Practical Formula for Tooth Deflection of Internal Spur Gear*

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In the present paper, the tooth deflections of internal spur gears with various numbers of teeth and addendum modification coefficients were calculated by 2-dimensional elasticity theory and mapping function. The results were compared with those by 2-dimensional finite element method (FEM) and the formula proposed by Hidaka, et al., and the validity of tooth deflection analysis of internal spur gear by the theory of elasticity was confirmed. The tooth deflections of internal spur gears due to Hertz contact calculated by the theory of elasticity were compared with those by Lundberg's and Weber's formula. Furthermore a practical formula for tooth deflection of internal spur gear was derived on the basis of these calculated results.

Key Words : Gear, Internal Gear, Tooth Deflection, Hertz Contact, Theory of Elasticity, Mapping, FEM, Practical Formula

1. Introduction

Recently with increased requirements for higher reduction ratio, size-down and lighter weight, planetary gear units have come increasingly into use and not only their load bearing capacity but also their dynamic behavior is now a matter of great concern. In order to advance the study on the dynamic behavior of planetary gear unit, it would be necessary, first of all, to clarify the tooth deflections of external and internal gears. So far, many studies on the tooth deflection of external gear have been published and some practical formulas^{(1)~(4)} for tooth deflection of external gear are derived, but the studies on the tooth deflection of internal gear are only a few⁽¹⁾⁽⁵⁾ and a practical formula for tooth deflection of internal gear is derived only for the rack tooth which is substituted for the internal gear⁽¹⁾. For the tooth deflection analysis of internal gear the finite element method (FEM) and the boundary element method (BEM) are available, but these analysis methods require much time and labor to prepare the models and the input data.

In the present paper, the tooth deflections of internal spur gears with various numbers of teeth and addendum modification coefficients were calculated by 2-

dimensional elasticity theory and mapping function, which Terauchi and Nagamura⁽⁶⁾ had used in the tooth deflection analysis of external spur gear. These calculated results were compared with those by FEM and the formula proposed by Hidaka, et al.⁽⁵⁾, and the validity of tooth deflection analysis of internal spur gear by the theory of elasticity was confirmed. The tooth deflections of internal spur gears due to Hertz contact calculated by the theory of elasticity were compared with those by Lundberg's⁽⁷⁾ and Weber's⁽⁸⁾ formula. Furthermore a practical formula for tooth deflection of internal spur gear was derived on the basis of these calculated results.

2. Tooth Deflection Analysis by 2-dimensional Elasticity Theory and Mapping Function

2.1 Fundamental equations⁽⁶⁾

In this section, fundamental equations for calculation of tooth deflection are explained.

By using $z = x + iy$, $\bar{z} = x - iy$ and analytic functions $\varphi_1(z)$, $\psi_2(z)$, an equation for deflection is given as follows⁽⁹⁾.

$$u + iv = \frac{1}{2G} \left\{ \frac{3-\nu}{1+\nu} \varphi_1(z) - z \bar{\varphi}_1'(\bar{z}) - \bar{\psi}_2'(\bar{z}) \right\} \dots (1)$$

where u , v : deflections in x and y directions, G : shear modulus, ν : Poisson's ratio and $\varphi_1(z)$, $\bar{\psi}_2(\bar{z})$ denote the conjugate functions of $\varphi_1(z)$, $\psi_2(z)$.

For the conformal mapping of a tooth profile of the semi-infinite plate (z -plane : $z = x + iy$) on a straight boundary (ζ -plane : $\zeta = \xi + i\eta$), a mapping function is given by Eq.(2). The mapped profile, which is obtained by choosing adequately the coefficients c , $a_1 \sim a_7$, b_1

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$\sim b_7$ in Eq.(2), fits well the tooth profile of internal spur gear with various numbers of teeth and addendum modification coefficients⁽¹⁰⁾.

$$z = \omega(\xi) = c\xi - \sum_{i=1}^7 \frac{b_i}{\xi + a_i} \quad \dots\dots\dots(2)$$

Substitution of Eq.(2) into Eq.(1) yields the following equation.

$$u + iv = \frac{1}{2G} \left\{ \frac{3-\nu}{1+\nu} \varphi(\xi) - \omega(\xi) \frac{\bar{\varphi}'(\xi)}{\bar{\omega}'(\xi)} - \bar{\psi}(\xi) \right\} \quad \dots\dots\dots(3)$$

where $\varphi_1(z) = \varphi_1(\omega(\xi)) = \varphi(\xi)$, $\bar{\varphi}_1(\bar{z}) = d\bar{\varphi}_1/d\bar{z} = \bar{\varphi}'(\xi)/\bar{\omega}'(\xi)$, $\bar{\psi}_2(\bar{z}) = \bar{\psi}_2(\bar{\omega}(\xi)) = \bar{\psi}(\xi)$.

Analytic function $\mathcal{Y}(\zeta)$ is given by the following equation⁽¹⁰⁾.

$$\left. \begin{aligned} \varphi(\xi) &= \varphi_0(\xi) + \frac{1}{2\pi i} \int_{\Gamma} \frac{\bar{\varphi}'(-\lambda)}{\bar{\omega}'(-\lambda)} \frac{c\lambda - \sum_{i=1}^7 \frac{b_i}{\lambda + a_i}}{\lambda - \xi} d\lambda \\ \varphi_0(\xi) &= -\frac{P}{2\pi} e^{i\theta} \ln(\xi - \mu) \end{aligned} \right\} \quad \dots\dots\dots(4)$$

Equation (5) is obtained by calculating the residue at singular points of the integral on right-hand side of Eq.(4).

$$\varphi(\xi) = \varphi_0(\xi) + \sum_{i=1}^7 \frac{b_i}{\xi + a_i} \frac{\bar{\varphi}'(a_i)}{\bar{\omega}'(a_i)} \quad \dots\dots\dots(5)$$

where

$$\bar{\omega}'(a_i) = c + \sum_{j=1}^7 \frac{b_j}{(a_i + a_j)^2} \quad \dots\dots\dots(6)$$

$$\varphi'(a_i) = \varphi'_0(a_i) - \sum_{j=1}^7 A_{ij} \bar{\varphi}'(a_j) \quad \dots\dots\dots(7)$$

$$A_{ij} = \frac{b_j}{(a_i + a_j)^2 \bar{\omega}'(a_j)} \quad \dots\dots\dots(8)$$

$\mathcal{Y}'(a_i)$ and $\bar{\mathcal{Y}}'(a_i)$ are obtained by solving a system of equations which consists of Eq.(7) and the conjugate equations of Eq.(7), and then $\mathcal{Y}(\zeta)$ is obtained from Eq.(5). Analytic function $\psi(\zeta)$ is given by the following equation⁽¹⁰⁾.

$$\psi(\zeta) = \frac{P}{2\pi} e^{-i\theta} \ln(\zeta - \mu) - \sum_{i=1}^7 \frac{b_i}{a_i - \zeta} \frac{\varphi'(a_i)}{\omega'(a_i)} - \frac{\bar{\omega}(-\xi)\varphi'(\xi)}{\bar{\omega}'(\xi)} \quad \dots\dots\dots(9)$$

It is assumed in the present deflection analysis that the direction of load application is normal to the tooth surface and the load takes a Hertz contact stress distribution, since the frictional

Fig.1 Approximation of contact stress distribution

force between meshing gear teeth is not considered. This contact stress distribution is approximated by a group of n concentrated loads as shown in Fig.1.

2.2 Mapped profiles fitting tooth profiles of internal spur gear

The mapped profiles used in the tooth deflection analysis of internal spur gears by the theory of elasticity are expressed by the mapping functions obtained in ref. (11). These mapped profiles correspond to eleven tooth profiles of internal spur gears whose dimensions consist of module $m = 4$, pressure angle $\alpha_0 = 20^\circ$, number of teeth $z_i = 45, 60, 90, 180$, addendum modification coefficient $\alpha = -0.5, 0, 0.5$. The mapped profile obtained fits well the tooth profile of internal spur gear at both tooth surface and root fillet as shown in Fig.2. The coefficients of mapping functions are described in ref.(11).

3. Calculated Results and Discussions

3.1 Comparison between tooth deflections by 2-dimensional elasticity theory, FEM and formula proposed by Hidaka, et al.⁽⁵⁾

Figure 3 shows the tooth deflections

$m = 4, \alpha_0 = 20^\circ, z_i = 45, \alpha = 0$

Fig.2 Internal gear tooth and mapped profiles

$m = 4, \alpha_0 = 20^\circ, z_i = 45, \alpha = 0$ (Pitch point loading)

Fig.3 Calculated tooth deflection [Pn/b = 98 N/mm {10 kgf/mm}]

of the internal spur gear of $m = 4$, $\alpha_0 = 20^\circ$, $z_i = 45$ under the pitch point load $P_n/b = 98 \text{ N/mm}$ $\{10 \text{ kgf/mm}\}$ (b : face width) calculated by 2-dimensional elasticity theory and FEM⁽²⁾. In the tooth deflection analysis by the theory of elasticity, the load at pitch point for the gear pair of $z_i/z_p = 45/25$ (z_i, z_p : numbers of teeth of internal and external spur gear) is approximated by a group of ten concentrated loads (Fig.1, $n = 10$). In the FEM analysis, the rim thickness of the model is chosen as $t_w = 8m$ (m : module), regarding the model as solid gear on the basis of the results of ref.(12) (the root stress for this model computed by FEM agrees well with the measured one). It is seen from Fig.3 that the tooth deflection by the theory of elasticity agrees well with that by FEM. The direction of displacement at the loading point due to the load application differs from that of the line of action. The deformation due to Hertz contact occurs near the loading point. The tooth deflections at the loading point due to a group of ten concentrated loads (Fig. 1, $n = 10$) are only 0.3% larger than those of thirty ones ($n = 30$) and fifty ones ($n = 50$). The following analyses are carried out for $n = 10$.

Figure 4 shows the tooth deflections δ_t , δ_c in the direction of the line of action at the loading point and at the intersection between center line of tooth and line of action obtained from the result of Fig.3. In Fig.4 the tooth deflections calculated by the formula proposed by Hidaka, et al. (deflection by Karas' formula⁽¹⁾ + deflection due to elastic deformation at built-in end by Ishikawa's formula⁽²⁾) are also indicated. The tooth deflections calculated by the formula proposed by Hidaka, et al. are $\delta_t = \delta_B + \delta_s + \delta_c + (1/2)\delta_P$ and $\delta_c = \delta_B + \delta_s + \delta_c$ ($\delta_B, \delta_s, \delta_c, \delta_P$: deflections due to bending, shearing, elastic deformation at built-in end and Hertz contact). It is seen from Fig.4 that δ_{IF} (FEM), δ_{IM} (theory of elasticity), δ_{IH} (Hidaka, et al.) for δ_t and δ_{CH} , δ_{CF} , δ_{CM} for δ_c become larger in the order mentioned. δ_{IF} , δ_{CF} are considered to approach δ_{IM} , δ_{CM} when using a finer mesh pattern for the model. Hence the deflection analysis method by 2-dimensional elasticity theory and mapping function is

$m = 4$, $\alpha_0 = 20^\circ$, $z_i = 45$, $\alpha = 0$, $b = 30 \text{ mm}$, $P_n/b = 98 \text{ N/mm}$ $\{10 \text{ kgf/mm}\}$

Fig.4 Calculated tooth deflections by various methods (pitch point loading)

considered valid for the calculation of tooth deflection of internal spur gear.

Figure 5 shows the relations between the tooth deflection of the gear pairs of $m = 4$, $\alpha_0 = 20^\circ$, $z_i/z_p = 45/25$, $90/25$, $90/72$, $\alpha = 0$, $b = 30 \text{ mm}$ and meshing position

(a) $z_i/z_p = 45/25$

(b) $z_i/z_p = 90/25$

(c) $z_i/z_p = 90/72$

Fig.5 Relation between tooth deflection and meshing position [$m = 4$, $\alpha_0 = 20^\circ$, $\alpha = 0$, $P_n/b = 98 \text{ N/mm}$ $\{10 \text{ kgf/mm}\}$]

calculated by the theory of elasticity and the formula proposed by Hidaka, et al. The abscissa denotes the meshing position l/s [l : distance from meshing position to position of tangential angle $\theta = 45^\circ$ as shown in Fig.5(a), s : tooth thickness at position of $\theta = 45^\circ$] and the ordinate the tooth deflection in the direction of line of action. δ_{IM} is a little smaller than δ_{IH} and δ_{CM} is a little larger than δ_{CH} . This might be because the tooth deflection $\delta_{IH} - \delta_{CH}$ due to Hertz contact is larger than $\delta_{IM} - \delta_{CM}$.

3.2 Tooth deflection due to Hertz contact

Figure 6 shows the relations between

(a) $z_i/z_p = 45/25$

(b) $z_i/z_p = 90/25$

(c) $z_i/z_p = 90/72$

Fig.6 Relation between tooth deflection due to Hertz contact and meshing position [$m = 4$, $\alpha_0 = 20^\circ$, $\alpha = 0$, $P_n/b = 98 \text{ N/mm}$ {10 kgf/mm}]

the tooth deflection of the gear pairs of $m = 4$, $\alpha_0 = 20^\circ$, $z_i/z_p = 45/25$, $90/25$, $90/72$, $\alpha = 0$, $b = 30 \text{ mm}$ due to Hertz contact and meshing position calculated by the theory of elasticity, Lundberg's⁽⁷⁾ and Weber's⁽⁸⁾ formula. $\delta_p = \delta_{IM} - \delta_{CM}$ by the theory of elasticity and the half values of the deflections calculated by Lundberg's and Weber's formula are taken as the tooth deflection due to Hertz contact. δ_p by the theory of elasticity is much smaller than that by Lundberg's formula and almost equal to that by Weber's formula as in the case of external spur gears⁽⁴⁾.

3.3 Relation between tooth deflection and normal tooth load

Figure 7 shows the relations between tooth deflection δ_{total} of the gear pair of $m = 4$, $\alpha_0 = 20^\circ$, $z_i/z_p = 45/25$ and the normal tooth load. δ_{total} increases proportionally to the normal tooth load as in

Fig.7 Relation between tooth deflection and normal tooth load

Fig.8 Relation between δ_c and l/s [$P_n/b = 98 \text{ N/mm}$ {10 kgf/mm}]

the case of external spur gears⁽⁶⁾. Hence in the vibration analysis under normal running condition, the bending stiffness of gear teeth is considered to be almost constant irrespective of the normal tooth load.

4. Practical Formula for Tooth Deflection of Internal Spur Gear

Figure 8 shows the relation between δ_c and l/s for various mapped profiles. The abscissa denotes the meshing position l/s [Fig.5(a)] and the ordinate the tooth deflection δ_c on a log scale. δ_c on a log scale increases proportionally to l/s . Approximation of the results of Fig.8 by a straight line yields the following equation.

$$\delta_c = 0.512e^{3.01(l/s)} \quad (10)$$

where e : base of natural logarithm. Equation (10) is obtained for $P_n/b = 98$ N/mm, $E = 206$ GPa and δ_c for various materials and normal tooth loads is expressed by the following equation on the basis of the results of Fig.7.

$$\delta_c = 1.07 \frac{P_n}{Eb} e^{3.01(l/s)} \quad (11)$$

Figure 9 shows the relation between δ_P and a for various mapped profiles. The abscissa denotes the half width a (Fig.1) of Hertz contact width and the ordinate

Fig.9 Relation between δ_P and a [$P_n/b = 98$ N/mm {10 kgf/mm}]

Fig.10 Tooth deflections of internal and external spur gear calculated by practical formulas [tip loading, $P_n/b = 98$ N/mm {10 kgf/mm}]

the tooth deflection due to Hertz contact. The results of Fig.9 are approximated by the following equation.

$$\delta_P = 0.717 - 0.209 \ln a \quad (12)$$

Equation (12) is obtained for $P_n/b = 98$ N/mm, $E = 206$ GPa and δ_P for various materials and normal tooth loads is expressed by the following equation on the basis of the results of Fig.7.

$$\delta_P = \frac{2P_n}{E\pi b} \{1.41 + 0.689 \ln(m/a)\} \quad (13)$$

The tooth deflection δ_t is obtained from Eqs.(11), (13) and expressed as follows.

$$\delta_t = 1.07 \frac{P_n}{Eb} e^{3.01(l/s)} + \frac{2P_n}{E\pi b} \{1.41 + 0.689 \ln(m/a)\} \quad (14)$$

Figure 10 shows the tooth deflections δ_c , δ_t of internal and external spur gears of various numbers of teeth under normal tip load $P_n/b = 98$ N/mm calculated by Eqs. (11), (14) and the formulas proposed by Terauchi and Nagamura⁽⁴⁾. It is seen from this figure that the tooth deflections of internal spur gears approach those of external spur gears with an increasing number of teeth.

5. Conclusions

The main results obtained from this investigation are summarized as follows.

- (1) The deflection analysis method by 2-dimensional elasticity theory and mapping function is considered valid also for the calculation of tooth deflection of internal spur gear.
- (2) A practical formula for tooth deflection of internal spur gear is derived.
- (3) The tooth deflection of internal spur gear due to Hertz contact by the 2-dimensional elasticity theory and mapping function is much smaller than that by Lundberg's formula and almost equal to that by Weber's formula as in the case of external spur gear.
- (4) A practical formula for tooth deflection of internal spur gear due to Hertz contact is derived.
- (5) The tooth deflection of internal spur gear increases proportionally to the normal tooth load as in the case of external spur gear.

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