An Implicit Finite-difference Method to Solve

Incompressible Fluid Flow*

By Nobumasa Takemitsu**

An implicit finite-difference method to solve an incompressible fluid flow is proposed. In the method primitive variables (velocity and pressure of the fluid) are used, and an idea that the Navier-Stokes equations have a certain invariant in each iteration is introduced. As a numerical example, a two-dimensional transient Womersley flow is calculated. It is shown that the method has an accuracy of $O(\Delta t)$ (Δt : discretized time increment), and the accuracy of the present method is found to be satisfactory by comparison with the analytic and the existing methods.

Key Words: Incompressible Flow, Numerical Analysis, Implicit Finite-difference Method, Unsteady Flow

1. Introduction

Recently, numerical analysis in fluid dynamics is very popular, and many books about the numerical methods has been published (see for example, Refs. (1) and (2)). If we confine ourselves to the finite-difference methods of two-dimensional incompressible fluid, after the introduction of numerical methods (Thom⁽³⁾ and Kawaguti⁽⁴⁾), stream function and vorticity (ψ - ω) method has been widely used by many authors.

Since the advent of electronic computer, numerical calculations have become popular very rapidly and now widely used to study fluid dynamics. The method, which uses primitive variables of the fluid (velocity and pressure of the fluid) (u , v - p method), has also been proposed by various authors with a remarkable success of the MAC⁽⁵⁾ (Marker-And-Cell) method. Another method, for example, which is formulated by integro-differential equation⁽⁶⁾, is also proposed, but we do not use the method in this paper.

As is well known, the ψ - ω method automatically satisfies the equation of continuity, and it needs a smaller number of variables and iterations to obtain solutions than the u , v - p method. However, it is not always easy to apply the method to the flow with free surface. On

the other hand, the u , v - p method is easy to apply to the flow with free surface and three-dimensional flow, but the equation of continuity is satisfied only after many iterations.⁽⁷⁾⁻⁽¹⁰⁾

Therefore a method, which gives accurate solution, and needs fewer iterations compared with the existing methods, and is simple with the primitive-variable formulation, would be useful one in actual calculations. Fortunately the present author has proposed such a method.⁽¹¹⁾ In this paper, the method is extended to an implicit finite-difference equation. As is also well known, the implicit method is accurate and makes it possible to take large discretized time increment unlike the explicit method. In the following sections, we first discuss the one-dimensional finite-difference solution which is obtainable analytically, and then present an implicit finite-difference equation. Finally, we discuss the solutions obtained by explicit and implicit finite-difference equations.

2. One-dimensional Finite-difference Solutions

2.1 Transient Poiseuille Flow

This flow is an example in which a flow starts impulsively from rest to finite velocity. The one-dimensional Navier-Stokes (N-S) equation of a viscous incompressible fluid is expressed in non-dimensional form as

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{1}{Re} \frac{\partial^2 u}{\partial y^2} \dots\dots\dots(1)$$

using a Cartesian velocity component u , time t , and pressure p . In the

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** Part-time Lecturer, Faculty of Engineering, Tokyo Denki University, (2-2 Nishiki-cho, Kanda, Chiyoda-ku, Tokyo, 101 Japan).

above equation $Re=UL/\nu$ is the Reynolds number, U the characteristic speed, L the characteristic length of the flow field, and ν the kinematic viscosity of the fluid. The vorticity transport equation is

$$\frac{\partial \omega}{\partial t} = \frac{1}{Re} \frac{\partial^2 \omega}{\partial y^2} \quad (2)$$

$$\omega = -(\partial^2 \psi / \partial y^2), \quad u = \partial \psi / \partial y \quad (3)$$

where ψ is a scalar stream function and ω the vorticity.

The initial and boundary conditions of this flow are

$$t < 0; u = \psi = \omega = 0 \quad (\text{everywhere}), \quad (4.a)$$

$$t > 0; \psi = \omega = 0 \quad \text{at } y = 0 \quad (\text{centerline}), \quad (4.b)$$

$$\psi = 1, \quad u = \partial \psi / \partial y = 0 \quad \text{at } y = 1 \quad (\text{upper wall}). \quad (4.c)$$

The analytic solution for this flow, obtained by the Laplace transform, is

$$u = \frac{3}{2}(-y^2 + 1) + \sum_{n=1}^{\infty} \frac{2}{\beta_n^2} \left(\frac{\beta_n \cos \beta_n y}{\sin \beta_n} - 1 \right) \times \exp(-\beta_n^2 t / Re) \quad (5.a)$$

$$\psi = \frac{1}{2}(-y^2 + 3y) + \sum_{n=1}^{\infty} \frac{2}{\beta_n^2} \left(\frac{\sin \beta_n y}{\sin \beta_n} - y \right) \times \exp(-\beta_n^2 t / Re) \quad (5.b)$$

$$\omega = 3y + \sum_{n=1}^{\infty} \frac{2 \sin \beta_n y}{\sin \beta_n} \exp(-\beta_n^2 t / Re) \quad (5.c)$$

$$\frac{\partial \psi}{\partial x} = -\frac{1}{Re} \left\{ 3 + 2 \sum_{n=1}^{\infty} \exp(-\beta_n^2 t / Re) \right\} \quad (5.d)$$

where β_n is the n th positive solution of $\beta_n = \tan \beta_n$ ($0 < \beta_1 < \beta_2 < \dots$). Numerically, 40 terms are required for the convergence in the above series.

The flow of $Re=5$ is studied. The velocity distribution obtained by Eq.(5.a) is shown in Fig.1.

The finite-difference equations for Eqs.(2) and (3) are given by

$$\omega_{i,j}^{t+\Delta t} = \omega_{i,j}^t + \frac{\Delta t}{s^2 Re} (\omega_{i,j+1} + \omega_{i,j-1} - 2\omega_{i,j})^t \quad (6.a)$$

$$2\psi_{i,j}^{t+\Delta t} = (\psi_{i,j+1} + \psi_{i,j-1} + s^2 \omega_{i,j})^{t+\Delta t} \quad (6.b)$$

$$u_{i,j}^{t+\Delta t} = (\psi_{i,j+1} - \psi_{i,j})^{t+\Delta t} / s \quad (6.c)$$

$$\left(\frac{\partial \psi}{\partial x} \right)_{i,j}^{t+\Delta t} = -\frac{u_{i,j}^{t+\Delta t} - u_{i,j}^t}{\Delta t} - \frac{\omega_{i,j+1} - \omega_{i,j}}{s Re} \quad (6.d)$$

and

Fig.1 Analytic solution of transient Poiseuille flow ($Re=5$)

$$2\left(1 + \frac{\Delta t}{s^2 Re}\right) \omega_{i,j}^{t+\Delta t} = \frac{\Delta t}{s^2 Re} (\omega_{i,j+1} + \omega_{i,j-1})^{t+\Delta t} + 2\omega_{i,j}^t + \frac{\Delta t}{s^2 Re} (\omega_{i,j+1} + \omega_{i,j-1} - 2\omega_{i,j})^t \quad (7.a)$$

$$2\psi_{i,j}^{t+\Delta t} = (\psi_{i,j+1} + \psi_{i,j-1} + s^2 \omega_{i,j})^{t+\Delta t} \quad (7.b)$$

$$u_{i,j}^{t+\Delta t} = (\psi_{i,j+1} - \psi_{i,j})^{t+\Delta t} / s \quad (7.c)$$

$$\left(\frac{\partial \psi}{\partial x} \right)_{i,j}^{t+\Delta t} = -2 \left(\frac{\partial u}{\partial t} \right)_{i,j}^{t+\Delta t/2} - \frac{2}{Re} \left(\frac{\partial \omega}{\partial y} \right)_{i,j}^{t+\Delta t/2} - \left(\frac{\partial \psi}{\partial x} \right)_{i,j}^t \quad (7.d)$$

The indices i and j count cell-center, cell-boundary, or cell-corner positions in the x and y directions, and s is the mesh size in y direction. The superscripts t and $t+\Delta t$ (Δt =time increment) show time steps. The velocity components u and v are given at cell boundaries; pressure p and dilatation D , which is required for later calculation, are defined at cell centers; the stream function ψ and the vorticity ω are defined at cell corners (see Fig.2). In this flow, the index i is not necessary and the position of pressure p coincides with that of velocity component u , and velocity component v is identically zero.

Equation(6) is obtained using forward-time and centered-space difference (Explicit) method, and Eq.(7) by Crank-Nicolson (Implicit) method. The accuracy in time of Eq.(6) is $O(\Delta t)$, and of Eq.(7) is $O(\Delta t^2)$. The accuracy in space of Eqs.(6) and (7) are both $O(s^2)$. In the implicit method the pressure is calculated by

$$\left(\frac{\partial \psi}{\partial x} \right)_{i,j}^{t+\Delta t} = -\left(\frac{\partial u}{\partial t} \right)_{i,j}^{t+\Delta t/2} - \frac{1}{Re} \left(\frac{\partial \omega}{\partial y} \right)_{i,j}^{t+\Delta t/2} \quad (7.e)$$

in addition to Eq.(7.d).

Solutions of Eqs.(6.b), (7.a) and (7.b) can be easily obtained because they are all tridiagonal matrices. Velocity and pressure at new-time level are obtained by Eqs.(6.c), (7.c), and (6.d), (7.d) and (7.e), respectively.

Fig.2 Placement of field variables

In the following calculations, mesh size s is chosen as 0.1, and time increment Δt is 1/100. Figure 3 shows temporal velocity at $y=0.05$. In the figure "Exact" means the solution by Eq.(5.a), "Explicit" means the solution by the forward-time and centered-space method (Eq.6), and "Implicit" means the solution by the Crank-Nicolson method (Eq.(7)). From the figure, we can see that the explicit method is more accurate than the implicit method except for very small time. This does not coincide with our common knowledge that the second-order finite-difference solution is more accurate than the first-order finite-difference solution. This is caused by the following reason.

Fig.3 Temporal variation of velocity at $y=0.05$; one-dimensional finite-difference solution of transient Poiseuille flow

Fig.4a Temporal variation of pressure gradient, one-dimensional finite-difference solution of transient Poiseuille flow

The value of $\partial^2\omega/\partial y^2$ is used not only at an old time but also at a new time in the Crank-Nicolson method (see Eq.(7.a)). Hence, if $\partial^2\omega/\partial y^2$ is not approximated properly for small time, this error remains in later calculations. Since the value of vorticity varies abruptly near the wall in the case of a transient Poiseuille flow, the solution obtained by the implicit method is less accurate than the explicit method. In the case of a transient Womersley flow, where an abrupt variation of the value of vorticity does not occur, the solution by the implicit method is better than by the explicit method, because $\partial^2\omega/\partial y^2$ is approximated accurately by the coarse mesh size (see 2.2).

Figure 4a shows temporal pressure gradients by various methods. In the figure, pressure gradient by the implicit method is calculated by Eq.(7.e). Pressure gradient by the explicit method becomes accurate as time proceeds, but the pressure gradient by the implicit method is more accurate than one by the explicit method. However both methods give less accurate pressure gradient for earlier time. Figure 4b shows temporal pressure gradient obtained by Eq.(7.d). This large oscillation is due to the fact that the influence of $(\partial p/\partial x)'$ remains when Eq.(7.d) is calculated.

From above discussions, it is found that the pressure gradient in the implicit method should be calculated by Eq.(7.e). However, it should be noted that the accuracy in time of Eq.(7.e) is $O(\Delta t)$, although the accuracy in time of Eqs.(7.a)-(7.d) is $O(\Delta t^2)$. Equation (7.e) is written as

Fig.4b Temporal variation of pressure gradient, one-dimensional full implicit finite-difference solution of transient Poiseuille flow (Eq.(7.d))

$$\begin{aligned}
\left(\frac{\partial u}{\partial t}\right)^{t+\Delta t/2} &= \left(-\frac{\partial p}{\partial x} + \frac{1}{Re} \frac{\partial^2 u}{\partial y^2}\right)^{t+\Delta t/2} \\
&- \left(\frac{\partial p}{\partial x}\right)^{t+\Delta t} + \left(\frac{\partial p}{\partial x}\right)^{t+\Delta t/2} \\
&= \left(-\frac{\partial p}{\partial x} + \frac{1}{Re} \frac{\partial^2 u}{\partial y^2}\right)^{t+\Delta t/2} \\
&- \frac{\partial}{\partial x} (p^{t+\Delta t} - p^{t+\Delta t/2}) \\
&= \left(-\frac{\partial p}{\partial x} + \frac{1}{Re} \frac{\partial^2 u}{\partial y^2}\right)^{t+\Delta t/2} \\
&- \frac{\partial}{\partial x} \left(\frac{\partial p}{\partial t}\right)^{t+\Delta t/4} \cdot \frac{\Delta t}{2}.
\end{aligned}
\tag{8}$$

Hence the accuracy of the implicit method is $O(\Delta t)$. It is commonly said that the finite-difference equation has first-order or second-order accuracy except for the pressure term. However, the author believes that not only velocity term but also pressure term must be considered when we discuss the accuracy of the finite-difference equation for the N-S equations. If we want to obtain a second-order finite difference equation, a different procedure, which may be complicated, must be considered.

2.2 Transient Womersley flow

In this flow, the fluid accelerates gradually from rest to finite velocity. The N-S equation is written as

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{2\pi}{\alpha^2} \frac{\partial^2 u}{\partial y^2} \tag{9}$$

and the vorticity transport equation is

$$\frac{\partial \omega}{\partial t} = \frac{2\pi}{\alpha^2} \frac{\partial^2 \omega}{\partial y^2} \tag{10}$$

$$\omega = -(\partial^2 \psi / \partial y^2), \quad u = \partial \psi / \partial y \tag{11}$$

where α is Womersley number, which is defined by $\alpha = L\sqrt{\Omega/\nu}$. Here, Ω is the frequency, and ν the kinematic viscosity of the fluid.

The initial and the boundary condition for this flow are

$$t < 0; \quad u = \psi = \omega = 0 \quad (\text{everywhere}) \tag{12.a}$$

$$t > 0; \quad \psi = \omega = 0 \quad \text{at } y = 0 \quad (\text{centerline}) \tag{12.b}$$

$$\psi = \sin 2\pi t, \quad u = 0 \quad \text{at } y = 1 \quad (\text{upper wall}). \tag{12.c}$$

The analytic solution for Eqs.(9)-(12) is also found by the Laplace transform

$$\begin{aligned}
u &= \text{Im} \beta_0 \frac{\cos \beta_0 y - \cos \beta_0}{\sin \beta_0 - \beta_0 \cos \beta_0} \exp(2\pi i t) \\
&- \sum_{n=1}^{\infty} \frac{2\alpha^2}{\alpha^4 + \beta_n^4} \left(\beta_n \frac{\cos \beta_n y}{\sin \beta_n} - 1 \right) \exp(-2\pi \beta_n^2 t / \alpha^2)
\end{aligned}
\tag{13.a}$$

$$\begin{aligned}
\phi &= \text{Im} \frac{\sin \beta_0 y - \beta_0 y \cos \beta_0}{\sin \beta_0 - \beta_0 \cos \beta_0} \exp(2\pi i t) \\
&- \sum_{n=1}^{\infty} \frac{2\alpha^2}{\alpha^4 + \beta_n^4} \left(\frac{\sin \beta_n y}{\sin \beta_n} - y \right) \exp(-2\pi \beta_n^2 t / \alpha^2)
\end{aligned}
\tag{13.b}$$

$$\begin{aligned}
\omega &= \text{Im} \beta_0^2 \frac{\sin \beta_0 y}{\sin \beta_0 - \beta_0 \cos \beta_0} \exp(2\pi i t) \\
&- \sum_{n=1}^{\infty} \frac{2\alpha^2 \beta_n^2}{\alpha^4 + \beta_n^4} \frac{\sin \beta_n y}{\sin \beta_n} \exp(-2\pi \beta_n^2 t / \alpha^2)
\end{aligned}
\tag{13.c}$$

$$\begin{aligned}
\frac{\partial p}{\partial x} &= \text{Im} \frac{2\pi i \beta_0 \cos \beta_0}{\sin \beta_0 - \beta_0 \cos \beta_0} \exp(2\pi i t) \\
&+ \sum_{n=1}^{\infty} \frac{4\pi \beta_n^2}{\alpha^4 + \beta_n^4} \exp(-2\pi \beta_n^2 t / \alpha^2)
\end{aligned}
\tag{13.d}$$

where $\beta_0 = \sqrt{-i}\alpha$ and β_n are also positive solutions of $\beta_n = \tan \beta_n$ as in 2.1.

The finite-difference solutions for Eqs.(9)-(11) are calculated by the explicit, implicit, and two-step Lax-Wendroff (Two-Step) methods. Explicit and implicit finite-difference equations are obtained by substituting $2\pi/\alpha^2$ instead of $1/Re$ into Eqs.(6) and (7). Two-step method is calculated by

i) First-step

$$\omega_{i,j}^{t+\Delta t/2} = \omega_{i,j}^t + \frac{\pi \Delta t}{\alpha^2} (\omega_{i,j+1} + \omega_{i,j-1} - 2\omega_{i,j})^t \tag{14.a}$$

ii) Second-step

$$\omega_{i,j}^{t+\Delta t} = \omega_{i,j}^t + \frac{2\pi \Delta t}{\alpha^2} (\omega_{i,j+1} + \omega_{i,j-1} - 2\omega_{i,j})^{t+\Delta t/2} \tag{14.b}$$

$$2\phi_{i,j}^{t+\Delta t} = (\phi_{i,j+1} + \phi_{i,j-1} + s^2 \omega_{i,j})^{t+\Delta t} \tag{14.c}$$

$$u_{i,j}^{t+\Delta t} = (\phi_{i,j+1} - \phi_{i,j})^{t+\Delta t} / s \tag{14.d}$$

$$\left(\frac{\partial p}{\partial x}\right)^{t+\Delta t} = -\left(\frac{\partial u}{\partial t}\right)^{t+\Delta t/2} - \frac{2\pi}{\alpha^2} \left(\frac{\partial \omega}{\partial y}\right)^{t+\Delta t/2} \tag{14.e}$$

In the example calculated below α is taken to be 4, and $s = 0.1$. Time increment Δt is taken as 1/200 in the explicit and the two-step methods, and $\Delta t = 1/200$ and 1/50 in the implicit method.

Figure 5 shows the temporal velocity variation at $y = 0.05$, and the temporal variation of pressure

Fig. 5 Temporal variation of velocity at $y = 0.05$, one-dimensional finite-difference solution of transient Womersley flow

Fig. 6 Temporal variation of pressure gradient, one-dimensional finite-difference solution of transient Womersley flow

gradient is shown in Fig. 6. From these figures, we can see that the implicit method ($\Delta t=1/200$) is the most accurate. The curve of velocity is not shown in Fig. 5, because it almost agrees with the exact solution. The accuracy of the implicit method would be satisfactory even if Δt were taken as $1/50$ in the actual calculation. It should be noted that the results of the two-step method are closer to those of the explicit method than to those of the implicit method. Thus, it can be said that the two-step (Lax-Wendroff) method is not so accurate as expected in the calculation of the incompressible fluid flow.

3. Formulation of an Implicit Finite-Difference Equation

Beam and Warming⁽¹³⁾ have proposed an effective method to solve viscous compressible N-S equations. We try to derive a method to solve viscous incompressible N-S equations in this paper.

The N-S equation of a viscous incompressible fluid is expressed in non-dimensional form as

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla P + A \nabla^2 v \quad (15)$$

using a velocity vector v (Cartesian components u, v, w), and $A=1/\text{Re}$ or $2\pi/\alpha^2$. The equation of continuity is

$$D = \nabla \cdot v = 0 \quad (16)$$

where D is dilatation. The vorticity transport equation is

$$\frac{\partial \omega}{\partial t} + \frac{\partial u \omega}{\partial x} + \frac{\partial v \omega}{\partial y} = A \nabla^2 \omega \quad (17)$$

in two-dimensional flow. The velocity components u , and v are obtained by

$$\omega = -\nabla^2 \psi \quad (18)$$

$$u = \partial \psi / \partial y, \quad v = -\partial \psi / \partial x. \quad (19)$$

In order to obtain numerical solutions of Eqs. (15) and (16), or of Eqs. (17)–(19) initial and boundary conditions are required. In both cases, the solution process is iterative.

As already mentioned, the ψ - ω method requires fewer iterations than the u, v - p method in two-dimensional flows. In the ψ - ω method, the equation of continuity is satisfied automatically (implicitly) by introducing the stream function, and the stream function is scalar in two-dimensional case. From these facts, the author believes that fewer iterations are necessary if some conditions are imposed implicitly in each iteration when Eqs. (15) and (16) are solved simultaneously. This is the basic idea of the author's method.

When this idea is applied to the explicit method, we can impose the vorticity as an implicit invariant at the reference time step.⁽¹¹⁾ In the case of the implicit method, simultaneous equations about the velocity and pressure must be solved at the reference time step. Vorticity does change its value in each iteration in this case.

However, if the following manipulation is made, we can introduce a potential function as in the explicit method. We denote the vorticity at the number of iteration k by $\omega^{(k)}$, and then we can introduce a fictitious vorticity ω such as

$$\omega = \omega^{(k)} \quad (20)$$

This is rewritten as

$$\nabla \times v = \nabla \times v^{(k)} \quad (21)$$

from which we have

$$v = v^{(k)} - \nabla \phi^{(k+1)} \quad (22)$$

where ϕ is a potential function. If we take the divergence of Eq. (22), we obtain

$$D = D^{(k)} - \nabla^2 \phi^{(k+1)} \quad (23)$$

Since the dilatation D is zero in each iteration, we obtain

$$D^{(k)} = \nabla^2 \phi^{(k+1)} \quad (24)$$

Thus, an invariant is introduced in each iteration if Eq. (24) is to be solved. This means that Eq. (20) is added implicitly in the present formulation in the same manner that Eq. (16) is added implicitly in the ψ - ω method.

The finite-difference equation by the Crank-Nicolson method for Eq. (15) is

$$\begin{aligned} & \left[2 + \Delta t \left\{ \mu \left(\frac{1}{h} + \frac{1}{s} \right) + 2A \left(\frac{1}{h^2} + \frac{1}{s^2} \right) \right\} \right] u_{i,j}^* \\ &= \Delta t \left[- \left(\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} \right)_{i,j} + \mu \left(\frac{1}{h} + \frac{1}{s} \right) u_{i,j} \right. \\ & \quad - 2 \left(\frac{\partial p}{\partial x} \right)_{i,j} + A \left(\frac{u_{i+1,j} + u_{i-1,j}}{h^2} \right. \\ & \quad \left. \left. + \frac{u_{i,j+1} + u_{i,j-1}}{s^2} \right) \right] + 2u_{i,j}^t \end{aligned}$$

$$+ \Delta t \left[- \left(\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} \right) + A \nabla^2 u \right]_{i,j}^t \quad \dots (25 \cdot a)$$

$$u_{i,j}^{(k+1)} = u_{i,j}^{(k)} + RF_u (u_{i,j}^* - u_{i,j}^{(k)}) \quad \dots (25 \cdot b)$$

and

$$\begin{aligned} & \left[2 + \Delta t \left\{ \mu \left(\frac{1}{h} + \frac{1}{s} \right) + 2A \left(\frac{1}{h^2} + \frac{1}{s^2} \right) \right\} \right] v_{i,j}^* \\ &= \Delta t \left[- \left(\frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} \right)_{i,j} + \mu \left(\frac{1}{h} + \frac{1}{s} \right) v_{i,j} \right. \\ & \quad - 2 \left(\frac{\partial p}{\partial y} \right)_{i,j} + A \left(\frac{v_{i+1,j} + v_{i-1,j}}{h^2} \right. \\ & \quad \left. \left. + \frac{v_{i,j+1} + v_{i,j-1}}{s^2} \right) \right]^{t+\Delta t} + 2v_{i,j}^t \\ & + \Delta t \left[- \left(\frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} \right) + A \nabla^2 v \right]_{i,j}^t \quad \dots (26 \cdot a) \end{aligned}$$

$$v_{i,j}^{(k+1)} = v_{i,j}^{(k)} + RF_v (v_{i,j}^* - v_{i,j}^{(k)}) \quad \dots (26 \cdot b)$$

where h and s are mesh sizes in x and y directions, respectively, and field variables are positioned as in Fig. 2. RF_u and RF_v are relaxation factors for u and v , respectively, and μ is a kind of relaxation factor, which is introduced in steady flow⁽¹⁴⁾ to stabilize the finite-difference equations. Similar stabilized finite-difference equations are presented by three authors⁽¹⁵⁾⁻⁽¹⁷⁾ based on different ideas. The method has second-order accuracy for the approximation of the space derivatives when solutions of the finite-difference equation are converged. The approximations for $\partial u^2 / \partial x$ etc. are

$$\frac{\partial u^2}{\partial x} = \frac{u_{i+1/2,j}^2 - u_{i-1/2,j}^2}{h} \quad \dots (27 \cdot a)$$

$$\frac{\partial uv}{\partial y} = \frac{u_{i,j+1/2} v_{i,j+1/2} - u_{i,j-1/2} v_{i,j-1/2}}{s} \quad \dots (27 \cdot b)$$

$$\frac{\partial p}{\partial x} = \frac{p_{i,j} - p_{i-1,j}}{h} \quad \dots (27 \cdot c)$$

$$\begin{aligned} \nabla^2 u &= \frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2} \\ &+ \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{s^2} \quad \dots (27 \cdot d) \end{aligned}$$

$$\frac{\partial uv}{\partial x} = \frac{u_{i+1,j-1/2} v_{i+1,j-1/2} - u_{i-1,j-1/2} v_{i-1,j-1/2}}{h} \quad \dots (28 \cdot a)$$

$$\frac{\partial v^2}{\partial y} = \frac{v_{i,j+1/2}^2 - v_{i,j-1/2}^2}{s} \quad \dots (28 \cdot b)$$

$$\frac{\partial p}{\partial y} = \frac{p_{i,j} - p_{i,j-1}}{s} \quad \dots (28 \cdot c)$$

$$\begin{aligned} \nabla^2 v &= \frac{v_{i+1,j} + v_{i-1,j} - 2v_{i,j}}{h^2} \\ &+ \frac{v_{i,j+1} + v_{i,j-1} - 2v_{i,j}}{s^2} \quad \dots (28 \cdot d) \end{aligned}$$

The quantities at all fractional indices are obtained as simple averages of two adjacent quantities, e.g.,

$$u_{i+1/2,j} = (u_{i,j} + u_{i+1,j}) / 2 \quad \dots (29)$$

Equation of continuity (16) is approximated by

$$D_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h} + \frac{v_{i,j+1} - v_{i,j}}{s} \quad \dots (30)$$

and Eq. (24) is approximated by

$$\begin{aligned} 2 \left(\frac{1}{h^2} + \frac{1}{s^2} \right) \varphi_{i,j}^* &= \frac{\varphi_{i+1,j} + \varphi_{i-1,j}}{h^2} + \frac{\varphi_{i,j+1} + \varphi_{i,j-1}}{s^2} \\ &- D_{i,j}^{(k)} \quad \dots (31 \cdot a) \end{aligned}$$

$$\varphi_{i,j}^{(k+1)} = \varphi_{i,j}^{(k)} + RF_\varphi (\varphi_{i,j}^* - \varphi_{i,j}^{(k)}) \quad \dots (31 \cdot b)$$

where RF_φ is a relaxation factor for ϕ . If there exists a corner in the fluid flow, the potential ϕ is calculated by the equations

$$\left(\frac{2}{h^2} + \frac{1}{s^2} \right) \varphi_{i,j}^* = \frac{\varphi_{i+1,j} + \varphi_{i-1,j}}{h^2} + \frac{\varphi_{i,j+1}}{s^2} - D_{i,j}^{(k)} \quad \dots (32 \cdot a)$$

$$\left(\frac{1}{h^2} + \frac{2}{s^2} \right) \varphi_{i,j}^* = \frac{\varphi_{i+1,j}}{h^2} + \frac{\varphi_{i,j+1} + \varphi_{i,j-1}}{s^2} - D_{i,j}^{(k)} \quad \dots (32 \cdot b)$$

instead of Eq. (31.a) at the corner C, since ϕ is multi-valued function (see Fig. 7). Similar manipulation is made, if the conditions $\varphi_{i,j} = \varphi_{i-1,j}$ or $\varphi_{i,j} = \varphi_{i,j-1}$ are assigned.

The numerical procedure per cycle is:

The main calculations proceed from left to right, and the notation $\leftarrow$ denotes iteration.

Since the iteration process of Eqs. (31.a) and (31.b) is not always stable, the values of ϕ are set equal to zero if $\varepsilon_b^{(k)} > \varepsilon_b^{(k-1)}$ and $\varepsilon_\varphi^{(k)} > \varepsilon_\varphi^{(k-1)}$. Here ε_D and ε_φ are defined by

$$\varepsilon_D = |D_{i,j}|_{\max} \quad \dots (33)$$

$$\varepsilon_\varphi = |\varphi_{i,j}^{(k)} - \varphi_{i,j}^{(k-1)}|_{\max} \quad \dots (34)$$

4. Numerical Example

Although the present method is applicable to three-dimensional flows, the paper deals with only the transient two-dimensional Womersley flow (2.2). Transient two-dimensional Poiseuille flow and a steady flow are studied in Ref. (11). Solutions are obtained by three methods; the explicit ψ - ω method, the present explicit method, and the present implicit method.

In the explicit ψ - ω method, vorticity and stream function are calculated by

Fig. 7 Corner cell, B=Boundary cell, C=Corner cell, Open box=Fluid cell

$$\begin{aligned} \omega_{i,j}^{t+\Delta t} = & \omega_{i,j}^t + \Delta t \left[- \left(\frac{u_{i+1,j-1/2} \omega_{i+1,j} - u_{i-1,j-1/2} \omega_{i-1,j}}{2h} \right. \right. \\ & + \left. \frac{v_{i-1/2,j+1} \omega_{i,j+1} - v_{i-1/2,j-1} \omega_{i,j-1}}{2s} \right) \\ & + A \left(\frac{\omega_{i+1,j} + \omega_{i-1,j} - 2\omega_{i,j}}{h^2} \right. \\ & \left. \left. + \frac{\omega_{i,j+1} + \omega_{i,j-1} - 2\omega_{i,j}}{s^2} \right) \right] \dots \dots \dots (35) \\ 2 \left(\frac{1}{h^2} + \frac{1}{s^2} \right) \phi_{i,j}^* = & \frac{\phi_{i+1,j} + \phi_{i-1,j}}{h^2} + \frac{\phi_{i,j+1} + \phi_{i,j-1}}{s^2} \\ & + \omega_{i,j} \dots \dots \dots (36 \cdot a) \end{aligned}$$

$$\phi_{i,j}^{(k+1)} = \phi_{i,j}^{(k)} + RF_{\phi} (\phi_{i,j}^* - \phi_{i,j}^{(k)}) \dots \dots \dots (36 \cdot b)$$

where RF_{ϕ} is a relaxation factor for ψ , and velocity components are calculated by

$$u_{i,j} = (\phi_{i,j+1} - \phi_{i,j})/s, \quad v_{i,j} = -(\phi_{i+1,j} - \phi_{i,j})/h, \dots \dots \dots (37)$$

In the present explicit method, velocity components u and v are calculated by

$$u_{i,j}^{t+\Delta t} = u_{i,j}^t + \Delta t \left[- \left(\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial p}{\partial x} \right) + A \nabla^2 u \right] \dots \dots \dots (38)$$

$$v_{i,j}^{t+\Delta t} = v_{i,j}^t + \Delta t \left[- \left(\frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial p}{\partial y} \right) + A \nabla^2 v \right] \dots \dots \dots (39)$$

The numerical procedures per cycle are:

i) Explicit ψ - ω method

ii) Present explicit method

In the above numerical procedures, Eqs.(31) and (36) are solved by the SLOR method,⁽¹²⁾ and Eqs.(25) and (26) are solved by the SOR method, the relaxation factors being $RF_{\phi}=1.8$ in the

explicit ψ - ω method, and $RF_{\omega}=RF_{\psi}$ in the present explicit and implicit methods.

Let $\alpha=4$, and divide the flow field as shown in Fig.8. Mesh sizes h and s are $h=s=0.1$, and $IC=61$, $JC=13$, and the time increment $\Delta t=1/200$ (explicit method), and $\Delta t=1/50$ in the implicit method.

The initial values for u , v , p , ψ , and ω are all zeros, and the boundary conditions are:

- 1) Upstream boundary
 u , ψ , ω : given by analytic solutions (13.a)-(13.c),
 $v=p=0$,
 $\phi_{i-1,j} = \phi_{i,j} \quad (i=2).$
- 2) Upper wall
 $u_{i,j+1} = -u_{i,j} \quad (j=JC-2), \quad v=0$,
 ψ : given by analytic solution (13.b),
 ω : $\omega_{B.C.} = -2(\phi_{i,j-1} - \phi_{i,j})/s^2 \quad (j=JC-1),$
 $\phi_{i,j+1} = \phi_{i,j} \quad (j=JC-2).$
- 3) Lower boundary (centerline)
 Symmetrical condition is imposed.
- 4) Downstream end
 $u_{i+1,j} = u_{i,j} - (h/s)(v_{i,j+1} - v_{i,j}) \quad (i=IC-1),$
 $\partial u/\partial x = \partial v/\partial x = 0$,
 $\partial^2 \phi/\partial x^2 = \partial \omega/\partial x = 0$,
 $\phi=0$.

Table I shows number of iterations, and CPU times by various methods. In the table, ϵ_{ϕ} is defined by

$$\epsilon_{\phi} = |\phi_{i,j}^{(k)} - \phi_{i,j}^{(k-1)}|_{\max} \dots \dots \dots (41)$$

and upper part of the table shows the results of $t=0.2$, and lower part those of $t=0.5$. From the table we can see that the present explicit method is the fastest one to obtain a converged solution ($\epsilon_{\phi} < 10^{-2}$), and the relaxation parameter μ , which is introduced to stabilize the calculation, is very effective in the present implicit method. However, the implicit method is not faster than the explicit method as far as the present calculation is concerned. This is true for the results of $t=0.5$. In the present calculation, a large time increment, which is 4 times larger compared with the explicit method, causes a large variation of velocity and pressure in the flow field. Hence it seems that the number of iterations in the implicit method becomes much larger than in the explicit method. However, in the calculations using a finer mesh, the implicit method would be useful because the time increment becomes smaller in proportion to the square mesh size in the explicit method.

Velocity distributions obtained by various methods are shown in Fig.9. From the figure, we can see that the criterion of convergence $\epsilon_{\phi} < 10^{-2}$ is sufficient in the actual calculations.

Fig.8 Discretion of the flow field, B=Boundary cell, Open box=Fluid cell

Table I Comparison of the number of iterations
by various methods

Methods	Criterion of Convergence	Number of Time Step	Total Number of Iterations	CPU Time (Approx.) (sec)	Comments
$\psi-\omega$	$\epsilon_\psi < 2.0 \times 10^{-4}$	40	752	51	Explicit, $RF_\psi = 1.8$
Present I	$\epsilon_D < 1.0 \times 10^{-2}$	40	329	40	Explicit
II	$\epsilon_D < 1.0 \times 10^{-3}$	40	849	72	Explicit
III	$\epsilon_D < 1.0 \times 10^{-2}$	10	372	102	Implicit ($\mu=0$)
IV	$\epsilon_D < 1.0 \times 10^{-2}$	10	295	89	Implicit ($\mu=1$)
Present V	$\epsilon_D < 1.0 \times 10^{-2}$	100	891	99	Explicit
VI	$\epsilon_D < 1.0 \times 10^{-2}$	25	978	253	Implicit ($\mu=1$)

Fig.9 Velocity distributions by various methods at downstream end, two-dimensional finite-difference solution of transient Womersley flow

Fig.10 Temporal variation of pressure gradient at downstream end, two-dimensional finite-difference solution of transient Womersley flow

Fig.11 Pressure distributions by various methods along center line ($t=0.2$), two-dimensional finite-difference solution of transient Womersley flow

Figure 10 shows temporal variations of pressure gradient at downstream end, and the smoothing for these results is not made. Large deviations of the results from the analytic solution are found in all methods for small t . As time proceeds, these deviations become small. If the criterion of convergence becomes severe ($\epsilon_D < 10^{-3}$), the deviation from the analytic solution becomes small in the present explicit method. Pressure distributions at $t=0.2$ by various methods are shown in Fig.11.

Figure 12 shows velocity distributions at $t=0.5$ by various methods. Here, we can see the point of inflection in the figure.

Fig.12 Velocity distributions by various methods at downstream end

5. Conclusions

An implicit method, which uses primitive variables, is proposed to calculate incompressible fluid flow. The method is composed of the Crank-Nicolson method and the author's method. Although the basic idea of the proposed method is applicable to another implicit method and to three-dimensional flows, simple unsteady two-dimensional flows are studied precisely in the present paper. As far as the present calculation is concerned, only the explicit method has advantage over the implicit method, and the present implicit method does not show dominant position compared with the explicit method. As a result of this study, we conclude:

- (1) Navier-Stokes equation with an invariant implicitly in each iteration produces efficient and accurate numerical solutions in the implicit method as well as in the explicit method.
- (2) If we include not only velocity term but also pressure term in the finite-difference equation, the finite-difference equation with an accuracy of $O(\Delta t^2)$ about the velocity term will in general be an approximation of $O(\Delta t)$ for the Navier-Stokes equation. If we want to obtain second-order finite-difference equation, a different procedure, which may be complicated, must be considered.
- (3) When we calculate the flow, which starts impulsively from rest to finite velocity, the finite-difference solution obtained by the implicit method is inaccurate, unless a finer mesh size near the wall is taken.

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