A Penalty Function Type of Virtual Work Principle
for Impact Contact Problems of Two Bodies *

by Naoaki ASANO **

This paper shows how an impact contact force can be expressed as a product series of penalty functions and subsidiary contact conditions having relative movement components between two bodies on a contact surface. A penalty function type of virtual work principle for various contact and separate states of two bodies is formulated by using the expression of the impact force. This principle is most effective for solving a general impact response of the alternation between contact and separation, and a stress wave propagation response of cracked structures with open and/or closed states on a cracked surface, because of both the use of the relative components and the relaxation of all the subsidiary conditions in this principle. A finite element method (FEM) based on this principle is applied to a two-dimensional analysis for longitudinal impact of two uniform rods. The mean value of impact force by FEM results coincides well with the value from the theory of one-dimensional elastic stress wave propagation.

Key words: Shock, Structural Analysis, Virtual Work Principle,
Penalty Function Method, Impact Contact Problem, FEM,
Variational Principle, Problems of Two Bodies, Elasticity.

1. Introduction

In recent years, composite material such as fiber-reinforced plastics and metals is widely used as an industrial material. But, the composite material sometimes causes a crack between its matrix and fibers. Then, crack propagation and open/closed behavior must be taken into consideration in the use of the composite material. It is very difficult to formulate the equation of motion of open/closed behavior for cracked structures in various shapes, but the use of variational principle for two bodies in Table 1 makes it easy to formulate the equation [1-8].

This paper presents a penalty function type of virtual work principle for impact contact/separate problems of two bodies. A finite element method (FEM) based on this principle is applied to a two-dimensional response for longitudinal elastic impact of two uniform rods with different lengths, in order to examine the validity of this principle. Moreover, FEM equations for open/closed behavior of cracked structures are derived from the various principles in Table 1, and it is proved that this principle is most effective in FEM analysis of the open/closed behavior.

2. On the Expression of Impact Contact Force Using Penalty Functions

For simplicity of treatment, we first consider the motion of two bodies, \( a = 1, 2 \), which collide on a contact area \( S_c \) in Fig.1. Rectangular Cartesian coordinates are taken in \( n \)-direction normal to \( S_c \), and in both \( t_a \)- and \( t_c \)-directions tangential to \( S_c \). From the continuity of displacement, subsidiary contact conditions on \( S_c \) between the two bodies are given by

\[
\begin{align*}
\mathbf{u}_a &= \mathbf{u}_b + \mathbf{r}_a, & (1) \\
\mathbf{u}_a' &= \mathbf{u}_b' + \mathbf{r}_a', & (2) \\
\mathbf{u}_a'' &= \mathbf{u}_b'' + \mathbf{r}_a'', & (3)
\end{align*}
\]

where \( u_{c_a}(a), u_{c_b}(a), \) and \( u_{c_c}(a) \) denote the displacement, velocity, and acceleration vectors on \( S_c \) of body \( a \) in a fixed coordinate system \( (O-X_1X_2X_3) \), respectively. \( r_a, r_c, \) and \( r_b \) are also the relative displacement, velocity, and acceleration vectors on \( S_c \) [1-5]. Subscript \( c \) is a direction of the local contact coordinates on \( S_c \).

In a stick state in which no slip occurs on \( S_c \) between the two bodies, the

![Fig.1 Rectangular cartesian coordinates on a contact area.](image)

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** Professor, Faculty of Engineering, Tamagawa University, Machida, Tokyo 194, JAPAN.
relative components of Eqs.(1)-(3) are expressed by
\[ r_c = \dot{r}_c, \dot{r}_c = 0, c = n, \bar{u}, b \ on \ S_c, \]  
```
(4)
```

2.1 Impact force using penalty functions and relative movement components

Impact force in a stick state is given by
\[
R_c = \beta_c (\dot{u}_c^2 - \dot{u}_c^2) + \phi_c (\dot{u}_c^2 - \dot{u}_c^2) + \gamma_c (\dot{u}_c^2 - \dot{u}_c^2), c = n, \bar{u}, b \ on \ S_c, \]  
```
(5)
```
where \( \beta_c, \phi_c, \) and \( \gamma_c \) are penalty functions in c-direction \([4,5,11,12].\)

Using Eqs.(1) to (4), Eqs.(5) is represented by
\[
R_c = G_c + P_c, \]  
```
(6)
```
where \( \xi_c, \eta_c, \) and \( \zeta_c \) are also penalty functions in c-direction. If \( \xi_c = \beta_c, \eta_c = \phi_c \) and \( \zeta_c = \gamma_c, \) Eqs.(6) is equal to Eqs.(5).

2.2 Virtual work principle in both stick and separate states

If the damping and thermal effects of the two bodies is neglected, a virtual work principle for the bodies is generally represented by
\[
\int_{T_1}^{T_2} \left[ \sum_{c=1}^{N_c} \left( \delta T^{(c)} + \delta E^{(c)} - \delta W^{(c)} \right) + \delta Q_c \right] dt = 0, \]  
```
(7)
```
sum for \( c = n, \bar{u}, b \ on \ S_c, \)
where \( T_1 \) and \( T_2 \) are time instants, \( \delta \) is a variational operator, and \( \delta Q_c \) is a virtual contact work on \( S_c, \) \( \delta T^{(c)} \), \( \delta E^{(c)} \), and \( \delta W^{(c)} \) are virtual kinetic, internal, and external works of body \( c \) respectively \([1-5].\) These virtual works are given by
\[
\delta E^{(c)} = \int_{V_c} \rho_c \delta \dot{u}^{(c)} \dot{u}^{(c)} dV, \]  
```
(8)
```
\[
\delta T^{(c)} = \int_{V_c} \rho_c \dot{u}^{(c)} \delta \dot{u}^{(c)} dV \]  
```
\[
\delta W^{(c)} = \int_{V_c} \sigma_{ij}^{(c)} \delta u_{ij}^{(c)} dV + \int_{S_c} \sigma_{ij}^{(c)} \delta u_{ij}^{(c)} \delta S_c \]  
```

where \( V^{(c)} \) and \( \rho^{(c)} \) are the volume and the density of body \( c \) respectively, \( \sigma_{ij}^{(c)} \) is the stress tensor, and \( \delta u_{ij}^{(c)} \)

the virtual strain tensor. Also \( u^{(n)}, \dot{u}^{(n)}, \) and \( \dot{u}^{(n)} \) are, respectively, the displacement, the acceleration, and the virtual displacement tensors in the fixed coordinates. \( F_l^{(c)} \) is a body force tensor, \( P_l^{(c)} \) a traction tensor, and \( S_l^{(c)} \) the area on which \( P_l^{(c)} \) acts. From Eqs.(6), \( \delta Q_c \) in Eq.(7) is represented by
\[
\delta Q_c = \int_{T_1}^{T_2} G_c (\delta u^{(n)} - \delta u^{(n)} - \delta r_c) dt + \int_{S_c} P_c \cdot \delta r_c dS_c, \]  
```
(9)
```
c = n, \bar{u}, b \ on \ S_c, \]
\[
\delta Q_c = \int_{T_1}^{T_2} G_c (\delta u^{(n)} - \delta u^{(n)} - \delta r_c) dt, \]  
```
(10)
```
c = n, \bar{u}, b \ on \ S_c, \]
where the derivation of Eqs.(9) and (10) will be described in appendix.

2.3 Features of virtual work principle using penalty functions

When numerical values of all the penalty functions in Eq.(6) approach infinity, the functions can be defined as a new penalty function \( \Omega_c \). Using \( \Omega_c, \) Eq.(7) is rewritten as
\[
\int_{T_1}^{T_2} \left[ \delta Q_c / \Omega_c \right] dt = -\left[ \int_{T_1}^{T_2} \left( \delta E^{(n)} - \delta E^{(n)} - \delta W^{(n)} \right) \right] / \Omega_c, \]  
```
(11)
```
sum for \( c = n, \bar{u}, b \ on \ S_c, \)
When \( \Omega_c \) reaches infinity, the right hand side of Eq.(11) becomes zero. Then, we obtain
\[
\delta Q_c / \Omega_c = 0, \ c = n, \bar{u}, b \ on \ S_c, \]  
```
(12)
```
Equation (12) is also rewritten as
\[
G_c / \Omega_c = 0, \ P_c / \Omega_c = 0, \ c = n, \bar{u}, b \ on \ S_c, \]  
```
(13)
```
in a stick state, and
\[
G_c / \Omega_c = 0, \ c = n, \bar{u}, b \ on \ S_c, \]  
```
(14)
```
in a separate state.

If \( \Omega_c \) tends to infinity, Eqs.(1) to (4) are derived from Eqs.(13) and (14). Therefore, Eq.(7) can include the conditions of both the stick and separate states, using Eqs.(9) and (10).

Table 1 Variational principles for impact contact of two bodies.

<table>
<thead>
<tr>
<th>Variational principles</th>
<th>Non-hybrid type of displacement method</th>
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3. A Virtual Work Principle for Contact and Separate States

This chapter describes a non-incremental penalty type of virtual work principle for various impact contact states of two bodies. For simplicity of treatment, the damping and thermal effects of the two bodies are neglected in this paper. Various virtual works used here are calculated using Eq. (8).

The contact area SC of the bodies, as shown in Fig. 1, is represented by

$$
S_C = S_{CA} + S_{CB} + S_{CM} + S_{Co},
$$

where $S_{CA}$ is the stick area, $S_{CB}$ the slip area, $S_{CM}$ the mixed area, and $S_{Co}$ the separate area.

A virtual work principle for the two bodies is formulated by

$$
\int \left( \sum_{a} (\delta T^{(a)} + \delta E^{(a)} + \delta W^{(a)}) + \delta Q_{ma} + \delta Q_{e} \right) dt = 0,
$$

where $\delta Q_{ma}$, $\delta Q_{e}$, and $\delta Q_{sa}$ are, respectively, the virtual works corresponding to $S_{CA}$, $S_{CB}$, $S_{CM}$, and $S_{Co}$. The details of various impact contact states will be described in sections 3.1 to 3.4.

3.1 Stick state of two bodies

We consider the motion of two bodies interconnected on a local contact area $S_C$. Assume that a stick state of the bodies is discriminated by Coulomb's law [1, 3-5, 7]. From Eq. (9), $\delta Q_{ma}$ in Eq. (16) is expressed by

$$
\delta Q_{ma} = \delta Q_{sa} + \delta Q_{va} + \delta Q_{ta},
$$

where $\delta Q_{sa}$, $\delta Q_{va}$, and $\delta Q_{ta}$ are, respectively, the virtual contact works in $n$-, $t_a$-, and $t_b$-directions in Fig. 1.

3.2 Slip state of two bodies

We consider the motion of two bodies slipping on a local contact area $S_C$. Slipping conditions in both $t_a$- and $t_b$-directions are given by Coulomb's law. Then, we obtain

$$
\left[ R_{ma} = \left[ \bar{u}_{ta} - R_{ta} \right] 
\right] = \left[ \bar{u}_{ta} - R_{ta} \right],
$$

where $R_{ma}$, $R_{ta}$, and $R_{tb}$ are impact contact forces in both $t_a$- and $t_b$-directions, respectively. $\bar{u}_{ta}$ and $\bar{u}_{tb}$ are frictional coefficients in both $t_a$- and $t_b$-directions.

From the continuity of contact displacement in $n$-direction, subsidiary contact conditions on $S_C$ are determined using Eqs. (1) to (4), as follows:

$$
\begin{align*}
\bar{u}_{ta} &= u_{ta} + r_{na}, \quad \bar{u}_{tb} = u_{tb} + r_{nb}, \\
\bar{u}_{ta} &= u_{ta} + \bar{r}_a, \quad r_{na} = \bar{r}_a = 0,
\end{align*}
$$

Subsidiary contact conditions in both $t_a$- and $t_b$-directions are treated in the same manner as in separate state [1-5]. From Eqs. (1) to (3), therefore, the conditions are represented by

$$
\begin{align*}
\bar{u}_{ta} &= u_{ta} + r_{na}, \quad \bar{u}_{tb} = u_{tb} + \bar{r}_b, \\
\bar{u}_{ta} &= u_{ta} + \bar{r}_a, \quad c = t_a, t_b on S_C.
\end{align*}
$$

From Eqs. (6) and (19), an impact contact force $R_{ma}$ in $n$-direction is expressed by

$$
\begin{align*}
R_{ma} &= G_a + P_a, \\
G_a &= \bar{G}_a (u_{ta} - u_{tb} - r_{na}) + \bar{G}_b (u_{ta} - u_{tb} - r_{nb}) + \bar{G}_c (u_{ta} - u_{tb} - \bar{r}_a), \\
P_a &= \bar{P}_a - r_{na} - \bar{r}_a.
\end{align*}
$$

where $\bar{G}_a$, $\bar{G}_b$, $\bar{G}_c$, $\bar{G}_n$, $\bar{G}_o$, $\bar{G}_s$, and $\bar{G}_t$ are penalty functions in $n$-direction, respectively.

A virtual friction work in a slip state [3-5] is given by

$$
\begin{align*}
\int_{S_C} R_{ma} \left( \delta u_{ta} - \delta u_{tb} - \delta r_{na} \right) ds \\
+ \int_{S_C} R_{ma} \left( \delta u_{ta} - \delta u_{tb} - \delta r_{nb} \right) ds.
\end{align*}
$$

In addition, Eq. (10) is added to Eq. (22) in both $t_a$- and $t_b$-directions, owing to the introduction of relative movement components.

Using Eqs. (10) and (18)-(22), $\delta Q_{ma}$ in Eq. (16) is represented by

$$
\begin{align*}
\delta Q_{ma} &= \int S_C \left[ \bar{G}_a (u_{ta} - u_{tb} - r_{na}) \\
&+ \bar{G}_b (u_{ta} - u_{tb} - r_{nb}) \\
&+ \bar{G}_c (u_{ta} - u_{tb} - \bar{r}_a) \\
&+ \bar{G}_n (u_{ta} - u_{tb} - \bar{r}_a) \\
&+ \bar{G}_o (u_{ta} - u_{tb} - \bar{r}_a) \\
&+ \bar{G}_s (u_{ta} - u_{tb} - \bar{r}_a) \\
&+ \bar{G}_t (u_{ta} - u_{tb} - \bar{r}_a) \\
&+ \bar{G}_u (u_{ta} - u_{tb} - \bar{r}_a) \\
&+ \bar{G}_v (u_{ta} - u_{tb} - \bar{r}_a) \\
&+ \bar{G}_w (u_{ta} - u_{tb} - \bar{r}_a) \\
&+ \bar{G}_x (u_{ta} - u_{tb} - \bar{r}_a) \\
&+ \bar{G}_y (u_{ta} - u_{tb} - \bar{r}_a) \\
&+ \bar{G}_z (u_{ta} - u_{tb} - \bar{r}_a) \right] ds,
\end{align*}
$$

where the terms of $\mu_n \xi_n - \bar{r}_a$, $\mu_n \eta_n - \bar{r}_a$, $\mu_n \xi_n - \bar{r}_a$, $\mu_n \xi_n - \bar{r}_a$.
are omittable, because \( r_h, \varepsilon_h, \) and \( \Gamma_h \) become equal to zero.

3.3 Mixed state of stick and slip

We consider the motion of two bodies in the mixed state of stick and slip on a local contact area \( c_a \), such as the lateral slipping behavior of a tire on the road. The conditions in both \( t_a \)- and \( t_b \)-directions are determined by Coulomb’s law. For example, the mixed state is treated as a stick state in \( t_a \)-direction and a slip state in \( t_b \)-direction. \( \delta Q_m \) in Eq.(16) is given by

\[
\delta Q_m = \delta Q_n + \delta Q_s + \delta Q_t, \quad \text{(24)}
\]

\[
\delta Q_n = \int_c \beta_1 (u^{(1)} - u^{(2)}) (\delta u^{(1)} - \delta u^{(2)} - \delta r_e) ds
\]

\[
+ \int_c \phi _1 (u^{(1)} - u^{(2)} - r_e) (\delta u^{(1)} - \delta u^{(2)} - \delta r_e) ds
\]

\[
+ \int_c n_e \cdot \varepsilon e \cdot \delta r e ds + \int_c n_e \cdot \varepsilon e \cdot \delta r ds
\]

\[
+ \int_c n_e \cdot \varepsilon e \cdot \delta r ds, \quad c = n, t_e, t_s \text{ on } C_a.
\]

\[
\delta Q_s = \int_c \beta_2 (u^{(1)} - u^{(2)} - r_s) (\delta u^{(1)} - \delta u^{(2)} - \delta r_s) ds
\]

\[
+ \int_c \phi _2 (u^{(1)} - u^{(2)} - r_s) (\delta u^{(1)} - \delta u^{(2)} - \delta r_s) ds
\]

\[
+ \int_c n_e \cdot \varepsilon e \cdot \delta r_s ds + \int_c n_e \cdot \varepsilon e \cdot \delta r_e ds
\]

\[
+ \int_c n_e \cdot \varepsilon e \cdot \delta r ds, \quad c = n, t_e, t_s \text{ on } C_a.
\]

3.4 Separate state of two bodies

When the whole or local area \( C_a \) between two bodies is in separation, impact contact forces reduce to zero. Consequently, \( \delta Q_o \) in Eq.(16) becomes equal to zero, so that the relation between the two bodies is independent, because there exist no subsidiary contact conditions.

We consider an alternating behavior between contact and separation, such as observed on a beam struck with a rod, on meshing gears, and so on. Then, the use of the clearance between the two bodies makes it easy to discriminate between contact and separation, and makes it possible to decrease the computing time in FEM analysis [2,4,5].

Relative movement components are equivalent to the clearance between the two bodies, and Eqs.(1) to (3) mean the subsidiary conditions. From Eq.(10), \( \delta Q_o \) in Eq.(16) is expressed by

\[
\delta Q_o = \delta Q_n + \delta Q_s + \delta Q_t, \quad \text{(25)}
\]

\[
\delta Q_n = \int_c \beta_1 (u^{(1)} - u^{(2)} - r_e) (\delta u^{(1)} - \delta u^{(2)} - \delta r_e) ds
\]

\[
+ \int_c \phi _1 (u^{(1)} - u^{(2)} - r_e) (\delta u^{(1)} - \delta u^{(2)} - \delta r_e) ds
\]

\[
+ \int_c n_e \cdot \varepsilon e \cdot \delta r e ds + \int_c n_e \cdot \varepsilon e \cdot \delta r ds
\]

\[
+ \int_c n_e \cdot \varepsilon e \cdot \delta r ds, \quad c = n, t_e, t_s \text{ on } C_a.
\]

4. Numerical Example

The validity of the impact contact force expressed in Eq.(6) is examined using a FEM based on the principle in Eq.(16). The FEM equation is applied to a two-dimensional response for longitudinal elastic impact of two uniform rods. However, the formulation of the FEM is not described here, owing to limited space.

4.1 Conditions for analysis

Figure 2 illustrates a two-dimensional finite element model for longitudinal elastic impact of two uniform rods. The material of the two rods is assumed to be mild steel. For simplicity of treatment, the damping effect of the rods is neglected, because of a short contact time of the rods. The end of rod (1) is fixed on a rigid wall, and rod (2) collides with the rod (1). The two elastic rods of different lengths are subjected to impact at their flat end faces. Assume that no plastic deformation occurs on a contact area. Since the contact area between the rods remains constant, calculations are carried out for an impact velocity \( V = 1 \text{ m/s} \) of rod (2). The frictional coefficient on the contact area is assumed to be 0.2.

Nodal points discretized on the contact area are arranged with a proper interval which makes it possible to investigate the variation of configuration and stress distributions in various contact states. The calculation is done for the upper halves of the rods on account of the symmetry of the rods in shape. The rods are divided into constant strain type triangular elements.

Runge-Kutta-Gill method is used to obtain numerical solutions of differential equation. A time interval for integration is 0.1 usec. A lumped mass matrix is used to decrease both the memory capacity and the computing time [5].

4.2 Numerical values of penalty functions

If the damping effect of the two rods is neglected, the collective expression of the FEM [5] is generally formulated by

![Fig.2 A finite element model for two-dimensional analysis of longitudinal impact.](image-url)
\[
\begin{bmatrix}
M_1 & 0 & 0 \\
0 & M_2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{X}_1 \\
\dot{X}_2 \\
\dot{X}_3
\end{bmatrix}
+ \begin{bmatrix}
-I_c & -I_c & -I_c \\
-I_c & I_c & I_c \\
-I_c & I_c & I_c + Z_c
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 \\
-\Phi_c & \Phi_c & -\Phi_c \\
-\Phi_c & \Phi_c & \Phi_c + Y_c
\end{bmatrix}
\begin{bmatrix}
\dot{X}_1 \\
\dot{X}_2 \\
\dot{X}_3
\end{bmatrix}
+ \begin{bmatrix}
K_1 & 0 & 0 \\
0 & K_2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
B_c & -B_c & -B_c \\
B_c & -B_c & B_c \\
B_c & B_c & B_c + Z_c
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
F_2 \\
0
\end{bmatrix}
\]

where \( M \) is the mass matrix, \( K \) the stiffness matrix, \( \dot{X} \), \( X \), and \( F \) are, respectively, the acceleration, the displacement, and the external force matrices. \( \delta, \lambda, \) and \( \Delta \) are the relative components corresponding to acceleration, velocity, and displacement matrices, respectively. Subscripts 1 and 2 refer to the rods (1) and (2) in Fig.3, respectively. \( \Gamma, \phi, B, Z, Y, \) and \( \Xi \) are penalty function matrices. Subscript \( c \) means the region of contact nodes discretized on the contact area.

When the numerical values of various penalty function matrices approach infinity, all the subsidiary contact conditions of Eqs.(1) to (4) are automatically satisfied. But the values cannot tend to infinity in numerical computation. The values should be determined by taking computational errors into consideration [9,10]. Then, we assume that

\[
\Gamma_c = M_{1c} \cdot a, B_c = K_{1c} \cdot a, \Phi_c = B_c, \quad \cdots\cdots(27)
\]

where \( a \) is the ratio of penalty matrix components to \( M_1 \) and \( K_1 \) matrices [5].

In a separate state, \( Z_c, Y_c, \) and \( \Xi_c \) are equal to zero, as found from Eqs.(10) and (25). In a stick state, \( Z_c, Y_c, \) and \( \Xi_c \) of Eq.(26) are determined by Eqs.(5), (6) and (9). Then, we obtain

\[
Z_c = \Gamma_c, Y_c = \Phi_c, \Xi_c = B_c, \quad \cdots\cdots(28)
\]

From the above description, the relation between both the stick and separate states is automatically discriminated by either \( Z_c = Y_c = \Xi_c = 0 \) or Eq.(28).

4.3 Behavior of impact contact force

Figure 3 illustrates the total elastic impact contact force on all contact nodes by the FEM in case of \( a = 10^6 \) in Eq.(27), and the value from the one-dimensional theory of propagation of elastic stress wave (ID theory) [2,7]. FEM results show a periodic change which depends on the deformability of elements due to the lateral effect of inertia as a two-dimensional stress wave propagation, and a high frequency fluctuation is due to a discretized error in element division [8]. However, the mean value of FEM results agrees well with the value from ID theory idealized. The impact force also agrees with that of other results in the range of \( a = 10^2 \) to \( 10^7 \)

The results in Fig.3 agree precisely with those of other FEMs [4,5,7]. Although the model in Fig.2 is wide compared to its length, the contact time of the FEM results is close to that of ID theory. Moreover, the contact time and the outside stress response apart from the impact face by this FEM agree well with experimental results [17]. Therefore, the validity of this FEM is recognized in comparison with ID theory, other FEMs, and experimental results.

5. Application of FEM to Open/Closed Behavior of Cracked Structures

The stiffness of cracked structures changes according to the open and/or closed behavior of cracked surfaces. For simplicity of treatment, we consider a cracked structure in Fig.4. Assume that either alternating force \( F(t) \) or displacement \( U(t) \) acts on the free end of the structure, and that an open and/or closed behavior appears on the cracked surface. The details of the cracked part are illustrated in Fig.5. The structure is divided into three regions.

![Fig.4 A finite element model for two-dimensional analysis of a cracked structure.](image)

![Fig.5 Nodal impact contact forces on a cracked surface.](image)
5.1 Problem of FEM applicable to open/closed behavior

In a closed state of the cracked surface, an impact contact force $R_i$ ($i=1, 2, \ldots$), in Fig. 5 acts on two contact areas between two regions (1) and (2). The force on region (1) also operates in the opposite direction on region (2). Assume that no slip occurs on the cracked surface, and the damping effect of cracked structures is negligible. Then, the collective expression of the FEM is generally formulated by

$$
\begin{bmatrix}
M_{11} & M_{12} & \cdots & M_{1n} & \chi_1 \\
M_{21} & M_{22} & \cdots & M_{2n} & \chi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
M_{n1} & M_{n2} & \cdots & M_{nn} & \chi_n
\end{bmatrix}
\begin{bmatrix}
\chi_1 \\
\chi_2 \\
\vdots \\
\chi_n
\end{bmatrix}
= 
\begin{bmatrix}
F_{1} \\
F_{2} \\
\vdots \\
F_{n}
\end{bmatrix}
$$

where $M$ is the mass matrix, $K$ the stiffness matrix, $\chi$, $\bar{\chi}$, and $F$ are, respectively, the displacement, the acceleration, and the external force matrices. Subscript $i$ to 3 indicate the regions in Fig.5, and double subscripts mean the relationship coupled between two regions. $K$ is an unknown matrix of the impact contact force, and is composed of $R_i$ on the cracked surface [2, 7, 12, 16].

Eliminating the $R$ matrix in Eq.(29), the equation of motion in a closed state of the cracked surface [7, 11, 12, 15, 16] is given by

$$
\begin{bmatrix}
M_{11} + M_{12} + M_{13} & M_{13} \\
M_{13} & M_{33}
\end{bmatrix}
\begin{bmatrix}
\chi_1 \\
\chi_3
\end{bmatrix}
+ 
\begin{bmatrix}
K_{11} + K_{12} + K_{13} & K_{13} \\
K_{13} & K_{33}
\end{bmatrix}
\begin{bmatrix}
\chi_1 \\
\chi_3
\end{bmatrix}
= 
\begin{bmatrix}
F_{1} + F_{2} \\
F_{3}
\end{bmatrix}
\begin{bmatrix}
\chi_1 \\
\chi_3
\end{bmatrix}
, 
\begin{bmatrix}
\bar{\chi}_1 \\
\bar{\chi}_3
\end{bmatrix} = \begin{bmatrix}
\chi_1 \\
\chi_3
\end{bmatrix}, 
\begin{bmatrix}
X_1 \\
X_3
\end{bmatrix} = \begin{bmatrix}
\chi_1 \\
\chi_3
\end{bmatrix}, 
\begin{bmatrix}
X_1 \\
X_3
\end{bmatrix} = \begin{bmatrix}
\bar{\chi}_1 \\
\bar{\chi}_3
\end{bmatrix}, 
\begin{bmatrix}
X_1 \\
X_3
\end{bmatrix} = \begin{bmatrix}
\bar{\chi}_1 \\
\bar{\chi}_3
\end{bmatrix}, 
\begin{bmatrix}
\bar{\chi}_1 \\
\bar{\chi}_3
\end{bmatrix} = \begin{bmatrix}
\chi_1 \\
\chi_3
\end{bmatrix}
$$

In an open state of the cracked surface, the impact force in Eq.(29) becomes zero.

The matrices degree of Eqs.(29) and (30) changes due to the alternation between the open and closed states. Then, the recomposition of the mass and stiffness matrices is required in the analysis of the open and/or close behavior. The recomposition takes much time for computation, and complicates the programming structure. Therefore, it is desirable that the matrices degree always remains constant in both the open and closed states.

On the other hand, the components of the mass and stiffness matrices change due to the alternation between the open and closed states, even if the matrices degree remains constant. When the number of varied components is as small as possible, the computing time becomes shorter. Therefore, it is desirable to decrease the number of the components.

When the mass and stiffness matrices become symmetric, the memory capacity to be used for the matrices is close to about half in comparison with the capacity of non-symmetric matrices. The decrease of the capacity makes it possible to increase the matrices degree in a limited memory capacity as in a small-scale computer. The programming algorithm for the symmetric matrices is simpler, and the computing time is shorter, as compared with those of the non-symmetric matrices. Therefore, it is desirable that the matrices become symmetric.

5.2 Comparison of various FEMs for open/closed behavior

From Eqs.(29) and (30), matrix equation of motion in both the open and closed states on a crack surface is collectively expressed by

$$
[M][\ddot{X}]+[K][X]=[F], 
\begin{bmatrix}
M & \vdots & K & \vdots & F
\end{bmatrix}
$$

where $M$ is the mass matrix, $K$ the stiffness matrix, $\chi$, $\bar{\chi}$, and $F$ are the displacement, the acceleration, and the external force matrices, respectively.

5.2.1 A FEM based on a hybrid type of virtual work principle

From the FEM [11,12] based on a hybrid type of virtual work principle for two bodies [4], various matrices in Eq.(31) are given by

$$
[M]=[L]_{kl=ms}, [K]=[L]_{kl=K}, 
[X]=\frac{d^2}{dt^2}[X]
$$

in a closed state:

$$
[L]=
\begin{bmatrix}
L_{11} & 0 & L_{13} \\
0 & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
, 
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix} = \begin{bmatrix}
\dot{F}_1 \\
\dot{F}_2 \\
\dot{F}_3
\end{bmatrix}
$$

in an open state:

$$
[L]=
\begin{bmatrix}
L_{11} & 0 & 0 \\
0 & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
, 
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix} = \begin{bmatrix}
\dot{F}_1 \\
\dot{F}_2 \\
\dot{F}_3
\end{bmatrix}
$$

where $Ic$ is the unit matrix related to all contact nodes, $a$ the clearance components, and $L^T$ means a transposed matrix.

Components within the dotted box of third column in $[L]$ matrix of Eq.(32) change due to both the open and closed states. Moreover, $[L]$ matrix becomes non-symmetric in the open state.
5.2.2 A FEM based on a virtual work principle using penalty functions

From the FEM [11,12] based on a virtual work principle using penalty functions for two bodies [5], various matrices in Eq.(31) are given by

\[
\begin{bmatrix}
X & F
\end{bmatrix}^T = \begin{bmatrix}
X_1 & X_2 & X_3 & X_4
\end{bmatrix}^T,
\]

\[
\begin{bmatrix}
F
\end{bmatrix} = \begin{bmatrix}
F_1 & F_2 & F_3 & F_4
\end{bmatrix}^T,
\]

\[
\begin{bmatrix}
X
\end{bmatrix} = \frac{d^2}{dt^2} \begin{bmatrix}
X
\end{bmatrix},
\]

\[
\begin{bmatrix}
L
\end{bmatrix} = \begin{bmatrix}
L_{11} + H & -H & L_{11} \\
-H & L_{22} + H & L_{22} \\
L_{11} & L_{22} & L_{33}
\end{bmatrix},
\]

\[
\begin{bmatrix}
y
\end{bmatrix} = \begin{bmatrix}
Y
\end{bmatrix},
\]

\[
\begin{bmatrix}
X
\end{bmatrix} = \begin{bmatrix}
X
\end{bmatrix},
\]

\[
\begin{bmatrix}
F
\end{bmatrix} = \begin{bmatrix}
F
\end{bmatrix},
\]

\[
\begin{bmatrix}
L
\end{bmatrix} = \begin{bmatrix}
L
\end{bmatrix},
\]

in a closed state:

\[
\begin{bmatrix}
M
\end{bmatrix} = \begin{bmatrix}
M
\end{bmatrix},
\]

where \( H \) represents the components of penalty functions.

\[ L \] matrix of Eq.(33) becomes symmetric in both the open and closed states. However, the value of \( H \) becomes zero in the open state. Therefore, the number of varied components in [\( L \)] matrix increases a little bit, as compared with those of Eqs.(32) and (34).

5.2.3 A FEM based on present principle

From the FEM [11,12] based on Eqs.(7) to (10), various matrices in Eq.(31) are given by

\[
\begin{bmatrix}
L
\end{bmatrix} = \begin{bmatrix}
L_{11} + H & -H & L_{11} \\
-H & L_{22} + H & L_{22} \\
L_{11} & L_{22} & L_{33}
\end{bmatrix},
\]

\[
\begin{bmatrix}
X
\end{bmatrix} = \begin{bmatrix}
X
\end{bmatrix},
\]

\[
\begin{bmatrix}
F
\end{bmatrix} = \begin{bmatrix}
F
\end{bmatrix},
\]

\[
\begin{bmatrix}
L
\end{bmatrix} = \begin{bmatrix}
L
\end{bmatrix},
\]

\[
\begin{bmatrix}
M
\end{bmatrix} = \begin{bmatrix}
M
\end{bmatrix},
\]

in an open state:

\[
\begin{bmatrix}
M
\end{bmatrix} = \begin{bmatrix}
M
\end{bmatrix},
\]

where \( J \) represents the components of penalty functions.

\[ L \] matrix becomes symmetric in both the open and closed states, but the matrices degree is larger than that of Eq.(33). However, the value of \( J \) becomes zero in the open state. Therefore, the number of varied components in \( L \) matrix is fewer in comparison with those of Eqs.(32) and (33). Hence, it is found that this FEM is most effective for solving the open and/or closed behavior of a cracked surface.

FEM results of the open/closed behavior of cracked structures are not described here, owing to limited space.

6. Conclusions

The paper presents a new penalty function type of virtual work principle for impact contact and separate states of two bodies. Table 1 systematically lists typical variational principles for two bodies.

The followings are summarized in this paper:

(1) A new virtual work principle for various impact contact and separate states of two bodies is formulated using an extended penalty function method [5]. This principle includes relative movement components corresponding to both the clearance in a separate state and the interference of a fit mechanism in a stick state, as compared with a principle [5] using the penalty function method.

(2) A FEM based on this principle is applied to a two-dimensional analysis for longitudinal elastic impact of two uniform rods. The validity of this principle is verified from both the impact contact force and the duration time between contact and separation. In addition, the results of the FEM agree precisely with those of other FEMs based on various principles in Table 1.

(3) This paper demonstrates on the applicability of various FEMs to the analysis of the open and/or closed behavior of cracked structures. The number of varied components of the mass and stiffness matrices in the FEM based on this principle is smallest in comparison with those of various FEMs based on other principles. Therefore, it is concluded that this principle is the most effective one for solving an impact contact response of the alternation between contact and separation, and a stress wave propagation response of cracked structures with open and/or closed states on a cracked surface.

APPENDIX

Derivation of Eqs.(9) and (10)

1. Hamilton's principle in a separate state of two bodies

Hamilton's principle for impact contact in a stick state of two bodies [4] is expressed as

\[
\delta \int_{t_1}^{t_2} \left( T_c - P_c \right) dt = 0.
\]

where \( \lambda_c \) is a Lagrange multiplier. \( T(a) \) and \( P(a) \) are, respectively, kinetic energy and potential one for the body \( (a) \). The last terms of \( T(a) \) and \( P(a) \) mean the energy difference depending on impact contact. \( T(a) \) and \( P(a) \) are calculated based on Eqs.(8)
Using the relative movement components of $r_C$ and $\dot{r}_C$, Eq.(35) is rewritten as

$$\delta \int_{\Sigma_0} (T_0 - P_0) d\sigma = 0,$$

(36)

where $T_0 = \sum_{\xi} T^{(a)} + \int_{\Sigma_0} \lambda_0 (u^{(a)} - u^{(a)} - r_0) \sigma d\sigma,$

$$+ \int_{\Sigma_0} \dot{\lambda}_0 \dot{r}_0 \sigma d\sigma,$$

$P_0 = \sum_{\xi} P^{(a)} + \int_{\Sigma_0} \lambda_0 (u^{(a)} - u^{(a)} - r_0) \sigma d\sigma,$

$$+ \int_{\Sigma_0} \dot{\lambda}_0 \dot{r}_0 \sigma d\sigma,$$

sum for $c = n, l, t$, on $S_r,$

where $\lambda_0$, $\gamma_0$, $\xi_0$, and $\xi_0$ are penalty functions.

Under the variational operation of Eq.(38), Eq.(39) is rewritten as

$$\int_{\Sigma_0} \left[ \sum_{\xi} T^{(a)} + \delta T^{(a)} - \delta W^{(a)} \right] d\sigma = 0,$$

(40)

where $G^{(a)} = G^{(a)} + \delta G^{(a)}$, $P^{(a)} = P^{(a)} + \delta P^{(a)}$, $\lambda^{(a)} = \lambda^{(a)} + \delta \lambda^{(a)}$.

As found from the form of $G^{(a)}$, an impact contact force is expressed using the combination of the difference of the contact displacements, the contact velocities, and the contact accelerations between two bodies, respectively [1,4,5,14]. Then, the components of the impact force corresponding to the difference of the contact velocities between the two bodies are represented by

$$\delta \int_{\Sigma_0} (T_0 - P_0) d\sigma = 0,$$

(37)

where $T_0 = \sum_{\xi} T^{(a)} + \int_{\Sigma_0} \lambda_0 (u^{(a)} - u^{(a)} - r_0) \sigma d\sigma,$

$$+ \int_{\Sigma_0} \dot{\lambda}_0 \dot{r}_0 \sigma d\sigma,$$

$P_0 = \sum_{\xi} P^{(a)} + \int_{\Sigma_0} \lambda_0 (u^{(a)} - u^{(a)} - r_0) \sigma d\sigma,$

$$+ \int_{\Sigma_0} \dot{\lambda}_0 \dot{r}_0 \sigma d\sigma,$$

sum for $c = n, l, t$, on $S_r.$

Variational operation of Eq.(37) is carried out under the following conditions [13]:

$$\delta u^{(a)}(r) = \delta u^{(a)}(r) = \delta \lambda_0 (r) = 0,$$

$$r = r_1, r_2, a = 1, 2, c = n, l, t, on S_r.$$

Rewriting Eq.(37) into a virtual work form by Eq.(35), we obtain Eq.(35) in Ref.[4]. Therefore, Eq.(37) is found to be Hamilton's principle in a separate state of two bodies.

2. Derivation of Eq.(9)

A penalty function method belongs to the concept of a least square method. Using the penalty function method, Eq.(36) is expressed by

$$\delta \int_{\Sigma_0} (T_0 - P_0) d\sigma = 0,$$

(39)

where $T_0 = \sum_{\xi} T^{(a)} + \int_{\Sigma_0} \lambda_0 (u^{(a)} - u^{(a)} - r_0) \sigma d\sigma,$

$$+ \int_{\Sigma_0} \dot{\lambda}_0 \dot{r}_0 \sigma d\sigma,$$

$P_0 = \sum_{\xi} P^{(a)} + \int_{\Sigma_0} \lambda_0 (u^{(a)} - u^{(a)} - r_0) \sigma d\sigma,$

$$+ \int_{\Sigma_0} \dot{\lambda}_0 \dot{r}_0 \sigma d\sigma,$$

sum for $c = n, l, t$, on $S_r.$

Using a penalty function method, Eq.(37) is rewritten as

$$\delta \int_{\Sigma_0} (T_0 - P_0) d\sigma = 0,$$

(42)

where $G^{(a)} = G^{(a)} + \delta G^{(a)}$, $P^{(a)} = P^{(a)} + \delta P^{(a)}$, $\lambda^{(a)} = \lambda^{(a)} + \delta \lambda^{(a)}$.

Under the variational operation of Eq.(38), Eq.(42) is expressed by

$$\int_{\Sigma_0} \left[ \sum_{\xi} T^{(a)} + \delta T^{(a)} - \delta W^{(a)} \right] d\sigma = 0,$$

(43)

where $G^{(a)} = G^{(a)} + \delta G^{(a)}$, $P^{(a)} = P^{(a)} + \delta P^{(a)}$, $\lambda^{(a)} = \lambda^{(a)} + \delta \lambda^{(a)}$.

3. Derivation of Eq.(10)

Using a penalty function method, Eq.(37) is rewritten as

$$\delta \int_{\Sigma_0} (T_0 - P_0) d\sigma = 0,$$

(35)

where $T_0 = \sum_{\xi} T^{(a)} + \int_{\Sigma_0} \lambda_0 (u^{(a)} - u^{(a)} - r_0) \sigma d\sigma,$

$$+ \int_{\Sigma_0} \dot{\lambda}_0 \dot{r}_0 \sigma d\sigma,$$

$P_0 = \sum_{\xi} P^{(a)} + \int_{\Sigma_0} \lambda_0 (u^{(a)} - u^{(a)} - r_0) \sigma d\sigma,$

$$+ \int_{\Sigma_0} \dot{\lambda}_0 \dot{r}_0 \sigma d\sigma,$$
\[ G_c = \gamma (\dot{\vec{a}}_{11} - \dot{\vec{a}}_{21} - \dot{\vec{r}}_c) + \beta (\dot{\vec{a}}_{12} - \dot{\vec{a}}_{22} - \dot{\vec{r}}_c) \]

sum for \( c = n, l a, l b \) on \( S_c \).

\( G_c' \) of Eq.(43) is composed of the difference of the contact displacements and that of the contact accelerations between two bodies. Then, the difference of the contact velocities between the two bodies \([5,14]\) is represented by

\[ \dot{\varphi}_c (\dot{\vec{a}}_{11} - \dot{\vec{a}}_{21} - \dot{\vec{r}}_c), c = n, la, lb \) on \( S_c \), \( \ldots \)(44)

\( G_c \) of Eq.(6) is obtained by introducing Eq.(44) into \( G_c' \) of Eq.(43). Therefore, Eq.(10) is expressed by using \( G_c \) of Eq.(6) instead of \( G_c' \) of Eq.(43).

References