On the Static Running Accuracy of Ball Bearings*
(2nd Report, Results and Discussions)

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A comprehensive expression of solutions is obtained for the ball passage vibration [running accuracy] problem of a ball bearing which is loaded in a radial direction and rotated slowly. The results offer the general relation between characteristics of the vibration [Fourier coefficients, peak to peak amplitudes, distortion of wave form] and dominant factors (specification of the bearing, radial clearance, and the load). A procedure is proposed to evaluate the relation between the clearance, and amplitudes in actual dimensions, using the chart of general expression.

Key Words: Vibration of Rotating Body, Bearing, Ball Bearing, Running Accuracy, Machine Element

1. Introduction

In a ball bearing, a number of balls (usually 6-14) are arranged discretely to rotate with the running of the inner ring (shaft), and to support the radial load as a rotor bearing. The number of effectively stressed balls and the load distribution fluctuate continuously and in a complicated fashion. The effect of the strongly nonlinear spring characteristic (1.5 power type due to Hertzian contact with clearance) between ball and inner/outer rings is added to this situation, and the inner ring executes a very complicated radial plane motion (though not very large), even if the radial load is constant. The problem is referred to as the running accuracy problem[1] of a ball bearing, which was raised by Perret and Meldau[2] in the 1950s. In this model, neither the inertia nor the frictional forces are taken into consideration, and so it should be referred to as a static one. An analysis considering these effects, i.e., the dynamic running accuracy problem [3], has been recently proposed and the static one is shown to play a central role in the dynamic one and is examined further. The dominant factors on this subject are the geometrical dimension reduced to and expressed by the number of balls and the elasticity factor[4], the radial clearance, and the radial load (constant) of the ball bearing. The phenomena and/or the results appear as the radial plane motions of the inner ring (expressed by the displacement components parallel with and perpendicular to the direction of the radial load), which are expressed as functions of revolving angle (measured from the radial load line) of a ball. Since the solution is defined by an implicit function of the nonlinear equation shown later, a series of complicated procedures must be employed in order to examine the effects of dominant factors on the solution.

Only some numerical examples of the solution of the equation are known today[4,5] and the effects of the dominant factors are only presumed to be complicated. Despite the many problems left to be solved, however, this phenomenon is considered to be the most serious of the vibration troubles[6] due to ball bearings and it is an established fact that vibration may occur catastrophically even in an ideal ball bearing. The course of solution is not easy because there is no way available to date other than solving the equation numerically by case, in order to estimate the magnitude of the vibration and to determine how the dominant factors govern the phenomena.

The authors, therefore, have set out to work out a comprehensive explicit expression showing clearly at a glance the corresponding relationship between characteristic parameters (p, q amplitudes, Fourier coefficients, and wave form factors) and the dominant factors. In the previous paper[7] therefore, they presented the solution and arranged some material expressing the results.

In this paper, as extensions of the previous study, further considerations are made and the results are presented to complete the solutions. Discussions are also made on a numerical example of a ball bearing type 6304/11S to analyse the complexity and the peculiarity of the phenomena as a whole.

The problem is stated as follows[4,5,7]: In the relationship:

\[
\begin{bmatrix}
X \\
Y
\end{bmatrix} = C \cdot \sum \left[ x \cos \theta_i + y \sin \theta_i - \gamma \sin \theta_i \right] \left[ \cos \theta_i \right] \\
\sin \theta_i
\]

where,
\[ \theta_j = \frac{\pi}{j - 1}, j = 1, 2, \ldots, N, \]
\[ N, C_\text{r}, \gamma_0 \text{ constants} \]
solve
\[ x, y = f(\theta_j; W, \gamma_0, N, C_\text{r}) \]
and solve also
\[ \frac{\partial X}{\partial x}, \frac{\partial Y}{\partial y}, \frac{\partial X}{\partial \theta_j}, \frac{\partial Y}{\partial x} \]
under the conditions of
\[ X = W > 0 \text{ constant}, Y = 0 \]

2. Expression of the Solutions

Nomenclature (see Fig. 1)
- \( C_\text{r} \): elasticity factor of the ball bearing
- \( N \): total number of balls
- \( W \): constant radial load acting on the inner ring from the exterior
- \( X, Y \): components of the force acting on the inner ring from the balls (including the restoring force of the bearing as a spring)
- \( x, y \): co-ordinates in the radial plane, and orthogonal components of the displacement of the inner ring

\[ f = \gamma_0(NC_\text{r}/W)^{\frac{1}{2}} \text{ non-dimensional characteristic value} \]
\[ \Gamma = N^{\frac{1}{2}}[\cos \pi(N - 1)/2] \text{ critical condition in } \Gamma, \text{ where only two balls are loaded} \]
\[ \gamma_0 \text{ mean radial clearance (defined in JIS)} \]
\[ \theta_j \text{ rotational angle of a ball } j \text{ [see Eq.(2)]} \]

\[ \Gamma > \Gamma = f(N) \text{ holds only one or two balls located below the inner ring are loaded and the solution is obtained in a closed form} \]

The solution of Eq.(1) is generally expressed as follows:

\[ f = \left( \frac{W}{NC_\text{r}} \right)^{\frac{1}{2}} f \left( \frac{\xi}{\eta} \right) \text{ where,} \]
\[ \xi = \left( \begin{array}{c} \xi_0 \\ \xi_x \\ \xi_0 \sin \theta_j \\ \xi_x \cos \theta_j \end{array} \right) \]
\[ \eta = \left( \begin{array}{c} \eta_0 \\ \eta_x \\ \eta_0 \sin \theta_j \\ \eta_x \cos \theta_j \end{array} \right) \]

The mean value of the displacement, \( \xi_x \) (non-dimensional) is a function of \( N, \Gamma \) strictly. However, a practical formula with sufficient accuracy, which is apparently independent of the number of balls \( N \), may be as follows:

\[ \xi_x = \Phi_1(\Gamma) \]

Spring stiffnesses (Eq.(4)) are also expressed in the Fourier series in terms of \( N\theta_0 \), which are similar to Eq.(5). The mean values in the Fourier series of stiffnesses may be essential and are expressed similarly to Eq.(6), as follows:

\[ \xi_x = \Phi_1(\Gamma) \]

The Fourier coefficients \( \xi_x, \eta_x \) are expressed as functions in terms of \( \Gamma \) for each \( N \). In addition, the following characteristic values and their functions of \( N, \Gamma \) are defined to express the salient features of the wave form.

\[ p-p \text{ values: } [\xi_{pp}] = \max_{\xi} - \min_{\xi} \]
\[ \text{Distortion factor: } [\xi_{ex}] = \frac{\sqrt{1 - \xi_0^2}}{\xi_0} \times 100\% \]

The method of obtaining \( C_\text{r} \), function tables and/or the approximation formula of \( \Phi_1(\Gamma), \Phi_0(\Gamma) \), numerical results of \( \xi_0, \xi_x, \eta_0, \eta_x, \xi_{pp}, \eta_{pp}, \xi_x, \eta_x \) for \( N = 6, 8, 10 \) and the wave forms, \( \xi(\theta_j), \eta(\theta_j) \) for \( N = 6 \) have already been presented in the previous paper [1].

3. Results of Computation and Discussion

Figure 2 shows ten typical sets of computed values of \( \xi(\theta_j), \eta(\theta_j) \) for the number of balls, \( N = 7 \).

Fig. 1 Geometry and co-ordinates of a ball bearing.

As in the previous paper [1], each set of four pictures corresponds to one value of \( \Gamma \). The upper and lower left hand pairs indicate \( \eta(\theta_j) \) and \( \xi(\theta_j) \), respectively, whose abscissa is \( N\theta_0/2\pi = \frac{1}{2} \). The lower right, shows Lissajou's pattern \( \eta - \xi \) in a regular square frame. These three are all the same magnitude for \( \xi \) as well as \( \eta \). The upper right figure is Lissajou's pattern \( -\xi - \eta \) in which p-p spans in \( \xi \) and \( \eta \) are shown with the same magnitude deliberately to illustrate the features of the pattern in detail. Figure 3 shows \( \xi_0, \xi_x, \eta_0, \eta_x, \xi_{pp}, \eta_{pp}, \xi_x, \eta_x \) as functions of \( \Gamma \) for the case of \( N = 7 \), from which the conditions in Fig.2 are selected. In the figure, the parts marked "s" correspond to Fig.2 in the \( \Gamma \) value. The full lines indicate \( \xi_0, \eta_0 \) of which those marked "n" correspond to the positive value and those without the mark to the negative value, the numerals beside the curves stand for the order number \( s \), and \( \varepsilon_x, \varepsilon_y \) mean the values in percentage. The case of \( N = 7 \) (odd number) is shown here while the case of \( N = 6 \) (even number) was shown in the previous paper [1].
the results it is difficult to find any difference in the inherent characteristics between odd and even numbers of balls. By examining all the results obtained here together with those in the previous paper, as well as those for \(N=11-14\) shown in Appendixes, the general characteristics of the static running accuracy problem (including the discussions in the previous paper) are clarified and summarised as follows:

1. There exist maxima and minima of the p-p value, \(\eta(p, q, p')\) with respect to \(p'\); the number of which is \((N - 3)\), where the parts of monotonic change in the re-

\[\text{Fig. 2 Radial motion of inner ring, vertical } \eta(\theta), \text{ horizontal } \xi(\theta), \text{ where number of balls } N=7\]
Fig. 3 Fourier coefficients $\xi, \xi_0, \eta_0$, p-p amplitudes ($\xi p, \eta p$), distortion factors ($\xi, \xi_0$) of inner ring motion, as functions of $\Gamma$ where $N=7$ (mark "*" corresponds to Fig. 2).

Fig. 4 p-p amplitudes $\xi p, \eta p$ of inner ring motion, as functions of $\Gamma$ ($N=6-14$).
gions $\Gamma \to \infty$ are not taken into account. The maxima and the minima of the dimensionless variables $\xi_{\Gamma}, \eta_{\Gamma}$ with respect to $\Gamma$, which are obtained back from $\xi_{\Gamma}, \eta_{\Gamma}$ do not always coincide with those of $\xi_{\Gamma}, \eta_{\Gamma}$ with respect to $\Gamma$.

(2) The wave form or the pattern (not the magnitude) of $\xi(\Gamma), \eta(\Gamma)$ where $\Gamma < \Gamma \to \infty$, on one or two balls may be loaded, approaches a particular shape. The shapes may be described as "a suspension bridge" [harmonic amplitude $\propto (\text{order})^2$], wave form with corners) to the motion perpendicular to the load (horizontal). These are typically represented by the case of $\Gamma = 10$ in Fig. 2. The sufficient features of the wave form appear clearly in Fig. 3 where $\xi, \eta < 0$ for $\Gamma = 10$. They form a family of almost parallel lines (on the quasi-logarithmic plane), both $\xi, \eta$ converge to constants, and their amplitudes are almost proportional to $\Gamma$.

(3) In accordance with the change of $\Gamma$ (where $\Gamma < \Gamma$), in particular, all the values of amplitudes change their sign periodically such as $\ldots, 0, 0, 0, \ldots$, and the changing frequency is almost proportional to the order number, $k$. The relations $\xi, \eta > 0$ are valid when $\Gamma = \Gamma$, and always $\xi, \eta > 0$ when $\Gamma > \Gamma$. Moreover, with an increase of $\Gamma$, $\Gamma < \Gamma \to \infty$, $\xi, \eta$ change their sign only once in order of $k = 1, 2, \ldots$, and finally settle to the state of $\xi, \eta < 0$. The minima of $\xi, \eta$ value almost coincide with the points of $\Gamma$ where the fundamentals $\xi, \eta$ vanish while $\xi, \eta$ reach 100% at the same points.

(4) As an approximate trend, magnitude levels of all amplitudes go down with an increase of the order number as well as with $\Gamma \to \infty$, although rather complicated relationships occur locally.

(5) $\xi_{\Gamma}, \eta_{\Gamma}$ decrease quickly and monotonically in the region where $\Gamma < 1.245$, and $\xi(\Gamma), \eta(\Gamma)$ approach simple harmonic waves (since $\xi, \eta \to 0$). This special value indicates a critical state in which all the balls are stressed despite $\Gamma = 0$ (stressed where $\Gamma < 1.245$), and it is approximately independent of $N$.

(6) In Figs. 3 and 6, the amplitudes of the inner ring motion $\xi_{\Gamma}, \eta_{\Gamma}$ are the most interesting from a practical point of view. Figure 4 shows, therefore, those values in gathered form of a chart for the cases of $N = 1$.

The results do not always indicate $\Gamma$ values with dimension, since a parameter $N$ is related to $\Gamma$ as well as the multiplier $(W/NC)^{1/2}$ which converts $\xi_{\Gamma}, \eta_{\Gamma}$ (non-dimensional) to $\xi_{\Gamma}, \eta_{\Gamma}$ (dimensional).

(7) The functions of the inner ring motion and the corresponding differential stiffness, in terms of ball rotation angles $\delta$, are governed by parameters $\Gamma$ and $N$, the functional magnitude of the former of which is proportional to $(W/NC)^{1/2}$ and the latter to $[W/NC]^2$. Since the effect of $W$ is rather complicated, it is more convenient to redraw the chart to estimate the effects of dominant factors on the results in detail, in the dimensional relationship. Figure 3 shows an example.

4. Relationship between Radial Load, Radial Clearance and the Inner Ring Motions (Numerical Examples)

Figure 3 has the merit of demonstrating all the conditions and the states in one chart. On the other hand, it may be inconvenient in that some procedures are necessary to obtain direct relationships between the dominant factors and the motions with dimensions, for a given actual bearing, since the chart is expressed in non-dimensional terms.

Figure 5 shows computed results of the inner ring motion, $\xi_{\Gamma}, \eta_{\Gamma}$ as functions of the radial clearance, $2\gamma$ μm, and the radial load, $W$, in which all of the values are dimensioned. The value of elasticity factor, $G = 11.9 N/mm^2$ is calculated by the formula shown in Appendix (2) using geometrical dimensions actually measured. The value $C$, and the number of balls $N$ may be generally variable into the manufacturer, the design and production lot, and may not be considered to have the values absolutely constant specific to the model No.6304/JIS. The ranges of the parameters are $W = 1~1000 N$, $2\gamma = 10~50$ μm and all the cases which might occur in practice are included completely in this chart. The value $2\gamma$ particularly covers the range of radial clearances in the standard, also the pre-load state (see Chapter 6). Comments are summarised as follows:

(1) In the case of $W = 1N$ almost all correspond to the state of only one/two balls loaded ($\Gamma = \Gamma$).

(2) In the case of $2\gamma = 0, \xi_{\Gamma}, \eta_{\Gamma}$ are proportional to $W$, and grow to a considerable magnitude as $\Gamma \to \infty$.

(3) It is rather difficult to state simply the condition of maxima/minima with respect to $W, 2\gamma$, and the overall trends, but the magnitude of the motion in the direction of radial load (vertical) $x, y, z$, seems to be about 1/10 of that in the perpendicular direction (horizontal) $\xi_{\Gamma}, \eta_{\Gamma}$.

(4) The thrust pre-load $(\delta \gamma < 0)$ may be quite effective to suppress the vibration due to ball passage, when radial load is not so large.

5. Law of Similarity

Figure 5 is so made that it corresponds with the bearing 6304/JIS and the patterns are appropriate to the case of $N = 7$. Keeping the condition $N = 7$ and making some modification to the scale value
of \( W \) and \( 2\gamma_s \), the chart may be applied for those cases in which the value of \( C_r \) is different from that for 6304/JIS. The conversion is accomplished according to the following rule.

1. Multiply the scale value of \( 2\gamma_s \) by \( m' \).
2. Multiply the factor \( C_r \) by \( m' \).
3. Multiply the scale value of \( W \) by \( n'^4m' \).
4. Multiply the scale value \( x_{pp}, y_{pp} \) by \( n' \).

and/or,

1. Multiply the scale value of \( W \) by \( n' \).
2. Multiply the factor \( C_r \) by \( m' \).
3. Multiply the scale value of \( 2\gamma_s \) by \( (n'/m')^{-3} \).
4. Multiply the scale value \( x_{pp}, y_{pp} \) by \( (n'/m')^{-3} \).

It is not always guaranteed that the converted chart correctly fits the suitable range in the scale of \( W \) and/or of \( 2\gamma_s \), \( 2\gamma_s \) (one has merely to employ the standard to set the range) monotonically increases with an increase of geometrical dimension of the bearing. The same holds for \( C_r \), and both are almost in accordance with their trends of magnitude.

6. The Case of Pre-load

The state of pre-load at zero contact angle implies \( \Gamma < 0 \). The ball bearing with single row deep groove is frequently installed in a similar state to the angular contact bearing by applying some axial thrust (pre-shift between outer/inner ring). The above mentioned results can be applied approximately to this situation.

\( \alpha \) : contact angle between ball/running surface (equally in the outer as well as inner ring).

\( K_a \) : thrust load.

\( \varepsilon \) : relative displacement between inner and outer rings in the axial direction.

\( d_a = d(A_1 + A_2)/2 \) : characteristic length of ball bearing [see Appendix (2) for \( d, A_1, A_2 \)].

\[ \text{(10)} \]

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**Fig. 5** Dimensional characteristics: \( x_{pp}, y_{pp} \) of inner ring motion, as functions of radial clearance \( 2\gamma_s \) and load \( W \) for 6304/JIS.
$d_0$ represents the distance between centres of curvature of respective running surfaces, when $a=0$. Assuming inherent radial clearance $\gamma_0$ to be zero as in a single row radial bearing, we have

$$
\varepsilon = d_0 \sin \alpha \approx d_0 a \tag{11}
$$

$$
K_a = NC \left( d_0 (1/\cos \alpha - 1) \right)^{1/2} \tan \alpha
\approx NC \left( \frac{d_0}{2} \right)^{1/2} a \sqrt{\frac{2a^2}{d_0}} \tag{12}
$$

Fig. 6 Characteristic values of inner ring motion $\xi_n, \xi_r, \eta_n, \eta_r, \eta_{\theta}, \xi, \xi_r$

(number of balls, $N=11, 12, 13, 14$)
Assuming $a > 0$, the effective radial clearance (always negative) is obtained as follows,

$$r_e = -a(1 + \csc \theta - 1) = -a \csc \theta / 2$$

(13)

The apparent clearance $r_e$ may be substituted into $r_e$ in the aforementioned formulae when $K_a$ or $z$ is given.

7. Conclusions

Continuing from the previous study, the so-called (static) running accuracy problem in a ball bearing is treated and a general expression is presented, in which the relationships between characteristic values (Fourier coefficients, p-p amplitudes, and wave form distortion factors) and the dominant factors (number of balls, elasticity factor, radial clearance, and radial load) are explicitly shown. The charts expressing the results together with the previous ones cover all cases from $N=6$ to $N=14$ (9 kinds), and they may be assumed to offer perfect materials for the problem in practice. Based on the chart (non-dimensional), illustrative figures are presented in which the correspondences between radial clearance, radial load, and p-p amplitudes in the actual dimensions are shown, and the complicated and peculiar nature of the phenomena is clarified. Furthermore, a law of similarity is recognized concerning the chart with actual dimensions and the chart is applied for all the cases of $N=7$ almost independently of the differences in $C_2$, only by adjusting parameter scales. This suggests the possibility of using the general chart with actual dimensions (rather than non-dimensional) and this problem will be discussed in detail in a separate paper. According to the paper [1], the results presented here play a central role in the theory of (dynamic) running accuracy problem of the ball bearing, in which the inertia as well as frictional forces are additionally taken into consideration.

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Appendix

(1) Characteristic values of the inner ring motion (Fig. 6)

Fourier coefficients ($\xi_0$, $\xi$, $\eta_0$, $\eta$), p-p amplitudes ($\xi_p$, $\eta_p$), distortion factor of wave form ($\xi_0$, $\xi$, $\eta$) are presented as functions of $N$, for the cases of number, $N = 11, 12, 13, 14$, similarity to Fig. 3.

(2) Computation of $C_2$

Revised version of the previous one [2], which may be more convenient for practical use.

Nomenclature

- $D$: diameter of pitch circle of ball revolution orbit, [mm]
- $d$: diameter of a ball, [mm]
- $E$: Young's modulus (+207700 MPa), [MPa]
- $r_i$: curvature radius of inner ring, [mm]
- $r_o$: curvature radius of outer ring, [mm]
- $\nu$: Poisson's ratio ($0.3$)

Suffixes:
- $i$: inner ring
- $o$: outer ring

$$\lambda = 2r_0/d - 1,$$
$$\rho = d/D,$$
$$\psi = (1 + \rho)/(1 + \lambda \rho),$$

where upper/lower signs correspond to suffixes.

$$\delta = \sqrt{\frac{1}{1 + 2\lambda \psi}}$$

For the function $\delta(F)$, see the previous paper [1],

$$C_2 = \frac{2E\sqrt{3} \left[ \frac{1}{(\psi_0 + \psi_0)^{1.5}} \right]}{3(1 + \rho^2)} N/mm^1$$

$$= 15020 \sqrt{\frac{d}{(\psi_0 + \psi_0)^{1.5}}} N/mm^1$$

$C_2$ is obtained approximately as follows:

$$\lambda = (\lambda_i + \lambda_o)/2, (0.002 < \lambda < 0.2),$$
$$\rho = \rho^{1.5}, (0 < \rho < 0.3),$$
$$\mu = \log(\lambda d / \lambda_i), (|\mu| < 1),$$
$$C_2 = \frac{3.84(1 + 0.17\mu^{0.17})}{d/\lambda^{1.5}} \left[ \frac{1 + 0.86\mu^{0.17}}{d/\lambda^{1.5}} \right]$$

The error increase will be the same as previously pointed out when the denominator is omitted.

References

(4) Meldau, E., Werkstatt Bctr., 44-7 (1951), 308.