Vibrations of a Forced Self-excited System with a Retarded Argument^{*}

(Subharmonic Vibrations of Order 1/2)

By Yoshihiro TSUDA,^{**} Atsuo SUEOKA,^{***} Yutaka YOSHITAKE,[†]

Hideyuki TAMURA^{***} and Junkichi INOUE^{††}

The subharmonic entrained vibrations of order 1/2 of a nonlinear retarded system with a soft spring type are investigated by making use of an averaging method, under the condition that both a periodic exciting force and a constant force are operative. The Routh-Hurwitz criterion provides the stability conditions of these steady oscillations. It is also revealed that almost periodic vibrations occur in the same frequency region in which the entrained vibrations appear. These two kinds of oscillations are studied numerically using mapping plots in the phase plane. The analytical results relatively agree with the numerical solutions and the results of an analog computer.

In addition, it is shown that the subharmonic oscillations of order 1/2 can take place in the same system, even though there is only a periodic force as the stimulating force.

Key Words: Theory of Vibrations, Nonlinear Oscillation, Retarded Argument, Forced Self-excited System, Entrainment, Subharmonic Vibration of Order 1/2, Almost Periodic Vibration, Mapping

1. Introduction

Certain vibrating systems are governed by differential equations with dependence on the past state. Such systems may occur in many areas such as control engineering, biology, social science, etc. It was Krasowsky that first considered the solution of differential equations with retardation, that is, the so-called retarded differential difference equation, in infinitely dimensional space (for example, Banach space).⁽¹⁾ He then studied the stability of systems expressed by such equations by making use of the direct method of the Liapunov theory.⁽²⁾ However, it seems to be not necessarily easy for this work to make clear the response characteristics, e.g., fundamental harmonic, higher harmonic, subharmonic response, etc., which a specific system may exhibit. Generally, it is very difficult to constitute Liapunov functions for even a simple type of a differential difference equation, compared with the system for an ordinary differential equation.⁽³⁾

Although such viewpoints are very im-

portant to support this sort of study [e.g., see references (4), (5) and (6)], judging from computational technique an analysis and consideration using an averaging method seem to be preferable to the procedure mentioned above. For this reason, there are many papers with application of the averaging.⁽¹⁾ For instance, Rubanik⁽⁷⁾ investigated the van der Pol equation with a retarded argument by extending the averaging method and analyzing the influence of the retarding on response characteristics of the fundamental and the subharmonic vibrations of order 1/3.

To make clear the mechanism of the development of chatter vibrations of a work piece during cutting, we analysed the harmonic entrainment oscillations, beat phenomena, and other aspects of a forced self-excited system with both a retarded argument and a symmetrical, soft-type restoring force in a preceding paper.⁽⁸⁾ We showed the qualitative correctness of the theoretical analyses by means of experiments with an external periodic force during chatter vibration.

However, we did not deal with the subharmonic entrainment vibrations of order 1/2 (henceforth, these vibrations are simply referred to as the subharmonic vibrations of order 1/2) which were observed experimentally.⁽⁸⁾ The subharmonic vibrations of order 1/2 may be expected to occur, since the work piece is pushed through a force in a constant direction. The force contains a constant force component, that is, the restoring force becomes asymmetrical.

Since the system under consideration is smooth enough (that is, a polynomial expression of displacement and velocity), it should satisfy conditions of the exist-

* Received 18th October, 1984.

** Associate Professor, Oita Univ., 700 Dannoharu, Oita-shi, 870-11 Japan.

*** Professor, Faculty of Engineering, Kyushu Univ., 6-10-1 Hakozaki, Higashi-ku, Fukuoka-shi, 812 Japan.

† Research Assistant, Faculty of engineering, Kyushu Univ..

†† President, Kyushu Insti. of Tech., 1-1 Sensui-cho, Tobata-ku, Kita-Kyushu-shi, 804 Japan

ence of a solution and so on.⁽¹⁰⁾ Accordingly, as to the initial value problems of such functional differential equations, it is possible to deal with the equations with retarding in the same way as with ordinary differential equations¹¹ if consideration is restricted to only their solutions which tend towards the right direction of the initial time.

This paper also applies the averaging method⁽¹⁾⁽⁷⁾ for an ordinary differential equation, and it mainly analyzes the subharmonic vibrations of order 1/2 in the entrainment frequency region of the oscillations of the preceding paper.⁽⁸⁾ In addition, the analytical results are verified with the results of numerical integrations (that is, the point mapping method) and an analog computer. The point mapping method is appropriate to find periodic solutions to nonlinear vibrating systems, which are even systems with strong nonlinearities.^{(12) ~ (15)}

2. Theoretical Analysis

2.1 Approximate solutions

Consider a fundamental equation which includes a constant force component in terms of an exciting force, governed by

$$\ddot{x}(t) + h\dot{x}(t) + x(t - \tau) + q \cdot x^3(t - \tau) = a \cos \nu t + G \quad (1)$$

where τ : retarded argument, i.e., time lag (> 0), h : coefficient of viscous damping (> 0), q ($= -1$), a (> 0), G (> 0), ν : angular frequency of exciting force (these are system parameters), $x(t)$ is displacement, t is nondimensional time, and $(\cdot)$ denotes $d(\cdot)/dt$. Normal expression yields:

$$\left. \begin{aligned} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= a \cos \nu t + G - hy(t) \\ &\quad - x(t - \tau) - qx^3(t - \tau) \end{aligned} \right\} \quad (2)$$

Assuming that parameters h and τ are sufficiently small and $\tau > h$,⁽⁸⁾ and applying the averaging method to Eq. (1), steady solutions are obtained. Namely, in the region where the frequency, ν , of the external force is equal to 2, an approximate solution of the subharmonic oscillations of order 1/2 is expressed as follows:

$$x(t) = R + C \cos\left(\frac{\nu}{n}t - \theta\right) + A \cos \nu t, \quad (3)$$

where $A \approx a/(1 - \nu^2)$ is the amplitude of the fundamental harmonic component and R is a unidirectional component of the response and C is the amplitude of the subharmonic vibrations of order 1/2 and θ is the phase angle of the subharmonics and $n = 2$.

Substituting Eq. (3) into Eq. (1) and finding the averaging equations by the averaging method yields the following expressions:

$$\left. \begin{aligned} \frac{\nu}{n}C\dot{\theta} &= -\frac{1}{2}C\left[\cos\left(\frac{\nu}{n}\tau\right) - \left(\frac{\nu}{n}\right)^2\right. \\ &\quad \left.+ 3q\left(R^2 + \frac{1}{4}C^2 + \frac{1}{2}A^2\right)\cos\left(\frac{\nu}{n}\tau\right)\right] \end{aligned} \right\}$$

$$\left. \begin{aligned} &+ 3qRA \cos\left(2\theta - \frac{\nu}{n}\tau\right), \\ \frac{\nu}{n}\dot{C} &= -\frac{1}{2}C\left[h\left(\frac{\nu}{n}\right) - \sin\left(\frac{\nu}{n}\tau\right)\right. \\ &\quad \left.- 3q\left(R^2 + \frac{1}{4}C^2 + \frac{1}{2}A^2\right)\sin\left(\frac{\nu}{n}\tau\right)\right. \\ &\quad \left.+ 3qRA \sin\left(2\theta - \frac{\nu}{n}\tau\right)\right], \\ -h\dot{R} &= -G + R \\ &+ qR\left(R^2 + \frac{3}{2}C^2 + \frac{3}{2}A^2\right) \\ &+ \frac{3}{4}qC^2A \cos 2\theta \end{aligned} \right\} \quad (4)$$

In the steady state of the oscillations, the conditions $\dot{\theta} = 0$, $\dot{C} = 0$ and $\dot{R} = 0$ hold. So the expressions which supply the steady solutions are as follows:

$$\left. \begin{aligned} &\left[1 + 3q\left(R^2 + \frac{1}{4}C^2 + \frac{1}{2}A^2\right) - \frac{\nu}{n}\left\{h\right. \right. \\ &\quad \times \sin\left(\frac{\nu}{n}\tau\right) + \frac{\nu}{n}\cos\left(\frac{\nu}{n}\tau\right)\left.\right\}\right]^2 \\ &+ \left(\frac{\nu}{n}\right)^2\left\{h\cos\left(\frac{\nu}{n}\tau\right) - \frac{\nu}{n}\sin\left(\frac{\nu}{n}\tau\right)\right\}^2 \\ &- (3qRA)^2 = 0, \\ R(R - G) + qR^2\left(R^2 + \frac{3}{2}C^2 + \frac{3}{2}A^2\right) \\ &- \frac{1}{4}C^2\left[1 + 3q\left(R^2 + \frac{1}{4}C^2 + \frac{1}{2}A^2\right)\right. \\ &\quad \left.- \frac{\nu}{n}\left\{h\sin\left(\frac{\nu}{n}\tau\right) + \frac{\nu}{n}\cos\left(\frac{\nu}{n}\tau\right)\right\}\right] = 0 \end{aligned} \right\} \quad (5)$$

If, for any G ($\neq 0$), R , C and θ are a set of steady solutions to the simultaneous equations (3), (4) and (5) [henceforth, let us abbreviate the set as (R, C, θ)], it is clear that the system $(R, -C, \theta)$ is another solution. Furthermore, if $x(\nu t)$ is a solution in the case of $G = 0$, $-x(\nu t + \pi)$, that is, $(R, \pm C, \theta)$ also satisfies the simultaneous equations (3), (4) and (5). These contents should be noted in the digital simulation as described later.

2.2 Stability of steady solutions

As for the steady solutions obtained in the previous section as θ_0 , C_0 and R_0 , their stability conditions are furnished from the Routh-Hurwitz criterion⁽²¹⁾ (it is very easy to directly find the real part of characteristic roots⁽²⁰⁾). Sufficiently small variations are denoted with $\tilde{\theta}$, $\tilde{C}$ and $\tilde{R}$, respectively. Expressing $\theta = \theta_0 + \tilde{\theta}$, $C = C_0 + \tilde{C}$ and $R = R_0 + \tilde{R}$ and substituting them into Eq. (4) yield a set of variational equations as follows:

$$\frac{\nu}{n} C \tilde{\theta} = a_{11} C_0 \tilde{\theta} + a_{12} \tilde{c} + a_{13} \tilde{r},$$

$$\frac{\nu}{n} \tilde{c} = a_{21} C_0 \tilde{\theta} + a_{22} \tilde{c} + a_{23} \tilde{r},$$

$$-h \tilde{r} = a_{31} C_0 \tilde{\theta} + a_{32} \tilde{c} + a_{33} \tilde{r},$$

where

$$\Theta_1 = \nu\tau/n, \quad \Theta_2 = 2\theta_0 - \nu\tau/n,$$

$$a_{11} = 3qR_0 A \sin \Theta_2,$$

$$a_{12} = -(3/4)qC_0^2 \cos \Theta_1,$$

$$a_{13} = -(3/2)qC_0$$

$$\times (2R_0 \cos \Theta_1 + A \cos \Theta_2),$$

$$a_{21} = -3qR_0 A \cos \Theta_2,$$

$$a_{22} = (3/4)qC_0^2 \sin \Theta_1,$$

$$a_{23} = (3/2)qC_0$$

$$\times (2R_0 \sin \Theta_1 - A \sin \Theta_2),$$

$$a_{31} = -(3/2)qC_0 A \sin 2\theta_0,$$

$$a_{32} = (3/2)qC_0(2R_0 + A \cos 2\theta_0),$$

$$a_{33} = 1 + 3q(R_0^2 + C_0^2/2 + A^2/2)$$

..... (6)

Then, substituting $\tilde{r} = \tilde{r}_0 e^{\lambda t}$, $\tilde{c} = \tilde{c}_0 e^{\lambda t}$ and $\tilde{\theta} = \tilde{\theta}_0 e^{\lambda t}$ ($\tilde{r}_0$, $\tilde{c}_0$ and $\tilde{\theta}_0$ being constants), the characteristic equation of Eq.(6) is found and the Routh-Hurwitz criterion⁽²¹⁾ provides the stability conditions for the steady solutions R_0 , C_0 and θ_0 , that is,

$$\Psi_i > 0 \quad (i=0, 1, 2, 3),$$

$$\Psi_0 \Psi_3 < \Psi_1 \Psi_2,$$

where

$$b_1 = b_2 = \nu/n, \quad c_3 = -h,$$

$$\Psi_0 = -b_1 b_2 b_3,$$

$$\Psi_1 = a_{11} b_2 b_3 + a_{22} b_1 b_3 + a_{33} b_1 b_2,$$

$$\Psi_2 = a_{13} a_{31} b_2 + a_{12} a_{21} b_3$$

$$+ a_{32} a_{23} b_1 - (a_{11} a_{22} b_3$$

$$+ a_{11} a_{33} b_2 + a_{22} a_{33} b_1),$$

$$\Psi_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

..... (7)

3. Numerical Examples and Discussion

In Fig.1, the response curves vs. frequency of external force ν are illustrated. They are computed from Eq.(5). In the figure, solid lines correspond to stable and dotted lines correspond to unstable states, respectively, and the system parameters are designated as follows: $h=0.008$, $\tau=0.01$, $\alpha=0.1$ and $G=0.1$. The stable region is to be the frequency region $1.53 \leq \nu \leq 1.90$. In addition, the cases: $(h, \tau) = (0.01, 0.02)$ and $(0.02,$

0.03) are computed under the conditions of $\alpha=0.1$ and $G=0.1$. Although very slight differences exist in the stable regions between the three cases, these three response curves almost overlap and it is difficult to distinguish them in the figure. All the lower branches of the C curves are unstable, while a stable state exists on its upper branch. Let us refer to the upper branch which includes the stable state as the stable branch and to the lower branch as the unstable branch. The two R curves almost overlap each other in the high frequency region. Note that in the system with $G \neq 0$ there exist two kinds of subharmonic oscillations of order 1/2 which are different only in the phase property.

3.1 Result of numerical

integration method (i.e., mapping method^{(13)~(16) (19)})

Let $[x(t), y(t)]$ be the solution to Eq.(2) which passes through an arbitrary point $P_0(x_0, y_0)$ of $x-y$ plane at $t=0$. Also, assume that L is the period of the external force and

$$\left. \begin{aligned} x_i &= x(iL), \quad i=0, 1, 2, \dots \\ y_i &= y(iL) \end{aligned} \right\} \dots\dots\dots (8)$$

Then, it is possible to define the mapping T which maps point $P_0(x_0, y_0)$ to $P_1(x_1, y_1)$ in $x-y$ plane and to denote $P_i = T^i P_0$ [see, for example, reference (16)]. Therefore, the characteristics of a motion which starts from a given initial point $P_0(x_0, y_0)$ can be expressed with this sequence of points:

$$P_1, P_2, \dots, P_i, \dots \dots\dots (9)$$

— : stable } : present theory
 : unstable }
 0 : Numerical integration, Δ : Analog computer

Fig.1 Response curves of subharmonic vibrations of order 1/2 (the system with $G=0.1$)

The invariant point of this mapping T , i.e., $P_1 = T P_0 = P_0$ is a fixed point and the point which satisfies both $P_\ell \neq P_0$ ($\ell = 1, 2, \dots, m-1$) and $P_m = P_0 (= T^m P_0)$ is an m -periodic point. In this paper, the numerical integration of Eq.(2) is carried out by the Runge-Kutta method in which the time interval size Δt is given as $\Delta t = 2L/2^{11}$ or $2L/2^{12}$. The asymptotic behavior of the solution $[x(t), y(t)]$ at $t \rightarrow \infty$ is estimated with its characteristics after more than 4000 cycles of the exciting force.

As a result, it is definitely shown that an almost periodic motion occurs besides the subharmonic vibrations of order $1/2$. As regards the almost periodic vibrations, the trajectories obtained through numerical integration only are presented here.

Small points "." in $x-y$ plane as shown in Fig.2 [to Fig.4], correspond to 2-period point [see Eq.(9)] of Eq.(2), where the parameters are $h = 0.008$, $q = -1.0$, $\tau = 0.01$ and $G = 0.1$, and indicate the behavior of the solution of Eq.(2) under the given frequency ν . In these graphs, the points P_0 's correspond to the mark " $\odot$ ", and the point where the solution converges corresponds to the mark " $\times$ ". The curves

marked with AP correspond to the state considered to be almost periodic (hereafter, the mark AP is used in the same manner).

Figs.2(a) and (b) with $\nu = 1.75$ suggest that there exist two subharmonic vibrations of order $1/2$. Since the theory of functional differential equations, e.g. limit set (in the positive direction) and so on, is considered to similarly apply to the system under consideration⁽¹⁷⁾, the solution corresponding to these two points S_1 and S_2 in the figure is subharmonic oscillations of order $1/2$, whose phase angles only are different. Unlike Figs.2(a) and (b), the trajectories of Fig.2(c), which is also the case of $\nu = 1.75$, asymptotically tend towards some closed orbit (denoted by AP in the figure) and Figs. 2(d) and (e) are the case of $\nu = 1.90$. The trajectories of Fig.2(d) show the existence of subharmonic oscillations of order $1/2$, but the convergence to the steady state, characteristically, is extremely slow compared with the cases of Figs.2(a) and (b). Fig.2(e) tends to a closed trajectory asymptotically. Furthermore, when the frequency reaches $\nu = 1.94$, the characteristic of the trajectory results in a different one as shown in Fig.2(h), and it is no longer possible to expect the subharmonic oscillations of order $1/2$ to appear. In Fig.2(f) with $\nu = 1.85$, both two

Fig.2 Integral curves of subharmonic vibrations of order $1/2$ and almost periodic vibrations (the system with $G = 0.1$) (part 1)

(e) $\nu = 1.90$ (f) $\nu = 1.85$ (g) details of region D in Fig.2 (f) ($\nu = 1.85$) (h) $\nu = 1.94$

Fig.2 Integral curves of subharmonic vibrations of order 1/2 and almost periodic vibrations (the system with $G = 0.1$) (part 2)

subharmonic vibrations of order 1/2 (corresponding to the states S_1 and S_2 in the graph), and the outside closed orbit AP, are plotted.

In general, it follows that a bounded and uniformly stable solution tends towards an almost periodic solution asymptotically⁽¹⁸⁾ (although the authors are not able to concretely prove this uniform stability). Therefore, it is possible to conclude that this closed trajectory AP corresponds to the almost periodic solution of Eq.(2). Also Fig.2(g) shows the details of the region D in Fig.2(f), and it corresponds to the region of a solution on the unstable branch. The trajectories in the regions of AP_1 and AP_2 approach AP of Fig.2(f) and the orbits in the areas S_{21} and S_{22} tend to S_2 of Fig.2(f).

Consequently, the solutions in the shaded area of Fig.2(f) asymptotically tend to almost periodic solutions. Fig.2 (e) is a case in point. In Fig.2(h) with $\nu = 1.94$, the subharmonic vibrations of order 1/2 can not take place, but an almost periodic oscillation can occur. In Fig.2 (g), the orbits which tend to S_1 do not appear. The trajectories which tend to S_1 exist on the left of the area AP_1 and ap-

— : Present theory,
 • : Numerical integration

(a) External force

(b) Waveform of response

Fig.3 Waveform of subharmonic vibrations of order 1/2 (the system with $G = 0.1$ and $\nu = 1.70$)

pear in the region $x < 0.09$, if shown with x -coordinate.

In Fig.1, the amplitudes obtained by the digital simulation mentioned above are indicated with the mark "o", and the results by an analog computer with the mark " Δ ". It may be concluded that these three results qualitatively agree well. Regarding the upper limits of the stability, they agree well, while there are slight differences between the approximate solu-

Fig.4 Waveform of almost periodic vibrations (the system with $G=0.1$)

Fig.5 Response curves of subharmonic vibrations of order 1/2 (the system with $G=0$)

Fig.6 Integral curves of subharmonic vibrations of order 1/2 and almost periodic vibrations (the system with $G=0$)

Fig.7 Waveform of subharmonic vibrations of order 1/2 (the system with $G=0$ and $\nu=1.75$)

tions and the others as to their lower limits.

In fig.3, the waveforms of the approximate solution (solid line) and the numerical integration (points) are illustrated when $\nu=1.70$, where Fig.3(a) is an external force. These results agree well. Fig.4 shows the waveform of the almost periodic motion obtained from numerical integration when $\nu=1.88$. The mark "o" corresponds to the 2-period point of the exciting force.

From the above-mentioned analytical results, it is concluded that not only subharmonic oscillations of order 1/2, but also almost periodic vibrations occur in the frequency region under consideration. It is dependent on the initial conditions (i.e., initial value and function) which of them takes place actually. However, it is impossible to simply predict the asymptotic behavior of trajectories in the region surrounded with the closed orbit (i.e., the almost periodic solution). On the other hand, the orbits of Eq.(2) tend to this closed trajectory asymptotically outside of the closed area.

3.2 Case without constant external force (i.e., $G=0$)

The foregoing is the result of a system in which there exists a constant component in the external force. Let us now refer to the results of a case without a constant force. Fig.5 shows the response curves of a system with $G=0$. In the graph, the marks "o" and "Δ" correspond to the results of the numerical integration and those of an analog computer, respectively. There are more differences between the analytical result and the other two simulation results than in the case of Fig.1. Like Fig.1, satisfactory results are obtained with the upper limit of the stable regions, although there are some differences between these results as to

the lower limit of the stable regions. Fig.6 shows the numerically integrated results of the system with $G=0$, whose parameters are $h=0.008$, $\tau=0.01$, $\alpha=0.1$ and $q=-1.0$. Figs.6(a) and (b) are the trajectories of the solutions, whose phase angles only are different [i.e., $(R, \pm C, \theta)$, where $R > 0$]. In Fig.6(c), the solution corresponding to $(-R, C, \theta)$ (in this case, another solution out of phase naturally exists) and the almost periodic solutions are illustrated when $\nu=1.75$. Like the case with $G \neq 0$, it may be expected that there is a region in which the solution tends to the almost periodic oscillation on the area surrounded with this closed orbit. Fig.6(d) is a case where the subharmonic vibration of order 1/2 can not take place when $\nu=1.85$.

In Fig.7, the waveform of the response with $\nu=1.75$ and $G=0$ is plotted, where Fig.7(a) is an external force, and Fig.7(b) corresponds to the case of the solution (R, C, θ) , where $R > 0$. The waveforms corresponding to the rest of the solutions, i.e., $(-R, \pm C, \theta)$ and $(R, -C, \theta)$ may be plotted in the same manner. The points "•" in the figure correspond to the results of numerical integration; therefore, these two results are in good agreement.

4. Conclusions

The subharmonic vibrations of order 1/2 [confirmed experimentally in the preceding paper (8)] and almost periodic oscillations which occur in the entrainment region of subharmonic vibrations of order 1/2 of a forced self-excited system with a retarded argument have been investigated by making use of an averaging method. The results have been verified with the results of numerical integration and analog computer. From these considerations, we draw the following conclusions:

[1] Subharmonic vibrations of order 1/2 take place in the region under study and it may be possible to estimate an outline of their characteristics by an averaging method.

[2] Almost periodic oscillations can also take place in the same frequency region.

[3] In the region (i.e., phase plane) which is surrounded by the closed trajectory corresponding to an almost periodic solution, there exist two kinds of areas which tend to the subharmonic vibrations of order 1/2 and tend to almost periodic oscillations, respectively.

[4] Even in a system without a constant force, subharmonic oscillations of order 1/2 can occur.

[5] In case [4] above, the system has two kinds of solutions; one is the solution with (R, C, θ) , and the other with $(-R, C, \theta)$.

[6] There are slight differences as to the lower limit of the stable region, but the response characteristics of the subharmonic oscillations of order 1/2 of the approximate analysis, numerical integration, and analog computer agree well.

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