Flow of Non-Newtonian Fluids in Curved Pipes*

(1st Report, Numerical Analysis for Laminar Flow of Power-law Fluids)

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A steady fully-developed laminar flow of power-law fluids through curved pipes has been solved numerically under Dean numbers $D \leq 1000$, power-indices $n=1 \sim 0.5$ and curvature radius ratios $R_c=10 \sim 100$. Velocity profiles of a primary flow, stream lines of a secondary flow, distributions of wall shear stresses and friction factors are obtained. Effects of D , n and R_c upon flow characteristics are made clear.

Key Words: Fluid Mechanics, Non-Newtonian Fluid, Curved Pipe, Numerical Analysis, Secondary Flow, Power-law Fluid

1. Introduction

Curved pipes are utilized for heat exchangers, chemical reactors etc. Therefore, many theoretical and experimental investigations have been made about flows in curved pipes ranging widely from a laminar to a turbulent flow, and many results have been reported on a pressure loss and a flow pattern of the curved pipe flow. However, most of the investigations are about a Newtonian fluid such as water and air. These results cannot be applied to a flow of a non-Newtonian fluid such as a polymer liquid and blood.

The history of the investigation about a curved pipe flow of the non-Newtonian fluid is so short that only a few results have been reported. Of the investigations performed until today, analytical ones (1)~(4) have given an approximate solution of the visco-elastic fluid by the perturbation method. However, the applicable region of the analytical result is restricted to a small flow rate, and therefore more studies are needed for its practical application. On the other hand, experimental ones (5)~(8) have presented the friction coefficient of the curved pipe flow of a pseudo-plastic fluid which obeys the power-law model. However, effects of governing parameters are not so clear because the tested range of parameters (power-index n , Dean number D and curvature radius ratio R_c) is narrow, and therefore the flow characteristics are not enough grasped.

In the present study, a steady, fully-developed and laminar flow of the power-law fluid through a curved pipe is computed by a numerical method. By this analysis, the velocity profile of a primary flow and the flow pattern of a secondary flow are made clear. Also, the wall shear stress and the friction coefficient are obtained.

Besides, the power-law model is known as a useful model that correlates well the

flow behavior of many non-Newtonian fluids. The computational range of the present analysis is the power-index $n=1 \sim 0.5$, Dean number $D \leq 1000$ and the curvature radius ratio $R_c=10 \sim 100$.

2. Nomenclature

- a ; radius of a curved pipe
- D ; Dean number $= Re/\sqrt{R_c}$
- $e_{\alpha\beta}$; component of a nondimensional deformation rate $= E_{\alpha\beta}/(\rho a^2/K)^{1/(n-2)}$
- n, K ; power-index, material constant of a power-law fluid
- p ; nondimensional pressure $= P/(\rho a^{2n}/K^2)^{1/(n-2)}$
- (r, θ, φ) ; torus coordinate (see Fig.1)
- R_c ; curvature radius ratio $= R_c/a$
- Re ; Reynolds number $= \rho Wa^{2-n} (2a)^n / K$
- t ; nondimensional time $= T/(K/\rho a^2)^{1/(n-2)}$
- $T_{\alpha\beta}$; component of a shearing stress $= K \Phi E_{\alpha\beta}$
- (u, v, w) ; nondimensional velocity in (r, θ, φ) direction $= (U, V, W)/(\rho a^n/K)^{1/(n-2)}$
- w_a ; nondimensional average velocity $= Wa/(\rho a^n/K)^{1/(n-2)}$
- Wa ; cross-sectional average velocity
- λc ; friction coefficient for a pipe flow $= (-\frac{1}{R_c} \frac{\partial p}{\partial \varphi}) (2a)/\frac{1}{2} \rho Wa^2$
- ρ ; density
- $\tau_{\alpha\beta}$; component of nondimensional shearing stress $= T_{\alpha\beta}/(\rho a^{2n}/K^2)^{1/(n-2)}$
- ϕ ; $= \Phi/(\rho a^2/K)^{(n-1)/(n-2)}$ (see Eq.(4))
- ψ ; nondimensional stream function $= \Psi/(\rho a^{2n-4}/K)$
- ω ; nondimensional vorticity $= \Omega/(\rho a^2/K)^{1/(n-2)}$

Fig.1 Torus Coordinate System

* Received 3rd December, 1984.

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3. Analysis

The coordinate system is shown in Fig.1. For an incompressible, steady, fully-developed and laminar flow of the power-law fluid through the curved pipe, basic equations of the axial velocity w , the vorticity ω and the stream function ψ of the secondary flow are respectively

$$\begin{aligned} \frac{\partial w}{\partial t} = & \frac{\partial}{\partial r} \left(\phi \frac{\partial w}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\phi \frac{1}{r} \frac{\partial w}{\partial \theta} \right) \\ & + \left\{ \left(A + \frac{1}{r} \right) \phi + \frac{1}{hr} \frac{\partial \phi}{\partial \theta} \right\} \frac{\partial w}{\partial r} + \left(B \phi - \frac{1}{h} \frac{\partial \phi}{\partial r} \right) \\ & \times \frac{1}{r} \frac{\partial w}{\partial \theta} - \left\{ \frac{\phi}{h^2} + \frac{1}{h} \left(B \frac{\partial \phi}{\partial r} - \frac{A}{r} \frac{\partial \phi}{\partial \theta} \right) + A \frac{\partial \phi}{\partial r} \right. \\ & \left. + B \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right\} w - \frac{1}{h} \frac{\partial \phi}{\partial \theta} \dots \dots \dots (1) \end{aligned}$$

$$\begin{aligned} \frac{\partial \omega}{\partial t} = & \frac{\partial}{\partial r} \left(\phi \frac{\partial \omega}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\phi \frac{1}{r} \frac{\partial \omega}{\partial \theta} \right) \\ & + \left\{ \left(A + \frac{1}{r} \right) \phi + \frac{1}{hr} \frac{\partial \phi}{\partial \theta} \right\} \frac{\partial \omega}{\partial r} + \left(B \phi - \frac{1}{h} \frac{\partial \phi}{\partial r} \right) \\ & \times \frac{1}{r} \frac{\partial \omega}{\partial \theta} - \left\{ \frac{\phi}{h^2} + \frac{1}{h} \left(B \frac{\partial \phi}{\partial r} - \frac{A}{r} \frac{\partial \phi}{\partial \theta} \right) \right\} \omega \\ & + 2w \left(B \frac{\partial w}{\partial r} - \frac{A}{r} \frac{\partial w}{\partial \theta} \right) + f_\omega \dots \dots \dots (2) \end{aligned}$$

$$h\omega = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \left(\frac{1}{r} - A \right) \frac{\partial \psi}{\partial r} - \frac{B}{r} \frac{\partial \psi}{\partial \theta} \dots \dots \dots (3)$$

Here, ϕ of Eqs.(1),(2) and f_ω of Eq.(2) are expressed respectively as follows;

$$\phi = \left\{ \frac{1}{2} (e_{rr}^2 + e_{\theta\theta}^2 + e_{\varphi\varphi}^2) + (e_{r\theta}^2 + e_{\theta\varphi}^2 + e_{\varphi r}^2) \right\}^{(n-1)/2} \dots \dots \dots (4)$$

$$\begin{aligned} f_\omega = & \left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) e_{r\theta} + \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta \partial r} \\ & \times (e_{\theta\theta} - e_{rr}) + \frac{\partial \phi}{\partial r} \left\{ \frac{\partial e_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial e_{\theta\theta}}{\partial \theta} + \left(A + \frac{1}{r} \right) \right. \\ & \times e_{r\theta} + B(e_{\theta\theta} - e_{\varphi\varphi}) \left. \right\} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \left\{ \frac{\partial e_{rr}}{\partial r} + \frac{1}{r} \frac{\partial e_{r\theta}}{\partial \theta} \right. \\ & \left. + A(e_{rr} - e_{\varphi\varphi}) + B e_{r\theta} \right\} \dots \dots \dots (5) \end{aligned}$$

$e_{\alpha\beta}$ of Eqs.(4),(5) are

$$\begin{aligned} e_{rr} = & \frac{2}{h} \left\{ \left(A + \frac{1}{r} \right) \frac{1}{r} \frac{\partial \psi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta \partial r} \right\} \\ e_{\theta\theta} = & \frac{2}{h} \left\{ \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta \partial r} - B \frac{\partial \psi}{\partial r} - \frac{1}{r^2} \frac{\partial \psi}{\partial \theta} \right\} \\ e_{\varphi\varphi} = & \frac{2}{h} \left\{ B \frac{\partial \psi}{\partial r} - \frac{A}{r} \frac{\partial \psi}{\partial \theta} \right\} \\ e_{r\theta} = & \frac{1}{h} \left\{ \frac{\partial^2 \psi}{\partial r^2} - \left(A + \frac{1}{r} \right) \frac{\partial \psi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{B}{r} \frac{\partial \psi}{\partial \theta} \right\} \\ e_{\theta\varphi} = & \frac{1}{r} \frac{\partial w}{\partial \theta} - Bw \\ e_{r\varphi} = & \frac{\partial w}{\partial r} - Aw \dots \dots \dots (6) \end{aligned}$$

and h , A and B are

$$h = R_c + r \sin \theta, \quad A = \frac{\sin \theta}{h}, \quad B = \frac{\cos \theta}{h} \dots \dots \dots (7)$$

The boundary condition is set as follows;

$$\begin{aligned} r=1: & w=0, \quad \omega = \frac{1}{h} \frac{\partial^2 \psi}{\partial r^2}, \quad \psi=0 \\ \theta=\pm 90^\circ: & \frac{\partial w}{\partial \theta}=0, \quad \omega=0, \quad \psi=0 \\ r=0: & \theta: \frac{\partial w}{\partial r}=0 \end{aligned} \dots \dots \dots (8)$$

by considering that the flow is symmetric about the $\theta=\pm 90^\circ$ plane. Consequently, finite-difference equations are derived from Eqs.(1),(2) and (3) only in a half section of pipe where grid points are distributed with geometric intervals in a radial direction and with arithmetic intervals in a circular direction. The numerical calculation is carried out by the time-marching method⁽¹⁰⁾. The computation is terminated when the following convergence criterion is satisfied: the sectional average residuals of three variables w , ω and ψ are respectively less than 10^{-3} in each iteration. The total number of iterations was about 300 ~ 1800 in the present analysis.

From the above mentioned solution, we can estimate Reynolds number Re and the friction coefficient λ_c by the following expressions.

$$Re = 2^n \left\{ \frac{1}{\pi} \int_0^{2\pi} \int_0^1 (rw) dr d\theta \right\}^{2-n} \dots \dots \dots (9)$$

$$\lambda_c = 2^{4/(2-n)} Re^{2/(n-2)} \left(-\frac{1}{R_c} \frac{\partial \psi}{\partial \theta} \right) \dots \dots \dots (10)$$

Fig.2 Flow Pattern ($D=60$, $R_c=30$)

4. Result and Discussion

4.1 Flow pattern and velocity profile

The numerical solution in case of the curvature radius ratio $Rc=30$ is illustrated as an example. The flow pattern in the pipe cross section, namely the isotach distributions of the primary flow and the stream line of the secondary flow are shown in Figs.2~4, and the velocity profile of the axial velocity w at $\theta=0^\circ (\pm 180^\circ)$ and $\pm 90^\circ$ is shown in Fig.5. Numerical values in the figure are values of the axial velocity divided by the cross-sectional average and that of the stream function.

From the analytical result for a low Dean number ($D=60$; Fig.2), a middle Dean number ($D=300$; Fig.3) and a high Dean number ($D=1000$; Fig.4), it can be seen that the flow pattern of the pseudo-plastic fluid ($n < 1$) has the same tendency as that of the

Newtonian fluid ($n=1$). As Dean number increases, the location of the maximum axial velocity shifts toward the outside wall on the symmetric plane, the weak region of a viscous action is developed in the core, and there the transverse line to the symmetric plane becomes clear. Also, in the strong region of the viscous action near the wall, a tongue-like fast velocity region encroaches upon the inside wall portion along the pipe wall. The secondary flow, forming a pair of circulating vortices in the cross section, moves from the inside wall to the outside wall along the symmetric plane, but the stream line is heavily distorted as Dean number increases. At the same time, the vortex center moves into the above mentioned fast velocity region.

Next, the influence of the power- index n at a certain Dean number, will be discussed. As the pseudo-plasticity increases, namely n becomes smaller than unity, the viscous boundary layer becomes more limited to near the pipe wall in the outside wall and its thickness reduces as shown in Fig.5. At the same time, the tongue-like fast velocity region encroaches more and more on the inside wall portion, and the velocity gradient is larger in the boundary layer except a part of $\theta=-90^\circ$ neighborhood. Besides, the result of the dilatant fluid ($n=1.5$) is shown in Fig.5 for reference. As n is larger than unity, it can be seen that the viscous region becomes more thickened, the maximum velocity becomes faster, and its point shifts to the core region.

On the other hand, the secondary flow becomes weak and the absolute value of the stream function becomes small as the power-

(a) $n=1.0$

(b) $n=0.7$

(c) $n=0.5$

Fig.3 Flow Pattern ($D=300$, $Rc=30$)

index n decreases. So, as a measure to indicate the intensity of the secondary flow, the average energy of the secondary motion over the cross section is defined by the following expression

$$e = \frac{2}{\pi} \int_0^\pi \int_0^1 r(u^2 + v^2) dr d\theta \quad (11)$$

and the result is shown in Fig.6. It can be seen that the intensity of the secondary flow reduces as n decreases, but its dependence on Dean number is not so different depending on the power-index. Moreover, $n=1.2$ and 1.5 in Fig.6 indicate the result of the dilatant fluid obtained for reference. The dependence on Dean number varies with n in case of the dilatant fluid. Also, as shown in Fig.7, the vortex center of the secondary flow moves towards the inside wall corresponding to the fast region of the axial velocity as n decreases.

4.2 Distribution of wall shear stresses
Wall shear stresses of the curved pipe flow are the axial shear stress $(\tau_{r\varphi})_w$ related to the primary flow and the circular shear stress $(\tau_{r\theta})_w$ related to the secondary flow.

$$(\tau_{r\varphi})_w = \left(\phi \left| \frac{\partial v}{\partial r} \right| \right)_{r=1} \quad (12 \cdot a)$$

(a) $D=60$ (b) $D=300$ (c) $D=1000$ Fig.5 Axial Velocity Profile ($Rc=30$)

$$(\tau_{r\theta})_w = \left(\phi \left| \frac{1}{h} \frac{\partial^2 \psi}{\partial r^2} \right| \right)_{r=1} \quad (12 \cdot b)$$

Fig.8 shows the circumferential distribution of the nondimensional wall shear stresses in case of the curvature radius ratio $Rc=30$. Both wall shear stresses are divided by a half of the nondimensional pressure gradient.

At the low Dean number (Fig.8(a); $D=60$), the circumferential distribution is gentle. Comparing with the result at the

Fig.6 Kinematic Energy of Secondary Flow ($Rc=30$) Fig.7 Location of Vortex Center ($Rc=30$) (a) $D=60$ (b) $D=1000$ Fig.8 Distribution of Wall Shear Stresses ($Rc=30$)

high Dean number, both wall shear stresses are not so much influenced by the power-index n . At the high Dean number (Fig.8(b) : $D=1000$), the axial wall shear stress is smaller at the inside wall and larger at the outside wall than that at the low Dean number. And, the steep distribution appears near $\theta=-90^\circ \sim -60^\circ$. As the pseudo-plasticity increases, the axial wall shear stress decreases at the outside wall and increases at the inside wall except in $\theta=-90^\circ$ neighborhood. This phenomenon is due to the encroachment of the fast velocity region mentioned in the previous section. On the whole, the axial wall shear stress turns out to be flattened over the pipe wall as the pseudo-plasticity increases.

On the other hand, the circular wall shear stress of the secondary flow increases since the secondary flow comes to be intensified as Dean number increases. The circumferential distribution changes according to the power-index n , and it increases at the inside wall and decreases at the outside wall as n decreases. However, its maximum value is not so changed, whereas the maximum value of the axial wall shear stress is much affected by n .

Moreover, judging from the computational results, the effect of the curvature radius ratio is negligible on the velocity profile and the distribution of the wall shear stresses within $Rc=10 \sim 100$.

4.3 Friction coefficient The curved pipe flow of the power-law fluid can be described by three nondimensional parameters; the power-index n , the curvature radius ratio Rc , and Dean number D . So, in addition to $Rc=30$ in the foregoing section, the numerical calculation is performed for $Rc=10$ and 100 , and the friction coefficient λ_c of the curved pipe flow is obtained for three curvature radius ratios. The obtained λ_c , multiplied by Rc , is shown against Dean number in Fig.9. In the figure, the numerical result is indicated by a solid line and a broken line indicates the following friction coefficient λ_s

$$\lambda_s \sqrt{Rc} = \frac{8 \left\{ \frac{2(3n+1)}{n} \right\}^n}{D} \quad \dots\dots\dots (13)$$

in the laminar flow of the power-law fluid through a straight pipe. The analytical result can be expressed by a single curve of each power-index n with no dependence of the

Fig.9 Friction Coefficient of Curved Pipe Flow

curvature radius ratio Rc . The solid line of λ_c deviates from the broken line of λ_s since the effect of the centrifugal force is stronger as Dean number increases. Also, λ_c reduces owing to a decrement of the power-index for a certain Dean number. The result of the dilatant fluid ($n=1.2, 1.5$; Fig.9) is similar to that of the pseudo-plastic fluid. Dean number, when the curvature effect appears on the friction coefficient $\lambda_c \sqrt{Rc}$, is affected by n in case of the dilatant fluid, while it is not so dependent on n in case of the pseudo-plastic fluid.

5. Conclusions

A steady laminar flow of a pseudo-plastic fluid through a curved pipe was investigated analytically by the numerical calculation under the condition of the power-index $n=1 \sim 0.5$, the curvature radius ratio $Rc=10 \sim 100$, and Dean number $D \leq 1000$. From the results, the following conclusions were obtained:

- (1) The curvature effect on the flow pattern of the pseudo-plastic fluid is similar to that of the Newtonian fluid. A pair of circulating vortices is induced in the pipe cross section, and a weak region of the viscous action is developed in the pipe core as Dean number increases.
- (2) At the same Dean number, as the pseudo-plasticity increase, the location of the maximum axial velocity shifts toward the outside wall and the fast region of the axial velocity encroaches upon the inside wall along the pipe wall. Also, the vortex center of the secondary flow is moved to the inside wall, corresponding to the fast region of the axial velocity.
- (3) The secondary flow becomes weakened as the power-index decreases, but its dependence on Dean number is not so affected.
- (4) Both the axial and circular wall shear stresses decrease at the outside wall and increase at the inside wall as the pseudo-plasticity increases. Meanwhile, the circumferential distribution of the axial wall shear stresses becomes flattened.
- (5) The friction coefficient λ_c is reduced as the power-index n decreases. The relation between $\lambda_c \sqrt{Rc}$ and D can be expressed with the single curve which is dependent only on n .

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