Analytical Solution to Oseen's Equation for Creeping Flow around a Spherical Gas-bubble in Liquid*

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The first-order solution to Oseen's equation is determined for liquid flow around a spherical gas-bubble at low Reynolds numbers under the assumption that the velocities and the stress are continuous at the interface between gas and liquid. This solution leads to new drag-coefficients of a spherical gas-bubble rising in quiescent liquid as $C_d = 16(1+Re/8)/Re$, where Re is Reynolds number related to the diameter of a spherical gas-bubble. The measurements of hydrodynamic drags experienced by small gas-bubbles in quiescent liquid at the Reynolds numbers between 0.03 and 1.0 show good agreement with the new drag formula introduced in the present paper. The visualized streamlines around a spherical gas-bubble at the Reynolds numbers below 0.3 agree well with the streamlines predicted by the first-order solution to Oseen's equation.

Key Words : Fluid Mechanics, Gas-bubble, Creeping Flow, Oseen's Equation, Hydrodynamic Drag, Flow Visualization, Uncertainty Analysis

1. Introduction

Fluid dynamics of gas-bubbles in liquid flow is a classical subject related to the practical problems in boiling water or in liquid flows which include gas-bubbles. Thus excellent reviews⁽¹⁾⁽²⁾ have been presented. In particular, fluid dynamics for a single gas-bubble rising in quiescent liquid at low Reynolds numbers has been described in detail in the textbooks by many authors^{(3)~(7)}. However we feel that an analytical solution to Oseen's equation have received little attention.

The purposes of the present paper are to study the hydrodynamic drag of a single gas-bubble rising in quiescent liquid at low Reynolds numbers and to study liquid flow around a single gas-bubble at low Reynolds numbers analytically and experimentally. This is a classical subject dealt with in an extensive literature. First Stokes⁽⁸⁾ (1851) derived the drag formula for a single solid-sphere. Hadamard⁽⁹⁾ and Rybczynski⁽¹⁰⁾ (1911) presented the famous analytical solution to Stokes' equation assuming a circulating flow inside a gas-bubble. Bousinesq⁽¹¹⁾ (1913) presented an analytical solution to Stokes' equation assuming a slip at the interface between gas and liquid. Taylor & Acrivos⁽¹²⁾ (1964) obtained a solution based on the matched-asymptotic-expansions using Hadamard - Rybczynski's solution near a gas-bubble and a simple solution to Oseen's equation far from a gas-bubble. It should be noted that the transformation of the bubble shape is considered in Taylor-Acrivos's solution. Brabston & Keller⁽¹³⁾ (1975) obtained a numerical solution to the Navier-Stokes

equation using the series-truncation method. The hydrodynamic drag of a single gas-bubble rising slowly in various kinds of liquids has been measured by many authors, for example, by Allen⁽¹⁴⁾ (1900), Arnold⁽¹⁵⁾ (1911), Garner⁽¹⁶⁾ (1954), Kojima et al.⁽¹⁷⁾ (1975), Takahashi et al.⁽¹⁸⁾ (1976), Miyahara et al.⁽¹⁹⁾ (1983). These measurements show that the drag of a spherical gas-bubble in liquid can be represented as a function of Reynolds number and that the drag can be well represented as a Stokes formula or a Hadamard-Rybczynski formula at low Reynolds numbers. The scatters of the measurements between the Stokes and the Hadamard-Rybczynski formulas seem to come from the condition of the interface between gas and liquid⁽⁶⁾. We do not discuss this complicated problem in the present paper. In general, a small gas-bubble in pure liquid seems to take a perfectly spherical shape and to rise vertically. It seems that the flow inside and outside of a small gas-bubble is well approximated by Hadamard - Rybczynski's solution⁽¹⁸⁾.

Recently we have arrived at the following conclusions⁽²⁰⁾⁽²¹⁾ that uniform flow around a solid sphere is well approximated by a first-order solution to Oseen's equation, but not by a Stokes solution at the Reynolds numbers below 0.4. Furthermore the sphere drag is well approximated by the Oseen drag instead of the Stokes drag at $Re \rightarrow 0$. The same conclusion is expected to be valid for liquid flow around a gas-bubble at low Reynolds numbers. However there seems to have been reported no solution to Oseen's equation but the simple one used by Taylor & Acrivos⁽¹²⁾.

In the present paper a new analytical solution to Oseen's equation for a spherical gas-bubble in liquid is introduced, which satisfies the boundary conditions that the velocities and the stress are continuous at the interface between gas and liquid. The hydrodynamic drag and the

* Received 27th June, 1984.

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streamlines predicted by using this new solution are discussed with the measurements of the drag and visualized streamlines.

2. Nomenclature

a : radius of a spherical gas-bubble
 C_D : drag coefficient of gas-bubble
 D : hydrodynamic drag of gas-bubble
 d : diameter of a spherical gas-bubble
 g : acceleration due to gravity
 $k = U/(2\nu)$
 p : pressure considering the effect of gravity
 Re : Reynolds number related to d
 U : terminal velocity of gas-bubble in liquid or velocity of uniform flow
 $v = i_r v_r + i_\theta v_\theta$: velocity of axisymmetric flow

Δ_j : uncertainty interval in the variable j

(r, θ, ϕ) : spherical coordinates, of which the origin is placed at the center of a gas-bubble

$$Y = 1 + \sigma + Re(3 + 2\sigma)/16$$

μ : viscosity of fluid

ν : kinematic viscosity of fluid

ρ : density of fluid

$$\sigma = \mu_o/\mu_i$$

φ : Stokes' stream-function for axisymmetric flow

Subscripts

i : inside of gas-bubble

o : outside of gas-bubble

3. Analytical Solution to Oseen's Equation for Liquid Flow around a Spherical Gas-bubble

The continuity and Oseen's equations for steady flow are derived as

$$\text{div } \mathbf{v} = 0 \quad \dots\dots\dots (1)$$

and

$$U(\partial \mathbf{v} / \partial x) = -(1/\rho) \text{grad } p + \nu \nabla^2 \mathbf{v} \quad \dots\dots\dots (2)$$

respectively. When the Stokes stream-function φ is introduced for an axisymmetric flow as

$$v_r = -(1/r^2 \sin \theta) \cdot (\partial \varphi / \partial \theta) \quad \text{and} \quad v_\theta = (1/r \sin \theta) \cdot (\partial \varphi / \partial r),$$

a general solution to Eqs.(1) and (2) is given by Imai⁽⁴⁾ as

$$\varphi = \frac{U}{2} y^2 + m(1 - \cos \theta) + y^2 \left(B_0 + B_1 \frac{\partial}{\partial x} + B_2 \frac{\partial^2}{\partial x^2} + \dots \right) \frac{1}{r^3} + \left(A_0 + A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial^2}{\partial x^2} + \dots \right) (1 + \cos \theta) \times [1 - \exp\{-kr(1 - \cos \theta)\}], \quad \dots\dots\dots (3)$$

where $x = r \cos \theta$, $y = r \sin \theta$, $k = U/(2\nu)$ and the direction of a uniform flow is assumed to be along x-axis. The first term of the righthand side of Eq.(3) represents a uniform flow, the second term a source with intensity m , the third term a doublet and its derivatives, and the fourth term the Oseenlet and its derivatives respectively. The shape of a gas-bubble is assumed to be spherical and the interfacial tension around the gas-bubble is assumed to be constant. It is convenient to consider a uniform flow with velocity U past the gas-bubble fixed at the origin of the spherical coordinates, instead of considering an actual flow around the gas-bubble which rises in liquid at the terminal velocity U . The boundary conditions are represented as

(a) Uniform flow in the x-direction at large distances from the gas-bubble:

$$\varphi_o/r^2 \rightarrow (U/2) \cdot \sin^2 \theta \quad \text{as } r \rightarrow \infty; \quad \dots\dots\dots (4)$$

(b) Finite velocity at the center of the gas-bubble;

(c) No flow across the interface:

$$\frac{\partial \varphi_i}{\partial \theta} = \frac{\partial \varphi_o}{\partial \theta} = 0 \quad \text{at } r = a; \quad \dots\dots\dots (5)$$

(d) Continuity of tangential velocity across the interface:

$$\frac{\partial \varphi_i}{\partial r} = \frac{\partial \varphi_o}{\partial r} \quad \text{at } r = a; \quad \dots\dots\dots (6)$$

(e) Continuity of tangential stress across the interface:

$$\mu_i \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \varphi_i}{\partial r} \right) = \mu_o \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \varphi_o}{\partial r} \right) \quad \text{at } r = a; \quad \dots\dots\dots (7)$$

(f) Continuity of normal stress across the interface:

$$p_i - 2\mu_i \frac{\partial}{\partial r} \left(\frac{1}{r^2 \sin \theta} \frac{\partial \varphi_i}{\partial \theta} \right) = p_o - 2\mu_o \frac{\partial}{\partial r} \left(\frac{1}{r^2 \sin \theta} \frac{\partial \varphi_o}{\partial \theta} \right) + \frac{2T}{a} \quad \text{at } r = a, \quad \dots\dots\dots (8)$$

where p_i and p_o are modified pressures considering the acceleration due to gravity and the term $(2T/a)$ results from the pressure increment associated with the interfacial tension. The boundary condition (f) is known to be satisfied naturally for a small spherical gas-bubble⁽²²⁾. Since $kr \approx ka = Re/4$ at the vicinity of the bubble surface, we will consider each term to an order of $O(k^2)$ at low Reynolds numbers. The internal flow inside the gas-bubble is assumed to be represented as the well-known general solution⁽⁵⁾ to Stokes' equation. Under these assumptions, the stream-functions may be represented as

$$\varphi_o = \frac{U}{2} r^2 \sin^2 \theta + r^2 \sin^2 \theta \left(B_0 + B_1 \frac{\partial}{\partial x} \right) \frac{1}{r^3} + \left(A_0 + A_1 \frac{\partial}{\partial x} \right) (1 + \cos \theta) [1 - \exp\{-kr(1 - \cos \theta)\}] + \sin^2 \theta \left[\frac{U}{2} r^2 + \frac{B_0}{r} - \frac{3B_1 \cos \theta}{r^2} + A_0 \left\{ kr - \frac{k^2 r^2 (1 - \cos \theta)}{2} \right\} - A_1 k \cos \theta \right] + O(k^3) \quad \dots\dots\dots (9)$$

$$\varphi_i = \sin^2 \theta \left(\frac{E}{10} r^4 + \frac{F}{2} r + G r^2 + \frac{H}{r} \right), \quad (10)$$

$A_0, A_1, B_0, B_1, E, F, G$ and H are determined by using the boundary conditions from (a) to (e). Eq.(9) satisfies Eq.(4). Using Eq.(10) and the boundary condition (b), we obtain $F=H=0$. Substituting Eqs.(9) and (10) into Eqs.(5)~(7), we have

$$\frac{U}{2} a^2 + \frac{B_0}{a} + A_0 k a \left(1 - \frac{ka}{2} \right) + \frac{A_1 k^2 a}{2} = 0, \quad -\frac{3B_1}{a^2} + \frac{A_0 k^2 a^2}{2} - A_1 k = 0,$$

$$6B_1/a^3 + A_0 k^2 a = 0, \quad Ua - B_0/a^2 + A_0 k(1 - ka) + A_1 k^2/2 = 2Ea^3/5 + 2Ga,$$

and

$$\frac{2E}{5} - \frac{2G}{a^2} = \sigma \left\{ -\frac{U}{a^2} + \frac{4B_0}{a^5} + A_0 k \left(-\frac{2}{a^3} + \frac{k}{a^2} \right) - \frac{A_1 k^2}{a^3} \right\},$$

where $\sigma = \mu_o/\mu_i$. These equations yield

$$\left. \begin{aligned} A_0 &= -\frac{Ua}{4k} \cdot \frac{3+2\sigma}{Y}, \quad B_0 = \frac{Ua^3}{4} \cdot \frac{1+Re(3+2\sigma)/4}{Y}, \quad A_1 = -\frac{Ua^3}{4} \cdot \frac{3+2\sigma}{Y}, \\ B_1 &= \frac{Ua^4 \cdot Re}{96} \cdot \frac{3+2\sigma}{Y}, \quad E = 5\sigma U/(2a^2 Y), \quad F = 0, \quad G = -\sigma U/(4Y), \quad H = 0 \end{aligned} \right\}, \quad (11)$$

where $Y = 1 + \sigma - Re(3+2\sigma)/16$. The substitution of Eq.(11) into Eqs.(9) and (10) yields

$$\begin{aligned} \varphi_o &= \frac{U}{2} r^2 \sin^2 \theta + \frac{Ua^3}{4} \cdot \frac{1+Re(3+2\sigma)/4}{Y} \cdot \frac{\sin^2 \theta}{r} - \frac{Re}{96} Ua^4 \cdot \frac{3+2\sigma}{Y} \cdot \frac{3 \sin^2 \theta \cos \theta}{r^2} - \frac{Ua}{4k} \cdot \frac{3+2\sigma}{Y} (1 + \cos \theta) \\ &\quad \times \left\{ 1 - e^{-Re r(1-\cos \theta)/4a} \right\} - \frac{Ua^3}{4} \cdot \frac{3+2\sigma}{Y} \cdot \frac{\sin^2 \theta}{r} \left\{ 1 - \left(1 + \frac{Re r}{4a} \right) e^{-Re r(1-\cos \theta)/4a} \right\} \quad \text{at } r \geq a \quad (12) \end{aligned}$$

$$\begin{aligned} &\approx \frac{U}{2} r^2 \sin^2 \theta \left[1 + \frac{1}{2} \left(\frac{a}{r} \right)^3 \frac{1+Re(3+2\sigma)/4}{Y} - \frac{1}{2} \frac{a}{r} \frac{3+2\sigma}{Y} + \frac{Re}{16} \frac{3+2\sigma}{Y} \right. \\ &\quad \left. - \frac{Re}{16} \cdot \frac{3+2\sigma}{Y} \cos \theta \left\{ \left(\frac{a}{r} \right)^4 + 1 - 2 \left(\frac{a}{r} \right)^2 \right\} \right] \quad \text{at } r \geq a \quad \text{and} \quad \frac{Re r}{4a} \rightarrow 0 \quad (13) \end{aligned}$$

$$\varphi_i = \frac{U}{4} r^2 \frac{\sigma}{Y} \sin^2 \theta \left\{ \left(\frac{r}{a} \right)^2 - 1 \right\} \quad \text{at } r \leq a \quad (14)$$

The solutions given by putting $B_0=B_1=A_1=0$ in Eq.(9) are equivalent to the solutions to Oseen's equation which were used previously by Taylor & Acrivos⁽¹²⁾ in their analysis using matched-asymptotic-expansions. In the present paper this solution is named the basic solution to Oseen's equation. The solutions represented as Eqs.(12) and (13) are named the first-order solutions to Oseen's equation for a uniform flow around a spherical gas-bubble. Eqs.(13) and (14) approach the Hadamard-Rybczynski solutions at $Re \rightarrow 0$, and the first-order solution to Oseen's equation for a solid sphere at $Re \rightarrow 0$ and $\sigma \rightarrow 0$.

Substitution of Eqs.(12)~(14) into $v_r = -(1/r^2 \sin \theta) \cdot (\partial \varphi / \partial \theta)$ and $v_\theta = (1/r \sin \theta) \cdot (\partial \varphi / \partial r)$ yields

$$\begin{aligned} v_{ro} &= -U \cos \theta + \frac{U}{2} \cos \theta \left(\frac{a}{r} \right)^3 \cdot \frac{2+2\sigma-Re(3+2\sigma)/4}{Y} + \frac{Re U}{32} \left(\frac{a}{r} \right)^4 \frac{3+2\sigma}{Y} (3 \cos^2 \theta - 1) - \frac{U}{Re} \left(\frac{a}{r} \right)^2 \\ &\quad \times \frac{3+2\sigma}{Y} + \frac{3+2\sigma}{Y} \cdot e^{-Re r(1-\cos \theta)/4a} \left\{ \frac{U}{Re} \left(\frac{a}{r} \right)^2 + \frac{Ua}{4r} (1 + \cos \theta) - \frac{U}{2} \left(\frac{a}{r} \right)^3 \left(1 + \frac{Re r}{4a} \right) \cos \theta \right. \\ &\quad \left. + \frac{Re U}{16} \left(\frac{a}{r} \right)^2 \left(1 + \frac{Re r}{4a} \right) (1 - \cos^2 \theta) \right\} \quad \text{at } r \geq a \quad (15) \end{aligned}$$

$$\begin{aligned} &\approx -U \cos \theta \left[1 + \frac{1}{2} \left(\frac{a}{r} \right)^3 \frac{1+Re(3+2\sigma)/4}{Y} - \frac{1}{2} \frac{a}{r} \frac{3+2\sigma}{Y} + \frac{Re}{16} \frac{3+2\sigma}{Y} - \frac{3Re}{32} \frac{3+2\sigma}{Y} \left\{ \left(\frac{a}{r} \right)^4 + 1 - 2 \left(\frac{a}{r} \right)^2 \right\} \right. \\ &\quad \left. \times \cos \theta - \frac{Re U}{32} \frac{3+2\sigma}{Y} \left\{ \left(\frac{a}{r} \right)^4 + 1 - 2 \left(\frac{a}{r} \right)^2 \right\} \right] \quad \text{at } r \geq a \quad \text{and} \quad \frac{Re r}{4a} \rightarrow 0 \quad (16) \end{aligned}$$

$$\begin{aligned} v_{\theta o} &= U \sin \theta \left[1 - \frac{1}{4} \left(\frac{a}{r} \right)^3 \frac{1+Re(3+2\sigma)/4}{Y} + \frac{Re}{16} \left(\frac{a}{r} \right)^4 \frac{3+2\sigma}{Y} \cos \theta + \frac{1}{4} \left(\frac{a}{r} \right)^3 \frac{3+2\sigma}{Y} - \frac{1}{4} \frac{3+2\sigma}{Y} \right. \\ &\quad \left. \times \left\{ \frac{a}{r} + \left(\frac{a}{r} \right)^3 \left(1 + \frac{Re r}{4a} \right) - \frac{Re}{4} \left(\frac{a}{r} \right)^2 - \frac{Re}{4} \left(\frac{a}{r} \right)^2 \left(1 + \frac{Re r}{4a} \right) (1 - \cos \theta) \right\} e^{-Re r(1-\cos \theta)/4a} \right] \\ &\quad \text{at } r \geq a \quad (17) \end{aligned}$$

$$\begin{aligned} &\approx U \sin \theta \left[1 - \frac{1}{4} \left(\frac{a}{r} \right)^3 \frac{1+Re(3+2\sigma)/4}{Y} - \frac{1}{4} \frac{a}{r} \frac{3+2\sigma}{Y} + \frac{Re}{16} \frac{3+2\sigma}{Y} + \frac{Re}{16} \frac{3+2\sigma}{Y} \left\{ \left(\frac{a}{r} \right)^4 - 1 \right\} \cos \theta \right] \\ &\quad \text{at } r \geq a \quad \text{and} \quad Re r/4a \rightarrow 0 \quad (18) \end{aligned}$$

$$v_{ri} = -\frac{U}{2} \cos \theta \cdot \frac{\sigma}{Y} \left\{ \left(\frac{r}{a} \right)^2 - 1 \right\} \quad \text{at } r \leq a \quad (19)$$

$$v_{\theta i} = \frac{U}{2} \sin \theta \cdot \frac{\sigma}{Y} \left\{ 2 \left(\frac{r}{a} \right)^2 - 1 \right\} \quad \text{at } r \leq a \quad (20)$$

Eqs. (16) and (18) approach the velocities given by Hadamard & Rybczynski at $Re \rightarrow 0$, and the velocities by Stokes for a solid sphere at $Re \rightarrow 0$ and $\sigma \rightarrow 0$. It should be noted that the differences at the boundary surface are represented as

$$[v_{ro}]_{r=a} \approx -\frac{Re U \cos \theta}{8} \cdot \frac{3+2\sigma}{Y} = O(Re), \quad [v_{ri}]_{r=a} = 0,$$

$$\begin{aligned}
[v_{eo}]_{r=a} &= \frac{\sigma U \sin \theta}{2Y} \left\{ 1 - \frac{R_e(3+2\sigma)}{16Y\sigma} \right\} \\
&= [v_{ei}] \{1 - O(Re)\} \\
[v_{ei}]_{r=a} &= \frac{\sigma U \sin \theta}{2Y} \\
[\phi_o]_{r=a} &= \frac{R_e(3+2\sigma)Ur^2 \sin^2 \theta}{16Y} \\
[\phi_i]_{r=a} &= 0.
\end{aligned}$$

These differences at an order of $O(Re)$ of the present velocities from the exact velocities at the boundary conditions result from neglecting terms of $O(k^3)$ in Eq.(3). Including the terms of $O(k^n)$ in Eq.(3) may be expected to improve the boundary conditions up to an order of $O(Re^n)$. The higher-order solution to Oseen's equation can be easily derived by using the present procedures. However the higher-order solution to Oseen's equation takes a very complicated form. Furthermore the differences between the first- and the higher-order solutions are very small. It should be noted that these situations are similar to those described in our previous paper⁽²³⁾ for the case of a solid sphere. Thus in the present paper the first-order solution to Oseen's equation is adopted as the representative solution to Oseen's equation for a uniform flow past a spherical gas-bubble at low Reynolds numbers.

Since only the coefficient A_0 of the fourth term in the righthand side of Eq.(3) is known to contribute to the hydrodynamic drag⁽⁴⁾, the hydrodynamic drag D for a spherical gas-bubble is easily given by

$$D = -4\pi\sigma UA_0 = \frac{\pi\sigma U^2 a}{k} \cdot \frac{3+2\sigma}{Y} \quad (21)$$

Thus the drag coefficient is represented as

$$C_D = \frac{2D}{\rho U^2 \pi a^2} = \frac{8}{Re} \cdot \frac{3+2\sigma}{Y} \quad (22)$$

In general σ takes a large value for a gas-bubble in liquid, for example $\sigma \approx 55.4$ for an air-bubble in water at room temperature. Thus for the gas-bubble in liquid, C_D is well approximated by

$$\begin{aligned}
[C_D]_{\sigma \rightarrow \infty} &= \frac{8}{Re} \cdot \frac{2}{1-Re/8} \\
&\approx \frac{16}{Re} \left(1 + \frac{1}{8} Re \right) \quad (23)
\end{aligned}$$

In particular, at $Re \rightarrow 0$ and $\sigma \rightarrow 0$, C_D is represented as

$$\begin{aligned}
[C_D]_{\sigma \rightarrow 0} &= \frac{8}{Re} \cdot \frac{3}{1-3Re/16} \\
&\approx \frac{24}{Re} \left(1 + \frac{3}{16} Re \right) \quad (24)
\end{aligned}$$

Eq.(24) is the Oseen formula itself for a sphere drag.

4. Experimental Apparatus and Procedures

4.1 Measurement of hydrodynamic drag for a gas-bubble in liquid

When a gas-bubble rises at its terminal velocity in quiescent liquid, its hydrodynamic drag equals the net gravitational force considering its buoyancy. Thus the drag coefficient of the gas-bubble and the Reynolds number are determined experimentally by using the following equations:

$$C_D = \frac{4dg}{3U^2} \left(\frac{\rho_o - \rho_i}{\rho_o} \right) \approx \frac{4dg}{3U^2} \quad (25)$$

$$Re = \rho_o U d / \mu_o \quad (26)$$

Here each variable included in these equations is measured directly in the experiments. These experimental procedures were used in determining the sphere drag previously⁽²⁰⁾.

The experimental apparatus is schematically shown in Fig.1 and the specifications are summarized in Table 1. Experiments are carried out for an air-bubble in quiescent glycerol at the Reynolds numbers below 0.3 and for a hydrogen or an oxygen gas-bubble in quiescent water at the Reynolds numbers between 0.3 and 20. Ostwald's viscometer is used to measure the viscosity of glycerol. A single air-bubble is blown up at the exit of a capillary tube fixed on the bottom of the test tank. The test tank is filled with pure glycerol. A capillary tube is connected through a vinyl tube to an injection syringe at the outside of the test tank. When the piston is pushed

Fig.1 Experimental apparatus

1, Test tank; 2&3, Constant-temperature water tank; 4, Streak camera; 5, Electrode to generate H_2 gas-bubble; 6, Camera; 7&14, Slits; 8, Stroboscope; 9, Lens; 10, Sphere-releasing set-up; 11, Capillary tube; 12, Speed light; 13, 15 & 16, Lens; 17, Knife edge; 18, Camera body; 19, Synchronous cord; 20, Thin paper.

Table 1 Fluid, bubble and vessel.

	Fluid,	Kinematic viscosity	Bubble,	Diameter	Vessel
Experiment 1	Pure glycerol	$(1.0 \sim 3.0) \times 10^{-2} \text{ m}^2/\text{s}$	Air	0.5~3.0 mm	0.5×0.5×1.0 m
Experiment 2	Water	$(1.0 \sim 1.5) \times 10^{-5} \text{ m}^2/\text{s}$	H_2 or O_2 gas	0.05~0.2 mm	

slightly into the injection syringe, a single air-bubble is blown up at the exit of the capillary tube to rise in glycerol. When the size of the air-bubble becomes larger, its shape comes to deviate from a perfect spheroid. If the deviation from the spheroid is within 0.5 %, the bubble shape is assumed to be spherical in the present paper. From the preliminary experiment it was confirmed that the air-bubble, whose diameter was less than 3 mm, could be spherical in glycerol. The diameter of 3 mm corresponds to the Reynolds number approximately 0.3 at room temperature below 30°C. Therefore a single air-bubble in quiescent glycerol is used for the experiments at the Reynolds numbers below 0.3. At the Reynolds numbers above 0.3, an electrolytically-generated hydrogen or oxygen gas-bubble is used in quiescent water. The electrode to generate the hydrogen bubble is fabricated fixing a nichrome wire with epoxi resin in an acrylic tube. The electrode is polished finely. Another electrode, the anode, is a brass plate in the test tank. Hydrogen gas-bubbles are generated on the cathode. When switching on instantaneously, a single gas-bubble can be generated. The size of the single gas-bubble is controlled by changing the source voltage or the diameter of the cathode wire.

A single gas-bubble is photographed with a shadowgraph by using a stroboscope and a streak camera. The photograph is magnified on a white screen through a slide-projector. The location and the size of the single gas-bubble for a time lapse are measured on the screen. For a small gas-bubble in water the photograph is taken 10 times larger than the actual size of the gas-bubble on film. The test section of 10 mm is taken at the center of the test tank to avoid the effect of pressure change on the bubble size in its rising motion.

4.2 Uncertainty analysis

The uncertainties in the experimental determination of C_D and Re are summarized in Tables 2 and 3. The uncertainty interval $|W_c|$ and the uncertainty W_c/C_D may be well approximated by

$$|W_c| = \left\{ \left(\frac{\partial C_D}{\partial d} W_d \right)^2 + \left(\frac{\partial C_D}{\partial g} W_g \right)^2 + \left(\frac{\partial C_D}{\partial U} W_U \right)^2 \right\}^{1/2} \quad (27)$$

$$\frac{W_c}{C_D} = \pm \left\{ \left(\frac{W_d}{d} \right)^2 + \left(\frac{W_g}{g} \right)^2 + \left(\frac{2W_U}{U} \right)^2 \right\}^{1/2} \quad (28)$$

as is previously described in detail (20).

Substitution of numerical values given by Table 2 into Eq. (28) yields the total uncertainty $W_c/C_D = \pm 2.2\%$, whose value is larger than the one in the case of a solid sphere (20). This results from the errors in measuring the diameter of a gas-bubble photographically.

The uncertainty interval $|W_R|$ and the uncertainty W_R/Re may be well approximated by

$$|W_R| = \left\{ \left(\frac{\partial Re}{\partial \rho_0} W_{\rho_0} \right)^2 + \left(\frac{\partial Re}{\partial U} W_U \right)^2 + \left(\frac{\partial Re}{\partial \mu_0} W_{\mu_0} \right)^2 + \left(\frac{\partial Re}{\partial d} W_d \right)^2 \right\}^{1/2} \quad (29)$$

$$\frac{W_R}{Re} = \pm \left\{ \left(\frac{W_{\rho_0}}{\rho_0} \right)^2 + \left(\frac{W_U}{U} \right)^2 + \left(\frac{W_d}{d} \right)^2 + \left(\frac{W_{\mu_0}}{\mu_0} \right)^2 \right\}^{1/2} \quad (30)$$

The uncertainties included in ρ_0 and μ_0 are $\pm 1 \times 10^{-4}$ and $\pm 5 \times 10^{-3}$ respectively. The uncertainties in U and d are given in Table 3. Substitution of these values into Eq. (30) yields the total uncertainty in a measurement of Re as $\pm 1.6\%$.

4.3 Flow visualization

Visualization of streamlines around an air-bubble in glycerol at the Reynolds numbers below 0.3 is achieved by applying the visualization technique, which was developed previously for visualization of a uniform flow past a solid sphere (20). When several spheres are released one after another along a horizontal line through the sphere-releasing set-up shown in Fig. 1, columns of streaklines are formed in glycerol. After that an air-bubble is blown up at the exit of the capillary tube on the tank bottom. These streaklines move with the surrounding glycerol according as the air-bubble rises. In a steady motion of the air-bubble these streaklines coincide with the streamlines of a uniform flow past a fixed air-bubble. It makes a better photograph to prepare streaklines by using spheres coated with a mixture of black ink and glycerol. However since they stain glycerol and change the glycerol viscosity although slightly, the number of streaklines should be limited to an adequate small number. The agreement between the ink-mixed-glycerol streaklines and the water-laden-glycerol streaklines was satisfactory in our preliminary experiments. Examples of photographs taken by using the ink-mixed-glycerol streaklines are shown in Figs. 2 and 3.

Table 2 Results of uncertainty analysis for C_D .

Variable	Accuracy	Uncertainty	Total uncertainty
Liquid density ρ_0	$\rho_0 (1 \pm 10^{-4})$ g/cm ³	$W_{\rho_0}/\rho_0 = \pm 1 \times 10^{-4}$	$W_R/Re = \pm 1.6\%$
Bubble speed U	$U (1 \pm 10^{-2})$ mm/s	$W_U/U = \pm 1 \times 10^{-2}$	
Bubble diameter d	$d (1 \pm 10^{-2})$ mm	$W_d/d = \pm 1 \times 10^{-2}$	
Liquid viscosity μ_0	$\mu_0 (1 \pm 10^{-3})$ mPa s	$W_{\mu_0}/\mu_0 = \pm 7 \times 10^{-3}$	

Table 3 Results of uncertainty analysis for Re .

Variable	Instrumentation	Accuracy	Uncertainty	Total uncertainty
Diameter d	Photographic	$d (1 \pm 10^{-2})$ mm	$W_d/d = \pm 1 \times 10^{-2}$	$W_c/C_D = \pm 2.2\%$
Speed U	L/N		$W_U/U = \pm 1 \times 10^{-2}$	
Distance L	Photographic	$L (1 \pm 10^{-2})$ mm	$W_L/L = \pm 1 \times 10^{-2}$	
Time N	Stroboscope	$N (1 \pm 10^{-5})$ mm	$W_N/N = \pm 1 \times 10^{-5}$	

5. Results and Discussion

5.1 Drag coefficient of a gas-bubble

The present measurements of the drag coefficient C_D for a single gas-bubble in liquid are shown in Fig.4 on double logarithmic scale, where our previous measurements⁽²⁰⁾ for a solid sphere are also plotted for reference. Since the expressions using double logarithmic scale cannot represent the errors clearly, detailed discussions cannot be made by using this figure only. Thus Fig.5 shows the previous and the present measurements of C_D according to Maeda's expression⁽¹⁾ as C_{DS}/C_D vs. Reynolds number related to the diameter of a spherical gas-bubble. Here C_{DS} represents the Stokes formula for a solid sphere. In Fig.5 the line $C_{DS}/C_D=1$ means the Stokes formula itself, while the line $C_{DS}/C_D=1.5$ means the Hadamard-Rybczynski formula. The measurements show some scatters, but the measurements may be classified into two groups. The first group is composed of classical measurements by Allen, Arnold and Garner, which support the Stokes or the Oseen formula for a solid sphere. The second group is composed of more recent measurements, which support the Hadamard-Rybczynski formula, the formula given by Eq.(23), the analytical results by Taylor & Acrivos and the numerical results by Brabston & Keller. The classical measurements seem to include larger errors in the same manner as explained by Maxworthy⁽²⁴⁾ for the measurements of the sphere drag. Excepting these classical measurements, Fig.5 shows: (1) that the drag parts from the Hadamard-Rybczynski formula to approach the Stokes formula as $Re \rightarrow 0$ at $Re < 0.03$, (2) that the drag is well approximated by the Hadamard-Rybczynski formula at $Re = 0.03 \sim 0.05$, and (3) that the drag parts from the Hadamard-Rybczynski formula to approach the formula given by Eq.(23) as Re increases at $Re > 0.05$. Little difference is recognized among the Taylor - Acrivos results, the numerical results by Brabston & Keller, and the formula given by Eq.(23) at $0.03 < Re < 1.0$. Thus it is concluded that

Fig.2 Photograph
at $Re=0.01$.

Fig.3 Photograph
at $Re=0.3$.

the first-order solution to Oseen's equation for a spherical gas-bubble, which is derived assuming the continuity of the velocities and the stress at the interface between gas and liquid, represents the most simple and suitable analytical model at $0.03 < Re < 1.0$.

5.2 Streamlines around a spherical gas-bubble

Fig.4 Drag coefficient of a gas-bubble.

○, Air-bubbles in glycerol; ●, H_2 or O_2 gas-bubbles in water; ○, Solid spheres; —, Stokes formula for a solid sphere; ---, Oseen formula for a solid sphere; - · - ·, Hadamard-Rybczynski's formula; - - - -, Equation (23)

Fig.5 Drag coefficient of a gas-bubble.

○, Present measurements for air-bubbles in glycerol; ●, Present measurements for H_2 or O_2 gas-bubbles in water; +, Measurements in water by Allen; ○, Measurements in aniline by Allen; ▲, Measurements in linseed oil by Arnold; ○, Measurements in glycerol by Garner; ●, Measurements in 7% sodium chloride-water by Tadaki & Maeda⁽²³⁾; ■, Measurements in corn syrup by Kojima et al; ▽, Measurements in glycerol by Kojima et al; ▲, Measurements in glycerol-water by Miyahara & Takahashi; x, Numerical results by Brabston & Keller; - - - -, Equation (23); ---, Oseen formula for a solid sphere; - · - ·, Analytical results by Taylor & Acrivos.

The streamlines at $Re=0.01$ and $Re=0.3$ are shown in Figs.6 and 7 respectively. Here the visualized streamlines are reproduced by using the previous photographs of Figs.2 and 3, and each theoretical streamline is drawn by using the one which coincides with the visualized streamline at $x/a=-5$. At $Re=0.01$ no difference could be found between the theoretical and the visualized streamlines. However it should be noted that visualized streamlines support the validity of the Hadamard-Rybczynski streamlines. At $Re=0.3$ the visualized streamlines deviate from the Hadamard-Rybczynski streamlines near the gas-bubble, but show good agreement with the Taylor-Acrivos streamlines and with the streamlines given by Eq.(12). Since no difference is recognized between the Taylor - Acrivos streamlines and the streamlines given by Eq.(12), the transformation of the bubble shape may be vanishingly small at such low Reynolds number. These results confirm the validity of the Oseen formula for the drag of the spherical gas-bubble in liquid at the Reynolds numbers below 0.3 from a viewpoint of the flow field. For reference the velocities along x-axis around the spherical gas-bubble calculated by using Eqs.(15) and (17) are shown in Fig.8, where the velocities given by using the Stokes and the Hadamard-Rybczynski solution are also shown.

Fig.6 Streamlines at $Re=0.01$. Fig.7 Streamlines at $Re=0.3$.

—, Stokes streamlines for a solid sphere;
 ----, Hadamard-Rybczynski's streamlines;
 ---, Visualized streamlines, Equation (12)
 and Taylor-Acrivos's streamlines.

Fig.8 Velocity along x-axis at $Re=0.3$.

—, Stokes; ----, H-R; ---, Oseen.

6. Conclusions

The first-order solution to Oseen's equation is newly determined for liquid flow around a spherical gas-bubble at low Reynolds numbers under the assumption that the velocities and the stress are continuous at the interface between gas and liquid. The measurements of the drag for a small gas-bubble show that the Oseen formula assuming an internal flow within the gas-bubble is valid at $0.03 < Re < 1.0$. The flow visualization confirms that the first-order solution to Oseen's equation represents the most simple and suitable model at $Re \leq 0.3$.

The author wishes to thank Professors N.Okuda and A.Shima for their valuable discussions.

References

- (1) Maeda, S., Kagaku-kogaku (in Japanese), Vol.31, No.5 (1967), 438.
- (2) Harper, J.F., Adv. Appl. Mech., Vol.12 (1972), 59.
- (3) Lamb, H., Hydrodynamics, (1932), 601, Cambr. Univ. Press.
- (4) Imai, I., Fluid Dynamics (in Japanese), (1973), 385 and 355~360, Shokabo, Tokyo.
- (5) Happel, J. and Brenner, H., Low Reynolds Number Hydrodynamics, (1983), 127, Martinus Nijhoff Publishers.
- (6) Levich, V.G., Physicochemical Hydrodynamics, (1962), 434, Prentice-Hall.
- (7) Clift, R., Grace, J.R. and Weber, M.E., Bubbles, Drops and Particles, (1978), 30, Academic Press.
- (8) Stokes, G., Cambr. Trans., Vol.9, No.2 (1851), 8.
- (9) Hadamard, J., Compt. Rend., Vol.152 (1911), 152.
- (10) Rybczynski, W., Bull. Acad. Sci. Cracovie, Vol.1 (1911), 40.
- (11) Bousinesq, J., Compt. Rend., Vol.156 (1913), 1124.
- (12) Taylor, T.D. and Acrivos, A., J. Fluid Mech., Vol.18, No.3 (1964), 466.
- (13) Brabston, D.C. and Keller, H.B., J. Fluid Mech., Vol.69, No.1 (1975), 179.
- (14) Allen, H.S., Philos. Mag., Vol.50, No.304 (1900), 323.

- (15) Arnold, H. D., *Philos. Mag.*, Vol. 22 (1911), 755.
 - (16) Garner, F. H. and Hammerton, D., *Chem. Eng. Sci.*, Vol. 3, No. 1 (1954), 1.
 - (17) Kojima, E., Akehata, T. and Shirai, T., *J. Chem. Eng. Japan*, Vol. 8, No. 2 (1975), 108.
 - (18) Takahashi, T., Miyahara, T. and Izawa, H., *Kagaku Kogaku Ronbunshu* (in Japanese),
 - (19) Miyahara, T. and Takahashi, T., *Kagaku Kogaku Ronbunshu* (in Japanese), Vol. 9, No. 5 (1983), 593.
 - (20) Kumagai, T. and Fujiwara, J., *Trans. Japan. Soc. Mech. Engrs.* (in Japanese), Vol. 49, No. 439 (1983), 628.
 - (21) Kumagai, T. and Fujiwara, J., *Bull. of the JSME*, Vol. 26, No. 221 (1983), 1900.
 - (22) Imai, I., *Fluid Dynamics* (in Japanese), (1973), 355, Shokabo, Tokyo.
 - (23) Kumagai, T. and Kobayashi, A., *Bull. of the JSME*, Vol. 28, No. 243 (1985), 1925.
 - (24) Maxworthy, T., *J. Fluid Mech.*, Vol. 23, No. 2, (1965), 369.
 - (25) Tadaki, T. and Maeda, S., *Kagaku Kogaku* (in Japanese), Vol. 25, No. 4 (1961), 254.
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