The Prevention of Wave Motions in a Liquid Layer under a Vertical Periodic Motion Due to a Superposed Thin Viscous Liquid Layer

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Instabilities of a liquid layer, over which a thin viscous liquid layer is superposed, are investigated theoretically and experimentally under vertical periodic motion. It is assumed that the depth of the lower layer is finite and that of the upper one is very small. Thus, the motions of the lower and upper liquids are treated as an irrotational flow and a viscous flow, respectively. The neutral stability curves, which separate the stable region from the unstable one in the plane of the frequency and the amplitude of an externally imposed acceleration, are drawn by using a linear theory. The neutral curves depend on the Reynolds number, the depth ratio, the density ratio, the Weber number and the ratio of the surface tension coefficient. The motion of the ideal fluid layer is found to be stabilized by superposing a thin viscous liquid layer. The theoretical results are fully confirmed by the experimental ones obtained in the water-oil liquid layer.

Key words: Fluid vibration, Stability, Wave motion, Viscous liquid layer, Critical amplitude

1. Introduction

It has been well known that if an oil is sprinkled on the surface of a stormy sea, the wave motion generated by a wind is calmed down. The cause of such a phenomenon has not been made clear yet. Since it involves the viscous damping, the moving interface and the stratified fluid motion, it is very difficult to treat hydrodynamically the problem of this kind. The theoretical studies on a wave motion of the interface are usually done under the assumption that the fluid is ideal. In such cases, the equation governing the wave motion is derived from the condition that the normal forces are balanced at the interface and there holds a relation between the wavelength, the frequency and the phase velocity. Then the dispersion relation of the wave has no connection with viscosity except the damping coefficient of the wave (1). In a thin liquid layer, the wave motion on the free surface depends on the viscosity (2). However, this is a rare case.

In this paper, whether or not a wave motion on the water surface is damped down by superposing a viscous liquid layer over it is examined. The wave motions on the free surface can be induced by a wind (3), a stream of the liquid itself (4) and an externally applied oscillation (5) and so on. The problem considered here is one of the wave motion of a horizontal liquid layer in infinite extent with a finite depth under a vertically applied periodic motion. The condition that the wave motion can be depressed by superposing a thin viscous liquid layer over the free surface is sought theoretically. It is clear that such a problem depends largely on the viscosity (6). In this case, the wave motions are caused by the lower liquid layer with a comparatively large thickness rather than the upper viscous layer with a small thickness (7). Thus, the lower liquid is regarded as an ideal fluid and the upper one as a viscous fluid. On the other hand, the stability of the two liquid layers consisting of water and lubrication oil under a vertical oscillation is studied experimentally and the theoretical study is compared with the experimental one.

2. Governing Equations

An ideal liquid layer of the density $\rho_2$ with a mean finite depth $h_2$, over which a viscous liquid layer of the density $\rho_1$ with a small mean depth $h_1$ is superposed, is supposed to be vibrated vertically on the horizontal flat plate. The motion of the lower ideal liquid is described by using the velocity potential $\phi$ and the upper liquid has the kinematic viscosity $\nu$. $x, y, z$ coordinates are taken in Fig.1, in which $XY$-plane is at the mean position of the interface between the upper and lower layers and $z$-axis is upward vertically from the interface. The positions of the free surface with the surface tension coefficient $\sigma_1$ of the upper layer and of the interface with the interface tension coefficient $\sigma_2$ between the two layers are denoted by $\theta_1$ and $\theta_2$, respectively. If two liquid layers are vibrated together with the horizontal plate under the action
of gravity, the pressures $P_i$ and $P_j$ in the two liquid layers will be governed by the following equations:

$$-\frac{1}{\rho_i} \frac{\partial P_i}{\partial z} (g + f \cos \omega t) = 0 \quad (1)$$

$$-\frac{1}{\rho_j} \frac{\partial P_j}{\partial z} (g + f \cos \omega t) = 0 \quad (2)$$

$$z = 0: \quad P_i = P_j \quad (3)$$

$$z = h_i: \quad P_i = P_o \quad (4)$$

where, $g$ is the gravitational acceleration, and $P_o$ is the constant atmospheric pressure on the free surface of the upper liquid layer. In the disturbed state caused by an applied vertical oscillation of the horizontal plane, the pressures on the upper and lower layers are denoted by $P_i + p_i$ and $P_j + p_j$, and the $x$, $y$, and $z$ components of the velocity in the upper layer are denoted by $u$, $v$, and $w$. If the disturbed variables $u$, $v$, $w$, $\phi$, $p_i$, $p_j$, and $\eta_i$ are small and the quantities of the second order of them are neglected, the motions of the two liquid layers will be governed by the following equations:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_i} \frac{\partial p_i}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (5)$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_i} \frac{\partial p_i}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \quad (6)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_i} \frac{\partial p_i}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \quad (7)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (8)$$

$$z = 0: \quad -\rho_i (g + f \cos \omega t) \eta_i$$

$$-\rho_j \frac{\partial \phi}{\partial z} + 2 \rho_i \nu \frac{\partial^2 \phi}{\partial x^2}$$

$$+ \alpha \rho_j (\frac{\partial \eta_j}{\partial x^2} + \frac{\partial \eta_j}{\partial y^2}) = 0 \quad (9)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (10)$$

$$\frac{\partial \phi}{\partial z} = w = \frac{\partial \phi}{\partial z} \quad (11)$$

$$\eta = h_i$$

$$-\rho_i (g + f \cos \omega t) \eta_i$$

$$-2 \rho_i \nu \frac{\partial \phi}{\partial z} \left( \frac{\partial^2 \phi}{\partial x^2} - k^2 \right) \eta_i$$

$$-2 \rho_i \nu \frac{\partial \phi}{\partial z} \left( \frac{\partial^2 \phi}{\partial y^2} - k^2 \right) \eta_i$$

$$-2 \rho_i \nu \frac{\partial \phi}{\partial z} \left( \frac{\partial^2 \phi}{\partial z^2} - k^2 \right) \eta_i = 0 \quad (12)$$

Since the two liquid layers are assumed to be in infinite extent, the unknown variables are decomposed in the form of the Fourier components with respect to $X$ and $Y$ as follows:

$$u, v, w, \phi, p_i, p_j, \eta_1, \eta_2 = \hat{u}, \hat{v}, \hat{w}, \hat{\phi}, \hat{p}_i, \hat{p}_j, \hat{\eta}_1, \hat{\eta}_2$$

$$\phi = \hat{\phi}_1 \cos k(x + h_1) \quad (13)$$

The pressure is expressed by using Eqs. (5)-(7) as

$$\hat{p}_i = -\rho_i \frac{\partial^2 \phi}{\partial x^2} + \rho_i \nu \frac{\partial^2 \phi}{\partial x^2} (x^2 + k^2) \hat{w} \quad (14)$$

Then, Eqs. (5)-(16) are rewritten as the following equations governing the unknown variables $\hat{u}, \hat{v}, \hat{w}, \hat{\phi}, \hat{\eta}_1, \hat{\eta}_2$:

$$\hat{u} = \hat{u}_1 \cos k(x + h_1) \quad (15)$$

$$\hat{v} = \hat{v}_1 \cos k(x + h_1) \quad (16)$$

$$\hat{w} = \hat{w}_1 \cos k(x + h_1) \quad (17)$$

$$\hat{\phi} = \hat{\phi}_1 \cos k(x + h_1) \quad (18)$$

$$\hat{\eta}_1, \hat{\eta}_2 = \hat{\eta}_{11}, \hat{\eta}_{12} \quad (19)$$

$$\hat{u}_1, \hat{v}_1, \hat{w}_1 = \hat{u}_{11}, \hat{v}_{11}, \hat{w}_{11} \quad (20)$$

$$\hat{\phi}_1, \hat{\eta}_{11}, \hat{\eta}_{12} = \hat{\phi}_{11}, \hat{\eta}_{111}, \hat{\eta}_{121} \quad (21)$$

$$\hat{u}_{11}, \hat{v}_{11}, \hat{w}_{11} = \hat{u}_{111}, \hat{v}_{111}, \hat{w}_{111} \quad (22)$$

$$\hat{\phi}_{11}, \hat{\eta}_{111}, \hat{\eta}_{121} = \hat{\phi}_{111}, \hat{\eta}_{1111}, \hat{\eta}_{1211} \quad (23)$$

$$\hat{\eta}_{111}, \hat{\eta}_{1211} = \hat{\eta}_{1111}, \hat{\eta}_{12111} \quad (24)$$

Fig. 1 Schematic diagram and the coordinates
\[
\left( \frac{\partial^2}{\partial z^2} + k^2 \right) \bar{w} = 0 \quad \text{-------------------}(25)
\]

\[
\frac{d\bar{T}}{dt} = \bar{w} \quad \text{-------------------}(26)
\]

It should be noted that these equations contain variables with respect to only the upper liquid layer. In a differential system with the periodic time coefficient obtained above, the most dangerous mode can be supposed to be a disturbance having the frequency \(\sigma_f/2\). This fact has been known from the studies on a single oscillating liquid layer \(\sigma_f/2\). In this paper, the critical condition above which a wave motion arises is mainly discussed. The wave motion in this critical state is steady, that is, the amplitude of the wave does not grow nor does it dampen \(\sigma_f/2\). Thus, \(\bar{w}^*, \bar{\eta}^*\) and \(\bar{\eta}^T\) are assumed to be constant with respect to time as follows:

\[
\bar{w} = w^*(z \cos \frac{\sigma_f}{2} t + w^*(z) \sin \frac{\sigma_f}{2} t) \quad \text{-------------------}(27)
\]

\[
\bar{\eta} = \eta^* \cos \frac{\sigma_f}{2} t \quad \text{-------------------}(28)
\]

\[
\bar{\eta} = \eta^* \sin \frac{\sigma_f}{2} t \quad \text{-------------------}(29)
\]

Introducing the following dimensionless variables with the superscript *:

\[w^* = \sqrt{gh_1} \bar{w}, z = h_1 z^*, t = \frac{h_1}{\sqrt{gh_1}} t^*, \bar{\eta}^* = \frac{\bar{\eta}}{h_1}, \bar{\eta}^T = \frac{\bar{\eta}}{h_1} \]

\[k = \frac{1}{h_1} \sigma_f^*, \omega = \frac{\sqrt{gh_1}}{h_1} \sigma_f^* \]

and substituting Eqs.(27)-(29) into Eqs.(20)-(26), the governing equations can be rewritten as

\[w^{**} - 3H^* k^{*} w^{**} + \frac{1}{2} H^* R^* w^{**} = 0 \quad \text{-------------------}(30)\]

\[w^{**} - 2H^* k^{*} w^{**} + \frac{1}{2} H^* R^* w^{**} = 0 \quad \text{-------------------}(31)\]

\[z^* = 0: \]

\[w^{**} - 3H^* k^{*} w^{**} + \frac{1}{2} H^* R^* w^{**} = 0 \quad \text{-------------------}(32)\]

\[\frac{H^* R^*}{L} \left(1 - \frac{1}{2} \left(1 - \frac{1}{2} \right) k^* \right) \bar{\eta}^* = 0 \quad \text{-------------------}(33)\]

\[w^{**} + \frac{1}{2} H^* R^* w^{**} = 0 \quad \text{-------------------}(34)\]

\[w^{**} + \frac{1}{2} H^* R^* w^{**} = 0 \quad \text{-------------------}(35)\]

\[w^{**} + \frac{1}{2} H R^* w^{**} + \left(1 - \frac{1}{2} \right) k^* \bar{\eta}^* = 0 \quad \text{-------------------}(36)\]

\[w^{**} + \frac{1}{2} H R^* w^{**} = 0 \quad \text{-------------------}(37)\]

\[
\begin{align*}
&z^* = 1: \\
&w^{**} - 3H^* k^{*} w^{**} + \frac{1}{2} H^* R^* w^{**} = 0 \quad \text{-------------------}(38) \\
&- H^* R^* \left(1 + \frac{F^*}{2} + \frac{SW}{L} k^* \right) \bar{\eta}^* = 0 \quad \text{-------------------}(39) \\
&w^{**} + \frac{1}{2} H R^* w^{**} = 0 \quad \text{-------------------}(40) \\
&w^{**} + H^* k^{*} w^{**} = 0 \quad \text{-------------------}(41) \\
&w^{**} - \frac{1}{2} H R^* \bar{\eta}^* = 0 \quad \text{-------------------}(42) \\
&w^{**} + \frac{1}{2} H R^* \bar{\eta}^* = 0 \quad \text{-------------------}(43)
\end{align*}
\]

where, the prime denotes the differentiation with respect to \(z\). \(R^*\) is the Reynolds number \(\sqrt{gh_1 h_2/\nu}\) based on the depth of the lower liquid layer, \(L\) is the density ratio \(\rho_1/\rho_2\), \(H^*\) is the thickness ratio \(h_1/h_2\), \(S\) is the ratio of the surface tension coefficient \(\sigma_1/\sigma_2\), \(F\) is the ratio of the amplitude of the acceleration of the applied oscillation to the gravity acceleration \(g\) and \(W\) is the Weber number \(\sigma_1/\rho g h_2^2\).

Based on the lower liquid layer, Eqs.(30)-(43) are the final governing equations solved in the following sections.

3. Case of Both Layers Being Ideal

Both the lower liquid layer and the upper one are regarded as ideal. In this case, the critical condition above which a wave motion arises is firstly examined in the following. It will help understand the viscous effect on the stability of the system considered.

In case that the viscosity of the upper liquid layer can be neglected, only the terms proportional to the Reynolds number remain in the governing equations (30)-(43), and the boundary condition derived from the condition that the tangential forces balance each other at the interface is neglected. Thus, the governing equations in this case reduce to

\[w^{**} - \frac{1}{2} H^* k^{*} w^{**} = 0 \quad \text{-------------------}(44)\]

\[w^{**} - \frac{1}{2} H^* k^{*} w^{**} = 0 \quad \text{-------------------}(45)\]

\[z = 0: \]

\[\left( \frac{\omega^2}{2} \right)^{**} + \left(1 - \frac{1}{2} \right) k^* \bar{\eta}^* = 0 \quad \text{-------------------}(46)\]

\[\left( \frac{\omega^2}{2} \right)^{**} + \left(1 - \frac{1}{2} \right) k^* \bar{\eta}^* = 0 \quad \text{-------------------}(47)\]
\[ z = 1: \]
\[ \left( \frac{\omega^2}{2} \right) w'' - \left( 1 + \frac{F}{2} \right) \frac{SW^2}{L} k^2 w'''' = 0 \]
\[ \left( \frac{\omega^2}{2} \right) w'' - \left( 1 - \frac{F}{2} \right) \frac{SW^2}{L} k^2 w'' = 0 \]

(48)

(49)

\[ w'''' \] is found to be independent of \( w'' \). Then, if \( F \) is replaced by \(-F\) in the solution \( w'''' \), the solution \( w'' \) can be found. The following eigen-relations are obtained from two homogeneous differential systems:

\[ \left( \frac{\omega^2}{2} \right) - k^2 \left( 1 + \frac{SW^2}{L} k^2 \right) \tanh Hk^2 \]
\[ \times \left( 1 - L + \frac{F}{2} \right) \left( \frac{\omega^2}{2} \right) - k^2 \left( 1 + \frac{SW^2}{L} k^2 \right) \tanh Hk^2 \]
\[ = 0 \quad \left( 1 + \frac{SW^2}{L} k^2 \right) \left( \frac{\omega^2}{2} \right) \tanh Hk^2 \]

(50)

The relations (50) have two unstable regions in \( \omega^2 - F \) plane for a given wave number as shown in Fig.2. A 1/2 subharmonic resonance arises in the regions bounded by two curves which intersect on the \( \omega^2 \)-axis. In the problem considered here, the wave motion having a given wave number arises at two kinds of the frequency. This is different from the case of a single liquid layer. In the liquid layer in a vessel, there occurs only a wave motion with a particular wave number. But, in the liquid layer of infinite extent, there can occur wave motions with arbitrary wave numbers. The frequency \( \omega^2 \) at which the curves intersect on the \( \omega^2 \)-axis is found from Eq.(50) to be a continuous function of the wave number \( k^2 \) as shown in Fig.3. Each frequency \( \omega^2 \) has always such a corresponding unstable region as shown in Fig.2, and all points on the \( \omega^2 \)-axis in Fig.2 are equal to \( \omega^2 \) corresponding to a given wave number. Then all points in \( \omega^2 - F \) plane of Fig.2 are found to belong to the unstable region for a disturbance with a given wave number. Therefore, in the case of an ideal fluid, there always exists a disturbance under an externally applied oscillation. This fact is independent of the amplitude of the applied oscillation. It should be noted that \( \omega^2 - k^2 \) curve in Fig.3 corresponds to the upper unstable region in Fig.2, because the lower unstable region is unimportant practically as explained in the next section.

4. Critical Curve for a Given Wave Number

The homogeneous differential system, Eqs.(30)-(43), is numerically solved and the eigen-relation between parameters, \( \omega^2, F, k^2, H, R, L, W \) and \( S \) is found. As a result, the critical curves, which separate the unstable region from the stable one for a given wave number, are described in the \( \omega^2 - F \) plane as shown in Figs.4 and 5. Fig.4 is the case of a large Reynolds number and Fig.5 is the case of a small Reynolds number. When these critical curves are compared with those in the case of ideal fluid layers which are shown in Fig.2, it is found that the lower unstable region in Fig.2 diminishes rapidly as the Reynolds number decreases. Therefore, in practice, only the upper unstable region in Fig.2 plays an important role in the 1/2 subharmonic resonance. Then, the wave motion corresponding to the upper unstable region is examined in the following section.

5 Neutral Stability Curve

As pointed out in the section 3, there can exist disturbances with arbitrary wave numbers in the system considered here. Then, numbers of critical curves corresponding to various wave numbers can be described in \( \omega^2 - F \) plane. Thus, whether or not a disturbance arises is determined by a curve, called an envelope, in which a number of critical curves are enveloped in \( \omega^2 - F \) plane as shown in Fig.6. The envelope defined above separates the unstable region from the stable one in \( \omega^2 - F \) plane. Therefore, this envelope is called the neutral stability curve. On this curve, no disturbance grows and decays either. The 1/2 subharmonic waves with various wave
numbers are likely to arise on the right side of the neutral stability curve in Fig.6. The effect of the ratio $H$ of the layer thickness on the neutral stability curves is shown in Fig.7. The right side regions of the curves in Fig.7 correspond to the unstable regions. In the case of the ideal fluid layers, all regions of $a^+ - F$ plane belong to the unstable region. This fact is independent of the number of liquid layers. The stable region bounded by $a^-$-axis and the neutral stability curve is produced by superposing a thin viscous layer over the ideal liquid layer as shown in Fig.7. Then, the system is found to be stabilized by the viscosity of the upper liquid. Fig.7 shows that the system is fairly stabilized by increasing the thickness of the viscous layer. However, when the frequency of the applied oscillation is low, the stabilizing action due to the viscosity of the upper layer becomes weak.

The effect of the Reynolds number on the neutral stability curve is shown in Fig.8. It is found from this figure that the system is stabilized as the viscosity increases. If the density ratio $L$ of the upper layer to the lower one approaches unity, the system is stabilized as shown in Fig.9. The instability of the system considered largely depends on the existence of the interface between the lower and upper layers. If there is no interface in the system, it becomes perfectly stable. Then, it is inferred that if the density ratio tends to unity, the system is stabilized.

Finally, the effects of the surface tension and the interface tension on the neutral stability curve are shown in Figs.10 and 11. It is found that the system is excited as the surface and interface tensions increase.

6. Comparison Between Theory and Experiment

In order to confirm the theoretical results, the following experiments are undertaken by using two liquid layers: A thin liquid layer of the turbine oil is superposed over the water layer, and these layers are set in a square vessel made of acrylic resin with side length 20 cm and height 4 cm. The vessel is vibrated vertically by using the Matsudaire-type oscillator. The material properties of the turbine oil are the kinematic viscosity $\nu=1.69 \times 10^{-4}$ m$^2$/s and the density $\rho=0.88$ g/cm$^3$. The interface-tension coefficient of turbine oil to water, $\sigma$, is 15.0 mN/cm and the surface-tension coefficient of turbine oil to water, $\gamma$, is 0.0345 N/m.

![Fig.4](image1) The critical curve in the case of a comparatively large Reynolds number

![Fig.5](image2) The critical curve in the case of a comparatively small Reynolds number

![Fig.6](image3) The critical curve for various wave numbers and the envelope of them

![Fig.7](image4) The neutral stability curves (the effect of thickness ratio)
oil, $\sigma_1$, is 30.7 mN/cm. The critical conditions above which a disturbance arises in the vessel are experimentally searched for.

On the point slightly above the critical condition, the patterns of the wave excited are always two-dimensional as shown in Figs.12 and 13. Comparing of Figs.12 and 11, the wavelength is found to depend on the frequency of the oscillator. As the frequency decreases, the wavelength is made longer. When a stratified liquid layer composed of the water layer with depth 2 cm and the turbine oil layer with depth 3 mm is externally vibrated by the oscillator with various frequencies, disturbances with the wave numbers corresponding to the frequencies of the oscillator arise on the free surface of the turbine oil as shown in Fig.14. It is found from Fig.14 that the wave number of the disturbance is a continuous function of the frequency. The frequency $\omega_0$ of the applied oscillation corresponding to a minimum acceleration amplitude on the critical curve is expressed in the form of a solid line in Fig.14 as a function of the wave number by using the theoretical results. The relations between the frequency and the wave number based on the theory and the experiment fairly coincide with each other. However, making in detail a comparison of the theoretical and

Fig.8 The neutral stability curves (the effect of the Reynolds number)

Fig.9 The neutral stability curves (the effect of the density ratio)

Fig.10 The neutral stability curves (the effect of the interface tension)

Fig.11 The neutral stability curves (the effect of the Weber number)

Fig.12 The form of the wave motion in the exciting state on the free surface of water and turbine oil layers ($h_w=2cm$, $h_o=3mm$, $\omega=10.7Hz$, the amplitude of the oscillator=0.75mm)

Fig.13 The form of the wave motion in the exciting state on the free surface of water and turbine oil layers ($h_w=2cm$, $h_o=3mm$, $\omega=11.7Hz$, the amplitude of the oscillator=0.59mm)
experimental values, the former values are slightly smaller in the wavelength than the latter ones. The neutral stability curves are shown in Fig.15 comparing the theoretical and experimental values. Both of them are found from Fig.15 to agree excellently with each other. Thus, the theory mentioned in the previous sections is found to describe accurately the resonance phenomenon of the liquid layer in the practical engineering problem. The wave motions on the free surface of a water layer can be expected to be stabilized considerably by superposing over it a viscous liquid layer with a small thickness corresponding to the Reynolds number of the order of 10.

7. Conclusions

It is concluded from the stability theory that the wave motions on the free surface of a liquid layer with a comparatively small viscosity under a vertically applied vibration can be effectively depressed by superposing over it a thin liquid layer with a comparatively large viscosity. In the theory, the upper and lower layers are assumed to be viscous and ideal, respectively. The system considered is found to become more stable as the thickness ratio of the upper layer to the lower one increases, the Reynolds number becomes small and the density ratio of the upper layer to the lower one tends to unity. These results based on the theory are confirmed to fairly agree with those obtained experimentally by using a stratified liquid layer composed of water and turbine oil.

References

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