On the Stability of Turbulent Wall Jets Along Concave Surfaces*

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Stability analysis for the occurrence of Taylor-Görtler vortices in turbulent wall jets along concave surfaces is made in a comparative experiment. The eddy viscosity expression in the perturbation equations is derived from a modified version of a two-equation turbulence model including curvature, which predicts well the main features of the turbulent wall jets along concave surfaces. Through the stability analysis, it is found that the minimum curvature at the neutral stability state decreases slightly as the Reynolds number increases. Then, the theoretical minimum curvature comes to agree well with the experiment, when the eddy viscosity expression is treated as a function of curvature.

Key Words: Turbulence, Stability, Wall Jet, Numerical Analysis, Taylor-Görtler Vortex, Concave Surface, Turbulence Model

1. Introduction

Turbulent wall jets have been well-known as a complex turbulent shear flow comprising a wall region in which the flow exhibits structural similarities to a conventional boundary layer, and a jet region where the flow structure is more like that of a free shear flow. The flow structure of the turbulent wall jet has been investigated extensively, because the wall jet has applications in engineering to the boundary-layer control and thermal control of solid surfaces. Nevertheless, the detailed structure of a turbulence in the wall jets has not been made clear⁽¹⁾⁽²⁾.

One of the most important problems is the turbulent structure of wall jets along strong concave surfaces, in which the turbulence is affected not only by the streamline curvature but also by the production of Taylor-Görtler vortices due to the centrifugal instability in the wall region⁽³⁾. According to the previous experiments⁽⁴⁾⁽⁵⁾ about the turbulent wall jets along strong concave surfaces, the turbulent mixing in the wall jet increases when a secondary flow appears in the wall region. The curvature for the appearance of the secondary flow is considered to be $b/R=0.17-0.27$, where b is the half width and R is the local curvature of a solid surface.

On the other hand, the neutral stability conditions for the appearance of Taylor-Görtler vortices in turbulent wall jets along concave surfaces have been studied theoretically⁽³⁾⁽⁶⁾ on the

eddy viscosity concept, similar to the stability theory of a turbulent boundary layer along concave surface⁽⁶⁾⁽⁷⁾. However, the theoretical stability condition of the turbulent wall jets deviates from the experiment. It is because the eddy viscosity calculated from the mean velocity and the Reynolds stress cannot be accurate around the position of maximum velocity. Moreover, the curvature for the appearance of Taylor-Görtler vortices in the turbulent wall jet is much larger than that in the boundary layer. Therefore, the previous analysis using the eddy viscosity with curvature effect neglected could not predict the stability condition accurately.

In the present paper, the centrifugal instability of a turbulent wall jet along concave surfaces is described considering the above mentioned problem about the eddy viscosity, to compare the theoretical stability condition with the experimental one quantitatively.

2. Nomenclature

b : half width
 C : curvature = y_m/R
 k : turbulence energy = $(\overline{u^2} + \overline{v^2} + \overline{w^2})/2$
 M : nondimensional eddy viscosity
 p : pressure
 R : radius of surface curvature
 Re_m : Reynolds number = $U_m y_m / \nu$
 t : time
 U, V : mean velocity in x, y directions
 U_m : maximum velocity in wall jet
 U_τ : friction velocity
 u, v, w : velocity components in x, y and z directions
 u_1, v_1, w_1 : perturbed velocity components of Taylor-Görtler vortices in x, y and z directions
 $\bar{u}, \bar{v}, \bar{w}$: nondimensional perturbed velocity components of Taylor-Görtler vortices in x, y and z directions

* Received 20th May, 1985.

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x,y,z: curvilinear coordinate system
(Fig. 1)

ym : position of maximum velocity

β : growth factor of Taylor Görtler vortices

ε : dissipation

η : nondimensional distance in y direction

λ : wavelength of Taylor Görtler vortices

ν : kinematic viscosity of fluid

ν_t : eddy viscosity

ρ : density of fluid

σ : nondimensional wavelength of Taylor Görtler vortices $= 2\pi y_m / \lambda$

Subscripts

max: maximum value

min: minimum value

η : differentiation with respect to η

3. Theory

3.1 Basic equations

Let's consider a turbulent wall jet along a concave surface of constant radius R, as shown in Fig. 1. An orthogonal curvilinear coordinate system (x,y,z) is adopted with the origin fixed at an arbitrary position along x axis. The equations of motion and continuity are written for an incompressible turbulent flow as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{R}{R-y} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - \frac{uv}{R-y} \\ = \frac{1}{R-y} \left[\frac{R \partial T_{xx}}{\partial x} + \frac{\partial \{ (R-y) T_{yx} \}}{\partial y} \right. \\ \left. + \frac{\partial \{ (R-y) T_{zx} \}}{\partial z} - T_{xy} \right] \dots\dots\dots (1) \end{aligned}$$

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{R}{R-y} u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{u^2}{R-y} \\ = \frac{1}{R-y} \left[\frac{R \partial T_{xy}}{\partial x} + \frac{\partial \{ (R-y) T_{yy} \}}{\partial y} \right. \\ \left. + \frac{\partial \{ (R-y) T_{zy} \}}{\partial z} + T_{xx} \right] \dots\dots\dots (2) \end{aligned}$$

$$\begin{aligned} \frac{\partial w}{\partial t} + \frac{R}{R-y} u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \\ = \frac{1}{R-y} \left[\frac{R \partial T_{xz}}{\partial x} + \frac{\partial \{ (R-y) T_{yz} \}}{\partial y} \right. \\ \left. + \frac{\partial \{ (R-y) T_{zz} \}}{\partial z} \right] \dots\dots\dots (3) \end{aligned}$$

Fig. 1 Geometry of turbulent wall jet along concave surface

$$\frac{R}{R-y} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{v}{R-y} + \frac{\partial w}{\partial z} = 0 \dots\dots\dots (4)$$

where u, v and w are velocity components in x, y and z directions. The components of the stress tensor T in Eqs. (1) to (3) are expressed with the eddy viscosity ν_t :

$$T_{xx} = -\frac{p}{\rho} + 2(\nu + \nu_t) \left(\frac{R}{R-y} \frac{\partial u}{\partial x} - \frac{v}{R-y} \right) \dots\dots\dots (5)$$

$$T_{yy} = -\frac{p}{\rho} + 2(\nu + \nu_t) \frac{\partial v}{\partial y} \dots\dots\dots (6)$$

$$T_{zz} = -\frac{p}{\rho} + 2(\nu + \nu_t) \frac{\partial w}{\partial z} \dots\dots\dots (7)$$

$$T_{xy} = T_{yx} = (\nu + \nu_t) \left(\frac{R}{R-y} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{u}{R-y} \right) \dots\dots\dots (8)$$

$$T_{xz} = T_{zx} = (\nu + \nu_t) \left(\frac{R}{R-y} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \dots\dots\dots (9)$$

$$T_{yz} = T_{zy} = (\nu + \nu_t) \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \dots\dots\dots (10)$$

where p is pressure, ρ is density and ν is kinematic viscosity of fluid. In the present study, the same values of eddy viscosity ν_t are used in all directions⁽⁹⁾. This is based on the assumption that $w \ll u$.

The velocities u, v and w are treated as a sum of time mean velocities U, V and the perturbed velocity of Taylor-Görtler vortices as follows:

$$u = U(x, y) + u_1(y) \exp \left\{ \int \beta(x) dx \right\} \cos(2\pi z / \lambda) \dots\dots\dots (11)$$

$$v = V(x, y) + v_1(y) \exp \left\{ \int \beta(x) dx \right\} \cos(2\pi z / \lambda) \dots\dots\dots (12)$$

$$w = w_1(y) \exp \left\{ \int \beta(x) dx \right\} \sin(2\pi z / \lambda) \dots\dots\dots (13)$$

where $\beta(x)$ is a growth factor of Taylor-Görtler vortices in x direction⁽¹⁰⁾. Substituting Eqs. (11)-(13) into Eqs. (1)-(10), the nondimensional perturbation equations governing the neutral stability state ($\beta = 0$) are obtained, on the assumption that $u_1, v_1, w_1 \ll U, V$ and all terms involving $(y_m/R)^2$ can be neglected, as follows:

$$\begin{aligned} (M+1)\bar{u}_{\eta\eta} + \{M_\eta - C(M+1) - \bar{V}\} \bar{u}_\eta \\ + \{CM_\eta - \sigma^2(M+1) + \bar{V}_\eta\} \bar{u} = (\bar{U}_\eta - C\bar{U}) \bar{v} \dots\dots\dots (14) \end{aligned}$$

$$\begin{aligned} (M+1)\bar{v}_{\eta\eta\eta} + \{M_\eta - 2C(M+1) - \bar{V}\} \bar{v}_{\eta\eta} \\ + \{M_{\eta\eta} - CM_\eta - 2\sigma^2(M+1) - \bar{V}_\eta + C\bar{V}\} \bar{v}_{\eta\eta} \\ - \{CM_{\eta\eta} + \sigma^2 M_\eta - 2C\sigma^2(M+1) - C\bar{V}_\eta \\ - \sigma^2 \bar{V}\} \bar{v}_\eta + \sigma^2 \{M_{\eta\eta} + \sigma^2(M+1) + \bar{V}_\eta\} \bar{v} \\ = -2Re_m^2 C \sigma^2 \bar{U} \bar{u} \dots\dots\dots (15) \end{aligned}$$

$$\sigma \bar{w} = -v_\eta \dots\dots\dots (16)$$

where the subscript η implies a differentiation with respect to η . The nondimensional variables are $C=y_m/R$, $M=\nu_t/\nu$, $Re_m=Um y_m/\nu$, $\bar{U}=U/U_m$, $\bar{V}=Re_m V/U_m$, $\bar{u}_t=u_t/U_m$, $\bar{v}=Re_m v_t/U_m$, $\bar{w}=Re_m w_t/U_m$, $\eta=y/y_m$ and $\sigma=2\pi y_m/\lambda$. Here y_m is a position of maximum velocity, C is a nondimensional curvature on the basis of y_m and Re_m is a Reynolds number.

The boundary conditions for Eqs. (14) and (15) are expressed as $\bar{u}$, $\bar{v}$ and $\bar{w}=0$ at a surface and at a point far from the wall jet. In view of the continuity equation (16), these conditions are written as follows:

$$\eta=0, \infty: \bar{u}=\bar{v}=\bar{w}=0 \quad (17)$$

3.2 Parameters in perturbation equations

The eddy viscosity ν_t is calculated from the following equation with the experimental results for the Reynolds stress and the mean velocity.

$$-\overline{u'v'} = \nu_t \left\{ \left(1 - \frac{y}{R}\right) \frac{\partial U}{\partial y} + \frac{U}{R} \right\} \quad (18)$$

Generally speaking, the value of ν_t becomes infinity around the position of maximum velocity, where $\partial U/\partial y = 0$. Therefore, the eddy viscosity ν_t in the present study is obtained from the turbulence energy k and dissipation ε via the Kolmogorov-Prandtl expression:

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} \quad (19)$$

Prediction [—: $b/R=0$, ---: $b/R=0.086$, ---: $b/R=0.166$], Experiment [●: $b/R=0$, ○: $b/R=0.086$, ▲: $b/R=0.166$]

Fig. 2 Distribution of mean velocities

Fig. 3 Distribution of Reynolds stresses (For caption see Fig. 2)

where c_μ is determined from the Reynolds stress equations (shown in Appendix) as a function of the distance from a surface and a curvature, instead of a well-known constant $c_\mu=0.09$ ⁽¹¹⁾, to predict the flow characteristics of the concave wall jet.⁽¹²⁾ Then, the expression for c_μ is obtained as Eqs. (34)–(38). As accurate values of the dissipation ε are not available, k and ε in Eq. (11) are obtained from the following equation with curvature effect considered:

$$U \frac{\partial k}{\partial x} + \left(1 - \frac{y}{R}\right) V \frac{\partial k}{\partial y} = \frac{\partial}{\partial y} \left\{ \left(1 - \frac{y}{R}\right) \frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial y} \right\} + \nu_t \left\{ \left(1 - \frac{y}{R}\right) \frac{\partial U}{\partial y} + \frac{U}{R} \right\}^2 - \varepsilon \quad (20)$$

$$U \frac{\partial \varepsilon}{\partial x} + \left(1 - \frac{y}{R}\right) V \frac{\partial \varepsilon}{\partial y} = \frac{\partial}{\partial y} \left\{ \left(1 - \frac{y}{R}\right) \frac{\nu_t}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial y} \right\} + C_{\varepsilon 1} \frac{\varepsilon}{k} \nu_t \left\{ \left(1 - \frac{y}{R}\right) \frac{\partial U}{\partial y} + \frac{U}{R} \right\}^2 - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \quad (21)$$

where the empirical constants are selected as $\sigma_k=0.7$, $\sigma_\varepsilon=1.3$, $c_{\varepsilon 1}=1.44$ and $c_{\varepsilon 2}=1.92$ ⁽¹¹⁾⁽¹²⁾. The equations of motion (1) and (2) are reduced to the following expression by eliminating the pressure p , on the assumption that the flow is steady, two-dimensional and satisfies the thin shear-layer approximation:

$$U \frac{\partial U}{\partial x} + \left(1 - \frac{y}{R}\right) V \frac{\partial U}{\partial y} = \left(1 - \frac{y}{R}\right) \frac{\partial}{\partial y} \left\{ \nu_t \left(1 - \frac{y}{R}\right) \frac{\partial U}{\partial y} + \frac{UV}{R} \right\} + \frac{UV}{R} - \frac{2\nu_t}{R} \times \left\{ \left(1 - \frac{y}{R}\right) \frac{\partial U}{\partial y} + \frac{U}{R} \right\} - \frac{1}{(R-y)} \frac{d}{dx} \int_y^\infty U^2 dy \quad (22)$$

The following boundary conditions⁽¹²⁾ for Eqs. (20)–(22) are adopted, close to a wall surface ($U_\tau y/\nu=30-50$) where the turbulence is in local equilibrium:

$$k = U_\tau^2/0.239 \quad (23)$$

$$\varepsilon = U_\tau^3/(0.461y) \quad (24)$$

$$U/U_\tau = 5.0 \log_{10} (U_\tau y/\nu) + 4.9 \quad (25)$$

and at the outer edge of the wall jet:

Prediction [—], Experiment [●: Kobayashi and Fujisawa⁽¹⁶⁾⁽¹⁷⁾, □: Tailland et al.⁽¹³⁾, ○: Spettel et al.⁽¹⁴⁾, △: Wilson et al.⁽¹⁵⁾]

Fig. 4 Maximum Reynolds stress $(\overline{u'v'})_{\max}/U_m^2$ and curvature b/R

$$k = \epsilon = U = 0 \quad (26)$$

Eqs. (20)–(22) and the boundary conditions (23)–(26) are numerically solved by the finite difference procedure⁽¹²⁾, then k , ϵ , U and V are determined.

Figures 2–4 show comparative studies of prediction by the present turbulence model and experiment^{(13)–(17)}. Figure 2 shows a typical distribution of the mean velocities U . Predicted U on the concave wall jet agrees with the experiment. Figure 3 shows the distribution of Reynolds stresses $\overline{u'v'}/U_m^2$ in the wall jet calculated with Eqs. (18) and (19). The maximum value of the Reynolds stress $(\overline{u'v'})_{\max}/U_m^2$ is shown in Fig. 4, in relation to the curvature b/R . It is noted from Figs. 2–4 that the flow characteristics of the concave wall jet are well predicted in the wall region and in the jet region of $y/b \leq 0.8$, whereas the prediction deviates from the measurements in the jet region of $y/b \geq 0.8$. According to the preliminary calculations for this stability problem, these deviations of the parameters in the region of $y/b \geq 0.8$ could not much affect the stability condition. Therefore, it seems that the stability condition for turbulent wall jet can be well predicted from M , U and V calculated using the present turbulence model.

3.3 Numerical method

The perturbation equations (14), (15) and the boundary conditions (17) are treated as a set of simultaneous differential equations with the curvature C as an eigenvalue and the velocities $\bar{u}$, $\bar{v}$ as eigenfunctions. Here the nondimensional parameters M , U , V and their derivatives with respect to η are

Fig. 5 Neutral stability curve for turbulent wall jet along concave surface ($Re_m=10^4$)

Fig. 6 Minimum curvature $C_{\min}$ and Reynolds number Re_m

determined from the present turbulence model and values of Re_m and σ are given arbitrarily. In the numerical calculation, the velocities $\bar{u}$, $\bar{v}$ in Eqs. (14), (15) and (17) are solved iteratively by the successive approximation procedure, until they converge to some profiles. The convergence is judged from the uniformity of the C values in η direction.

4. Results and Discussion

4.1 Neutral stability curve

Figure 5 shows a relationship between the curvature C and the nondimensional wavenumber $\sigma (=2\pi y_m/\lambda)$ at the neutral stability condition for turbulent wall jets along concave surfaces, where the Reynolds number is set at $Re_m=10^4$. The upper side of the neutral stability curve is an unstable region for the occurrence of Taylor-Görtler vortices by the centrifugal instability. The stability curve in Fig. 5 takes a minimum $C_{\min} \approx 0.0325$ around $\sigma \approx 2.2$. Thus, the turbulent wall jet with concave curvature $C < C_{\min}$ is considered stable all the time.

Figure 6 shows the dependency of the minimum curvature $C_{\min}$ on the Reynolds number Re_m . It is noted that $C_{\min}$ decreases as Re_m increases. Therefore, the minimum curvature, below which the turbulent wall jet is stable all the time, becomes smaller for the wall jet with larger Reynolds numbers, under the condition of the same concave curvature.

4.2 Comparison with experiment

Figure 7 shows experimental results⁽⁵⁾⁽¹⁶⁾⁽¹⁷⁾ about the maximum value of turbulence energy $k_{\max}/U_m^2$ and the curvature b/R for turbulent wall jet along concave surfaces. In the range of curvatures $b/R \leq 0.17$, $k_{\max}/U_m^2$ decreases as b/R increases, which is predictable quantitatively by the present turbulence model described in section 3.2. On the contrary, $k_{\max}/U_m^2$ increases in the curvature range of $b/R \geq 0.27$, indicating an enhancement of turbulent mixing in a strong concave wall jet. At the same time, a secondary flow is observed in the flow visualization experiment with a

Prediction[—], Experiment[●, ○: Kobayashi and Fujisawa⁽⁵⁾⁽¹⁶⁾⁽¹⁷⁾], Theoretical minimum curvature $C_{\min}$ [---]

Fig. 7 Maximum turbulence energy $k_{\max}/U_m^2$ and curvature b/R

strong concave wall jet. Therefore, it is likely that the centrifugal instability occurs in turbulent wall jets along strong concave surfaces⁽⁵⁾. The symbol (o) in Fig. 7 indicates the appearance of a secondary flow in the flow visualization experiment. Thus, the experimental minimum curvature can be considered to be $(b/R)_{\min}=0.17-0.27$ in the Reynolds number range of $Re_m=(0.9-1.3)\times 10^4$. On the other hand, the theoretical minimum curvature in the same Reynolds number range is $C_{\min}=0.032-0.033$, which corresponds to $(b/R)_{\min}=0.20$ with the relation $y_m/b\approx 0.16$ for the concave wall jet⁽¹⁷⁾. Thus, the theoretical minimum curvature for the turbulent wall jet along concave surfaces agrees well with the experiment.

4.3 Discussion on the present theory

The theoretical minimum curvature is obtained as $(b/R)_{\min}=0.43$ at $Re_m=10^4$ assuming that the eddy viscosity ν_t is independent of the curvature. This is much larger than the above mentioned value of $(b/R)_{\min}=0.20$. Therefore, the curvature effect on the stability condition is important and it contributes to a decrease in the minimum curvature $(b/R)_{\min}$ for the occurrence of Taylor-Görtler vortices. The minimum curvature decreases not only when the growth factor of Taylor-Görtler vortices in Eqs. (11)-(13) is treated as time-dependent, but also when the first order terms of curvature C in Eqs. (14) and (15) are neglected, though their decrement is only less than 10 percent in $C_{\min}$. Moreover, the influence of the mean velocity V on the stability is calculated to be small, because the growth rate of the concave wall jet is much smaller than that of the plane wall jet.

5. Conclusions

Stability analysis for the occurrence of centrifugal instability in turbulent wall jets along concave surfaces is made in a comparative experiment. The results obtained are summarized as follows:

(1) The minimum curvature for the occurrence of centrifugal instability in the turbulent wall jet depends weakly on the Reynolds number. It decreases as the Reynolds number increases.

(2) The theoretical minimum curvature is in good agreement with the experiment, in which a secondary flow is produced and the turbulent mixing is enhanced in the wall jet.

(3) In the stability analysis of turbulent wall jet, it is important to treat the eddy viscosity as a function of curvature as well as to express it with the turbulence energy and dissipation.

The authors would like to express gratitude to Professor Kobayashi, R. of Tohoku University and also to Takakusagi, F. of Gunma University and also Sato, T. a student of Gunma University. The

authors gratefully acknowledge research support from Gunma University Foundation for Science and Technology.

Appendix

The equations for Reynolds stresses $(\overline{u_i'v_j'})$ are written for high Reynolds numbers as follows,

$$\begin{aligned} \text{Ad}(\overline{u_i'v_j'}) - \text{Dif}(\overline{u_i'v_j'}) = & \\ & -\overline{u_i'u_i'}\frac{\partial U_j}{\partial x_i} - \overline{u_j'u_i'}\frac{\partial U_i}{\partial x_i} \\ & + \frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) - \frac{2}{3}\varepsilon\delta_{ij} \quad \dots\dots\dots(27) \end{aligned}$$

where $\text{Ad}(\overline{u_i'v_j'})$ and $\text{Dif}(\overline{u_i'v_j'})$ are advection and diffusion of the Reynolds stresses $(\overline{u_i'v_j'})$, respectively. The local isotropic expression⁽¹⁸⁾ is adopted in the dissipation term in Eq. (27). $\text{Ad}(\overline{u_i'v_j'})$ and $\text{Dif}(\overline{u_i'v_j'})$ are approximated by the following relation⁽¹⁹⁾,

$$\begin{aligned} \text{Ad}(\overline{u_i'v_j'}) - \text{Dif}(\overline{u_i'v_j'}) = & \frac{\overline{u_i'u_j'}}{k}\{\text{Ad}(k) - \text{Dif}(k)\} \\ & \dots\dots\dots(28) \end{aligned}$$

where $\text{Ad}(k)$ and $\text{Dif}(k)$ are advection and diffusion of turbulence energy k . Using the near-wall correction formula⁽²⁰⁾⁻⁽²³⁾ to express the pressure-strain correlation (the third term of right hand side in Eq. (27)), the equation for the Reynolds stress $\overline{u_i'v_j'}$ is obtained from Eqs. (27) and (28) as follows:

$$\begin{aligned} -\overline{u_i'v_j'} = & \left\{ \frac{1 - C_2(1 - 1.5C_2f)}{C_1(1 + 1.5C_1f/C_1) + P_1/\varepsilon - 1} \right\} \frac{\overline{v'^2} \cdot k}{\varepsilon} \\ & \times \left\{ \left(1 - \frac{y}{R}\right) \frac{\partial U}{\partial y} + \frac{U}{R} + 2\left(\frac{\overline{v'^2}}{\overline{u'^2}} - 1\right) \frac{U}{R} \right\} \quad \dots\dots\dots(29) \end{aligned}$$

On the other hand, $\overline{u_i'v_j'}$ is written as follows using Eqs. (18) and (19),

$$-\overline{u_i'v_j'} = C_\mu \frac{k^2}{\varepsilon} \left\{ \left(1 - \frac{y}{R}\right) \frac{\partial U}{\partial y} + \frac{U}{R} \right\} \quad \dots\dots\dots(30)$$

Then, the following expression for c_μ is obtained from Eqs. (29) and (30):

$$\begin{aligned} C_\mu = & \frac{(1 - C_2)\{1 + 1.5C_2C_2f/(1 - C_2)\}}{(C_1 + P_1/\varepsilon - 1)\{1 + 1.5C_1f/(C_1 + P_1/\varepsilon - 1)\}} \\ & \times \frac{\overline{v'^2}}{k} \left\{ 1 + \frac{2U(\overline{u'^2}/\overline{v'^2} - 1)}{(R - y)\frac{\partial U}{\partial y} + U} \right\} \quad \dots\dots\dots(31) \end{aligned}$$

where,

$$P_1 = -\overline{u_i'v_j'} \left\{ \left(1 - \frac{y}{R}\right) \frac{\partial U}{\partial y} + \frac{U}{R} \right\} \quad \dots\dots\dots(32)$$

$$f = k^{1.5}/(C_w y \varepsilon) \quad \dots\dots\dots(33)$$

Here the empirical constants are $c_1=1.8$, $c_2=0.6$, $c_1'=0.6$, $c_2'=0.3$ and $c_w=3.72$ ⁽¹²⁾. Substituting the equations of $\overline{u'^2}$ and $\overline{v'^2}$ (obtained from Eqs. (27) and (28) in a similar way to $\overline{u_i'v_j'}$) into Eq. (31), the final expression for c_μ is obtained as follows:

$$C_\mu = 0.09 G_1 G_2 \{1 - (G_3 - 1) R_i\} \quad (34)$$

$$G_1 = \frac{1 + 1.5 C_2 C_3 f / (1 - C_2)}{1 + 1.5 C_1 f / (C_1 + P_1 / \varepsilon - 1)} \quad (35)$$

$$G_2 = \frac{1 - 2 C_2 C_3 f P_1 / \varepsilon / (C_1 - 1 + C_2 P_1 / \varepsilon)}{1 + 2 C_1 f / (C_1 - 1 + P_1 / \varepsilon)} \quad (36)$$

$$G_3 = C_3 \left\{ 1 + \frac{(C_2 C_3 P_1 / \varepsilon + 1.5 C_1 G_2) f}{C_1 - 1 + (3 - 2 C_2) P_1 / \varepsilon} \right\} / G_2 \quad (37)$$

$$R_i = \frac{-2U/R}{(1 - y/R) \partial U / \partial y + U/R} \quad (38)$$

where C_μ agrees with the empirical constant 0.09⁽¹¹⁾ for the free jet assuming that the wall function $f=0$ and $R \rightarrow \infty$. An empirical constant $C_3 = 1.2$ is determined from the experimental relation for u^2 and v^2 in plane wall jet⁽¹⁶⁾.

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