the variation of \( \mathbf{y} \) by \( \mathbf{\eta} \), the variational equations become as follows:

\[
d\mathbf{\eta}/d\tau = (T/2\pi)A\mathbf{\eta}
\]

(21)

where \( \mathbf{\eta} = (\eta_1, \ldots, \eta_n) \) : Jacobian matrix

(4) We assume \( T^* \) as an approximate value of \( T \) and initial value \( \mathbf{y}^* \) at \( \tau = 0 \). Then Eq.(20) is integrated numerically from \( \tau = 0 \) to about \( \tau = 2\pi \). When \( \mathbf{y}^*(\tau) \) becomes minimum during computation, let time \( \tau \) temporarily be the period \( T^* \) of Eq.(20) and set \( \mathbf{y}(\tau) = \mathbf{y}(\tau) \).

(5) Equation (21) is integrated numerically from \( \tau = 0 \) with the initial conditions \( \mathbf{\eta}(0) = (0, \ldots, 0) \) whose \( \tau \)-th element is unity, and \( \mathbf{\eta}(\tau) = (\eta_1(\tau), \ldots, \eta_n(\tau)) \) is obtained for \( i = 1, \ldots, n \). Then the fundamental solution matrix becomes as follows:

\[
B = (\mathbf{\eta}, \ldots, \mathbf{\eta})
\]

(22)

Since \( B = \partial y/\partial y \), we obtain the first approximate equation with respect to correction value \( \mathbf{y}^* \) for \( \mathbf{y}^* \) by the Newton-Raphson method,

\[
B^{-1} \mathbf{y} - \mathbf{y} - \mathbf{y}^* = \mathbf{0}
\]

(23)

where \( \mathbf{I} \) is an \( n \times n \) unit matrix.

As the phase is indefinite in an autonomous system \( B - \mathbf{I} \), is singular for a steady-state solution, it is fixed by putting \( \mathbf{y}^* = \mathbf{c} \) constant. Then Eq.(23) is rewritten as

\[
(B - \mathbf{I} - \mathbf{c}) \mathbf{y}^* = \mathbf{y} - \mathbf{y}^* = \mathbf{0}
\]

(24)

where \( B \) is \( (n - 1) \times (n - 1) \) matrix eliminating the \( k \)-th row and \( k \)-th column of \( B \). \( \mathbf{y}^* \), \( \mathbf{\eta}_k \) and \( \mathbf{\eta}_k^* \) are \( (n - 1) \)-dimensional vectors eliminating the \( k \)-th element of \( \mathbf{y}^* \), \( \mathbf{y} \) and \( \mathbf{\eta} \) respectively.

We solve Eq.(24) and replace \( \mathbf{y} = \mathbf{y}^* \) ( \( \mathbf{y}^* = \mathbf{c} \) is invariable) and \( T^*/2\pi \) by \( \mathbf{y} \) and \( T^* \) respectively, and then return to (4). The above procedure is repeated until the solution converges.

(6) We judge the stability of the convergent solutions. At least one of characteristic values of matrix \( B \) is unity for an autonomous system. If the absolute values of the characteristic values except for unity are all less than unity, then the corresponding characteristic solution is stable.

The present method is applied to van der Pol's equation

\[
y - y(1 - y^2) y + y = 0
\]

(25)

in order to confirm the accuracy of this method. The results are compared in Table 1 with those obtained by the harmonic balance method (the number in the parentheses indicates the maximum order in Fourier series) and by other investigators. The results of the present method and the harmonic balance method agree very well with each other.

3. Numerical Results and Discussion

In Fig.2 are shown the dimensions of the circular plate used in the numerical computation and in the experiment. The inner circumference of the circular plate is fixed by a bolt of 30mm diameter (see Fig.9). The theoretical natural frequencies and the experimental ones by hammering for \( (m, s) \)-mode (\( m \) is the number of nodal diameters and \( s \) is the number of nodal circles) are shown in Table 2.

In this report, the numerical computation and the experiment are carried out
for relatively weak normal loads of less than 30N. In the calculation, we set \( \varepsilon_{rs} = 0.003 \) constant which is common to each mode and the sweep velocities of the friction rods are the same at the two points, that is \( V_r = V_s = 0.085 \text{m/s} \). Furthermore, the characteristic of friction is the same at the two points and we assume the following two kinds:

(a) \( \varepsilon_I = 150 \text{ s}^2/\text{m}^4, \varepsilon_e = 4.5 \text{ s/m}, \varepsilon_s = 0.57 \)  
(b) \( \varepsilon_I = 200 \text{ s}^2/\text{m}^4, \varepsilon_e = 6.0 \text{ s/m}, \varepsilon_s = 0.7 \)

\[ \text{(26-a)} \]
\[ \text{(26-b)} \]

First the direct numerical integral method was applied to a system composed of 11 modes in ascending order from the lowest natural frequency. The results showed that there was no effect of the internal resonances in the circular plate illustrated in Fig.2 (the detailed discussion is found later). It was confirmed from the results by the averaging method that no stable self-excited vibrations with multiple modes existed and thus only vibrations of single mode were generated. In the vibration with multiple modes, there is no single period and more than two modes with different \( m \) or \( s \) occur simultaneously. From both methods, it was also made clear that there exist stable solutions of only the (1,0), (2,0) and (3,0)-single modes for the present range of normal loads. Therefore, taking into account the amount of calculation etc., we indicate the numerical computational results obtained by using only five modes from the lowest natural frequency.

In the figures below, the thick line shows the solution which has positive relative velocities at the two frictional points over one period, and the fine line the solution which has negative relative velocity at one or two points. The solid and the broken, dashed lines indicate the stable and unstable solutions respectively. From Eq.(14), if one set of \( \alpha_{as} \) and \( b_{as} \) is a solution, then the other set of \( -\alpha_{as} \) and \( -b_{as} \) is also a solution. Hence, either set of the solutions is indicated in the figures. The superscript 0 of the ordinates \( \alpha_{as} \) and \( b_{as} \) is omitted.

(1) In Fig.3 is shown the relationship between the velocity amplitudes \( \alpha_{as} \) and \( b_{as} \) and the angle \( \theta \). \( \theta \leq \theta_{90} \). The results are single-mode solutions derived from the averaging method for the characteristic of friction expressed by Eq.(26a) and with \( P_r = P_s = 19.6 \text{N} \). The solutions, except for broken lines, are either symmetric modes which have a loop between two frictional points or anti-symmetric modes which have a node between them. The former is conveniently called Type S and the latter called Type AS. If \( P_r = P_s \), then Type S has a loop at the centre of the frictional points \( (\alpha_{as} = 0, b_{as} = 0) \), and Type AS has a node at this centre \( (\alpha_{as} = 0, b_{as} = 0) \). On the other hand, the broken lines show the modes with \( \alpha_{as} = 0 \) and \( b_{as} = 0 \) but the corresponding steady-state solutions are all unstable. Therefore, stable solutions are always on the branches of Type S or AS. Types S and AS have a symmetric characteristic with respect to \( \theta = 180 \text{m} \) and \( \theta = 180(n + 1/2) \text{m} \) for integer \( n \) respectively. For example, Fig.3 shows that the (1,0)-mode of Types S and AS and the (2,0)-mode of Type AS can be generated at \( \theta = 45^\circ \). The results at \( \theta = 90^\circ \) correspond to those for only one frictional point.

The solid and thick lines correspond to the stable solutions which always have positive relative velocities. (We call the solutions type #1.) The parts shown by solid and fine lines are stable according to the averaging method but there are imaginary solutions because either or both of the relative velocities are negative. Therefore, not type #1 but a stick-slip vibration may be generated and hence the averaging method can not be applied to the corresponding regions. Making use of the result from the averaging method in which a single-mode vibration is generated in these regions, we approximate the system as a one-degree-of-freedom system and

\[ \text{Fig.2 Dimensions of circular plate (unit: mm)} \]
\[ \text{Fig.3 Characteristic of vibration } (P_r = P_s, \text{ characteristic of friction (a)}) \]
\[ \text{Fig.4 Characteristic of vibration } (P_r = P_s, \text{ characteristic of friction (b)}) \]
integrate numerically from the equilibrium state of static frictional force, restoring force and damping force. Consequently it is confirmed that these parts become stuck at the time that the relative velocities first become zero. (We call these solutions simply stick-slip.)

For the (1,0)- and (2,0)-modes, there are some regions of \( \theta \) where Types S and AS coexist and the regions depend on the generating mode. The (3,0)-mode has a small amplitude and is unstable. The (0,0)- and (4,0)-modes do not even have unstable solutions.

(2) Figure 4 shows results similar to Fig.3 for the characteristic of friction expressed by Eq.(26,b) and with \( P_1 = P_2 = 10.8\)N. The degree of falling gradient in the characteristic of friction becomes larger than that of Fig.3. The following are found: (i) The existing regions of solutions are expanded. The overlapping regions of Type S and Type AS are also expanded. (ii) The higher modes are stabilized and now they possess unstable solutions. [The (0,0)-mode has no unstable solutions yet.] (iii) The amplitudes become larger on the whole, and the stable type \#1 vibrations change gradually to the stick-slip.

(3) The results from the averaging method are illustrated for \( P_1 = 24.5\)N, \( P_2 = 14.7\)N and the frictional characteristics of Eqs.(26,a) and (26,b) in Figs.5 and 6 respectively. For \( P_1 + P_2 \), no solution in which either \( a_s \) or \( b_s \) vanishes independently of \( \theta \) exists. Hence the loop or the node of the vibration deviates from the centre of the two frictional points. This point is different from Figs.3 and 4. Comparison of the figures between Figs.3 and 4 shows obvious condensed vibration characteristic stemming from the equality of the two loads. As the degree of falling gradient in the characteristic of friction increases, (i) Types S and AS connect gradually with each other from the lower modes and the unstable solutions to higher modes begin to emerge and (ii) the amplitude of vibrations, that is, the regions of stick-slip become larger.

(4) At the angles \( \theta = 180(n + 1/2)/m \) and \( 180n/m \) (degrees) in Figs.3 to 6, when \( n \) is integer, Types AS and S having loops at the two frictional points are generated independently of the characteristic of friction and the normal loads.

(5) In general, which mode actually occurs, depends upon the magnitude of the external viscous damping for every mode, the magnitude of \( \omega_{s}(k) \) under the condition of Eq.(8) and the value of \( m\theta \) from Eq.(7).

The relationship between maximum

Fig.5 Characteristic of vibration \( [P_1 + P_2, \) characteristic of friction (b)]

Fig.6 Characteristic of vibration \( [P_1 + P_2, \) characteristic of friction (b)]

Fig.7 Relationship between loads and generating modes

Fig.8 Waveforms over one period derived from the direct numerical integral method
velocity amplitudes of the circular plate and the normal loads (assumed equal) for the characteristic of friction, expressed by Eq.(26a) at $\hat{\beta}=25^\circ$ is indicated in Fig.7 for Types S and AS. As the loads increase at this angle, the (1,0)-mode of Type S occurs and simultaneously becomes stable beyond certain loads. Then (2,0)AS, (3,0)AS and (2,0)S occur in this order. (At another angle the order is different.) The modes which have already become stable prevent the other modes from stabilizing. Only the modes whose amplitudes become so large as to dominate the stable modes appear in practice. The influence of the mutual interference among modes upon the stability (characteristic values of coefficient matrix of Eq.(18) for single-mode vibration) is regarded as important in this way, unlike the case of a single-degree-of-freedom system.

(6) As the loads and the degree of falling gradient in frictional characteristic decrease, the overlapped regions of Types S and AS disappear for even the lower modes of (1,0) and (2,0) and the corresponding solutions exist isolatedly only for restricted values of angle.

(7) We now compare the results of the direct numerical integral method with those of the averaging method. In Fig.8, the waveforms corresponding to the (2,0)-mode at $\hat{\beta}=21.5^\circ$ in Fig.6 are indicated. The frequency is not exactly coincident with $\omega_n$. The (2,0)-mode is dominant but other modes are present, all the modes being synchronous and forming a periodic solutions. Even if more than five modes are adopted, the pattern of the solution is not changed. This is a characteristic of higher approximated solutions to a multi-degree-of-freedom system. The averaging method gives the result of the (2,0)-single mode instead. The corresponding values of the main mode computed by the numerical integral method are shown as circles in Fig.3 and 5 in part. The white and black circles correspond to the stable and unstable solutions respectively. Comparison of the thick lines (solid, dashed and broken lines) of type # of derived from the averaging method with the results from the numerical integral method shows a fine agreement in the velocity amplitudes. By more detailed calculations, the boundaries between stable and unstable regions, and those between type #1 vibration and stick-slip are confirmed to be in good agreement with each other.

(8) The averaging method is applied on the assumption that no internal resonances occur in the circular plate, and a good quantitative agreement with the results from the direct numerical integral method is obtained. But the circular plate has an infinitely large number of natural frequencies. Therefore it is easily subjected to the effect of the internal resonances. The circular plate used in this report (Fig.2) is considered to avoid the internal resonances in its dimensions. Whether the internal resonances exist or not must be decided at the beginning of the calculation in applying the averaging method. Therefore, if the analysis is made on the assumption of no internal resonances even though they occur, then we obtain wrong results. In this report, the direct numerical integral method automatically gives the internal resonances and hence it is advantageous for the numerical computations not to require any information about the internal resonances at the beginning of the computation.

4. Experimental Results

The experimental apparatus is shown in Fig.9. Two friction rods in contact with the circular plate at two points (Points 1 and 2) on its outer circumference and a steel rod used as guide are mounted on the other plates. This set is called the slider. The normal loads are applied to the friction rods by load adjusters with springs through the bearings. A squeal of the circular plate is generated by sweeping the slider upwards at a constant speed. The material of the circular plate is SK4 and each friction rod is 20mm in diameter, 400mm in length and is made of S45C. Experimental values $V_s=14.5$, $F_s=4.4$ are adopted. The frictional surfaces are finished with emery paper #800, and ultrasonically cleaned before the experiment. During experiment, the frictional surfaces are held in the same relation to each other and also kept in the same state by wiping with a cloth containing acetone. The contacts between circular plate and friction rods in squeal are always observed electrically in the experiment. The frequencies of the generating modes agree with the experimental values in Table 2. It is very difficult to measure directly the characteristic of friction between circular plate and friction rods during vibration because of the elasticity, the inertia force, etc. of the circular plate. Hence the characteristic of friction was not measured here.

(1) Figure 10 shows Chladni's figures with sand after a few continuous occurrences for (2,0)-mode and the waveform of acceleration measured by the acceleration pick-ups placed at points A and C (cf. Fig.1b) in the vicinity of the outer circumference at $\hat{\beta}=17.5^\circ$ in (a) and at $\hat{\beta}=27.5^\circ$ in (b) when $P_1=P_2=9.8N$, (a) corresponds to Type S and (b) to Type AS. In each Type the positions of loop and node are easily distinguished (cf. Figs. 3 and 4). A similar example of the (1,0)-mode of Type S is shown in Fig.11 where $P_1=P_2=9.8N$ and $\hat{\beta}=17.5^\circ$. The vibration is identified as stick-slip by integrating the experimental waveform.
of acceleration (cf. Figs. 3 and 4).

(2) We examine the rate of occurrence of Type S and Type AS of the (2,0)-mode. To eliminate the effect of the pick-ups, displacement meters which do not touch the circular plate are used. The results are shown in Fig. 12(a) for \( P_1 = P_2 = 6.86 \) N and in Fig. 12(b) for \( P_1 = P_2 = 9.88 \) N. At \( \theta = 0^\circ \) only Type S occurs. Contrarily at \( \theta = 45^\circ \) only Type AS occurs (cf. Figs. 3 and 4).

However, Types S and AS can be generated for \( \theta = 17.5^\circ \sim 30^\circ \). Comparing (b) with (a) shows that the region in which Types S and AS coexist widens and the incidence of Type S in \( \theta \geq 22.5^\circ \) and that of Type AS in \( \theta \leq 22.5^\circ \) is increased. At \( \theta = 22.5^\circ \), Types S and AS are generated with almost the same incidence. The change from Type S to Type AS and vice versa were observed in the experiments. The incidence at each \( \theta \) is calculated statistically by using more than fifty sets of experimental data.

(3) The waveforms of acceleration for the (2,0)-mode of Type AS at points A and C are illustrated in Fig. 13 when \( P_1 \) is kept constant but \( P_2 \) is changed at \( \theta = 45^\circ \). Point A is always on a node and point C is always on a loop independent of changing \( P_2 \). [(d) corresponds to stick slip.] Figure 14 shows results similar to Fig. 13 for the (3,0)-mode of Type AS at \( \theta = 30^\circ \). The amplitudes increase with increasing loads [(a) and (b) belong to type 1 vibration]. These results mentioned above are in good agreement with the theoretical ones (cf. Figs. 3 and 4).

(4) Furthermore, the following is made clear from the experiment. Higher order modes can be easily generated with increasing loads (cf. Fig. 7). Which mode is more apt to occur depends upon the angle \( \theta \) [for example, the (1,0)-mode in the vicinity of \( \theta = 15^\circ \), (3,0)-mode in the vicinity of \( \theta = 30^\circ \) and so on]. The (0,0)-mode can not be generated experimentally and theoretically.

It is confirmed that the slider system used in the experiment has no natural frequencies near those of the circular plate determined by hammering.

S. Conclusions

By the averaging method and a direct

\[
\begin{align*}
\text{(a) } P_1 &= 4.9 \text{ N, } P_2 = 9.88 \text{ N} \\
\text{(b) } P_1 &= 9.88 \text{ N, } P_2 = 9.88 \text{ N} \\
\text{(c) } P_1 &= 14.7 \text{ N, } P_2 = 9.88 \text{ N} \\
\text{(d) } P_1 &= 19.6 \text{ N, } P_2 = 9.88 \text{ N}
\end{align*}
\]

Fig. 13 Waveforms of (2,0)-mode of Type AS at \( \theta = 45^\circ \)

\[
\begin{align*}
\text{(a) } P_1 &= 4.9 \text{ N, } P_2 = 9.88 \text{ N} \\
\text{(b) } P_1 &= 19.6 \text{ N, } P_2 = 9.88 \text{ N}
\end{align*}
\]

Fig. 14 Waveforms of (3,0)-mode of Type AS at \( \theta = 30^\circ \)
numerical integral method, an analysis was made of the self-excited vibration of a circular plate with no internal resonances, subjected to frictional forces at two points on its outer circumference. The experimental and the theoretical results agreed well qualitatively with each other.

Reference