A Comparative Study of Crack/Crack Interaction in Composite Materials Using Macro and Micromechanical Models*

By Hideaki KASANO**, Masaki NATORI***, Hiroyuki MATSUMOTO****,
and Ichiro NAKAHARA*****

The effects of interaction between crack-like defects in fiber reinforced composite materials on the local stress concentrations in their neighbourhoods are theoretically investigated from the macro and micromechanical points of view. The most simple models consisting of homogeneous and heterogeneous anisotropic elastic plates with two collinear or parallel cracks are used. Plane stress and plane strain conditions are assumed, and the mathematical formulations are made in terms of the Cauchy-type singular integral equations of the first kind by the methods of harmonic-type stress functions and Fourier transforms. Numerical calculations are carried out for the stress intensity factors, and the results thus obtained from these models are compared with each other.

Key Words: Elasticity, Composite Materials, Crack/Crack Interaction, Macro and Micromechanics

1. Introduction

In order to study the fracture and strength of materials from a fracture mechanics-point of view, many researchers have thus far investigated the stress analysis problems of an isotropic elastic body with one or more cracks^{(1),(2)}. Among them problems of an elastic body with two or more cracks, which are related to crack/crack interaction and are very interesting problems, have been analyzed by Willmore⁽³⁾, Erdogan⁽⁴⁾ and Yokobori et al.⁽⁵⁾ using a complex variable method or method of continuous distribution of dislocations. On the other hand, with the development and practical use of fiber reinforced composite materials such as FRP, similar problems of an anisotropic elastic body with cracks have become the center of interest. For example, Konishi⁽⁶⁾ studied two coplanar cracks in a transversely isotropic elastic body by assuming a plane strain state and using the method of finite Hilbert transforms, which was further extended to the case of three coplanar cracks by Dhawan⁽⁷⁾. Krenk⁽⁸⁾ analyzed a two dimensional problem of many collinear cracks in a general anisotropic elastic body by reducing it to a Hilbert problem and also Delate et al.⁽⁹⁾ did a problem of orthotropic strip with two straight cracks using a singular integral equation method. They are interested in collinear or coplanar cracks, and no parallel cracks. Now the methods of stress analysis of fiber reinforced composite materials with crack-like defects are divided into two main classes⁽¹⁰⁾.

One is a macromechanical approach which assumes a crack in a homogeneous anisotropic elastic body and the other a micromechanical one assuming a crack in a matrix or an inclusion and on an interface between the two phases of a heterogeneous elastic body. Sih et al.⁽¹¹⁾ analyzed a problem of a straight crack in a matrix by using these two models, while it seems very important from a practical view point to investigate a similar problem of two or more cracks. Therefore this paper considers a most fundamental problem of two collinear or two parallel cracks using the aforementioned two mechanical models and interaction effects of crack-like defects in a unidirectional fiber reinforced composite material are comparatively examined from macro and micromechanical points of view. In the analysis, the problem is formulated in a Cauchy-type singular integral equation of the first kind within the framework of two dimensional anisotropic elasticity theory and the effects of proximity of two cracks and elastic constants on the stress intensity factors at the crack-tips are elucidated.

Fig.1 Unbounded orthotropic plane with a crack

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** Research Associate, Tokyo Institute of Technology, 2-12-1, Ookayama, Meguro-ku, Tokyo, 152 Japan.

*** Graduate Student, Tokyo Institute of Technology.

**** Professor, Tokyo Institute of Technology.

2. Method of Solution

An orthotropic plane stress problem and a transversely isotropic plane strain problem can be solved by using the same procedure if the corresponding relations between their elastic constants are considered and so this paper deals with the former. In studying the crack/crack interaction problem, we first of all consider a stress state in an unbounded orthotropic plane with one straight crack under tension as shown in Fig.1. In general, such stress state may be obtained by use of superposition of a stress state in the same plane with no crack (fundamental solution) and that contributing to a stress perturbation due to the crack (perturbation solution), and, if the former is used, singular behaviour of stresses near at the crack-tips can be estimated directly from the latter. In this case the boundary conditions for the perturbation problem can be stated as

$$(\sigma_y)_{y=0} = -\sigma_0, \quad |\xi| < 1 \quad (1.a)$$

$$(\nu)_{y=0} = 0, \quad |\xi| \geq 1 \quad (1.b)$$

$$(\tau_{xy})_{y=0} = 0 \quad (1.c)$$

By taking account of the symmetry of stress state and the conditions at infinity, the stress functions ϕ_1, ϕ_2 may be taken as follows:

$$c_{66}\phi_i = \int_0^\infty g_i(\lambda) e^{-\lambda\eta/\sqrt{\mu_i}} \cos \lambda\xi d\lambda, \quad (i=1, 2) \quad (2)$$

where $\xi=x/c$, $\eta=y/c$ and $g_i(\lambda)$'s are unknown functions to be determined from the boundary conditions, and c_{66} and μ_i are an elastic constant of anisotropic plane and a constant expressed with its elastic constants⁽¹²⁾ assuming $\text{Re } \sqrt{\mu_i} > 0$, respectively.

Substituting Eq.(2) into the relations of stress function and stress or displacement and taking account of the boundary condition (1.c), we have the following stress and displacement in the y direction on the x axis:

$$\begin{aligned} \frac{(\sigma_y)_{y=0}}{\sigma_0} &= \int_0^\infty \lambda S(\lambda) \cos \lambda\xi d\lambda \quad (3) \\ \frac{c_{66}(\nu)_{y=0}}{\sigma_0 c} &= -\frac{(k_1 - k_2)}{(1+k_1)(1+k_2)(\sqrt{\mu_1} - \sqrt{\mu_2})} \\ &\times \int_0^\infty S(\lambda) \cos \lambda\xi d\lambda \quad (4) \end{aligned}$$

in which $S(\lambda)$ is an unknown function and it is given as follows using $g_1(\lambda)$:

$$\begin{aligned} S(\lambda) &= \frac{(1+k_1)(\sqrt{\mu_1} - \sqrt{\mu_2})}{\sigma_0 c^2 \sqrt{\mu_1}} \lambda g_1(\lambda), \\ g_2(\lambda) &= -\frac{(1+k_1)}{(1+k_2)} \sqrt{\frac{\mu_2}{\mu_1}} g_1(\lambda) \end{aligned} \quad (5)$$

Substitutions of Eqs.(3) and (4) into the remaining boundary conditions lead to a pair of dual integral equations

$$\int_0^\infty \lambda S(\lambda) \cos \lambda\xi d\lambda = -1, \quad 0 \leq \xi < 1 \quad (6.a)$$

$$\int_0^\infty S(\lambda) \cos \lambda\xi d\lambda = 0, \quad \xi \geq 1 \quad (6.b)$$

In order to transform the above equations into a singular integral equation, here is defined a new unknown function $h(\xi)$ by

$$\begin{aligned} h(\xi) &= \frac{\partial}{\partial \xi} \int_0^\infty S(\lambda) \cos \lambda\xi d\lambda \\ &= -\int_0^\infty \lambda S(\lambda) \sin \lambda\xi d\lambda \quad (7) \end{aligned}$$

and applying the Fourier transforms to the above equation by taking account of Eq.(6.b), we have

$$\lambda S(\lambda) = -(2/\pi) \int_0^1 h(t) \sin \lambda t dt \quad (8)$$

and, substituting Eq.(8) into Eq.(6.a), we reduce it as follows after some manipulation:

$$\frac{1}{\pi} \int_{-1}^1 \frac{h(t) dt}{t - \xi} = 1, \quad |\xi| < 1 \quad (9)$$

In order to ensure the uniqueness of displacement, $h(t)$ must satisfy

$$\int_{-1}^1 h(t) dt = 0 \quad (10)$$

By use of finite Hilbert transforms⁽¹³⁾ and taking account of Eq.(10), we have the solution of Eq.(9) as follows

$$\begin{aligned} h(\xi) &= -\frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1-t^2}{1-\xi^2}} \cdot \frac{1}{(t-\xi)} dt = \frac{\xi}{\sqrt{1-\xi^2}}, \\ &|\xi| < 1 \quad (11) \end{aligned}$$

Substituting Eq.(11) into Eq.(8) and the results into Eq.(3) we have the following stress on the x axis:

$$\frac{(\sigma_y)_{y=0}}{\sigma_0} = -\frac{1}{\pi} \int_{-1}^1 \frac{t dt}{\sqrt{1-t^2}(t-\xi)} \quad (12)$$

and, particularly outside the crack region ($|\xi| \geq 1$), the right side of the above equation reduces to

$$\frac{(\sigma_y)_{y=0}}{\sigma_0} = \frac{\xi - \sqrt{\xi^2 - 1}}{\sqrt{\xi^2 - 1}}, \quad |\xi| \geq 1 \quad (13)$$

So defining the mode I stress intensity factor by

$$K_I = \lim_{\xi \rightarrow 1} \sqrt{2\pi c(\xi - 1)} \cdot (\sigma_y)_{y=0} \quad (14)$$

and then substituting Eq.(13) into the above expression we have

$$K_I = \sigma_0 \sqrt{\pi c} \quad (15)$$

Fig.2 Two collinear cracks with equal lengths

which is independent of elastic constants and coincides with that of an isotropic elastic plane.

Next we consider two collinear cracks with equal lengths as shown in Fig.2. Here the region (I) containing two cracks is an orthotropic strip and its outer region (II) an orthotropic half plane, where both material directions coincide but elastic constants differ from each other in general. We call it a macro-mechanical model when the elastic constants of both regions are the same and coincide with those of composite material, whereas, when the elastic constants of both regions coincide with those of resin matrix and composite material, respectively, we call it a micromechanical one. Assuming a perfect bond of the interface between the regions (I) and (II) the boundary conditions in this case can be stated as follows:

$$(u^I)_{\eta=\gamma} = (u^{II})_{\eta=\gamma} \quad (16 \cdot a)$$

$$(v^I)_{\eta=\gamma} = (v^{II})_{\eta=\gamma} \quad (16 \cdot b)$$

$$(\sigma_y^I)_{\eta=\gamma} = (\sigma_y^{II})_{\eta=\gamma} \quad (16 \cdot c)$$

$$(\tau_{xy}^I)_{\eta=\gamma} = (\tau_{xy}^{II})_{\eta=\gamma} \quad (16 \cdot d)$$

where $r=h/c$ and the superscripts "I" and "II" refer to the corresponding regions.

On the other hand the boundary conditions on the line where the cracks lie are given as follows:

$$(\sigma_y^I)_{\eta=0} = -\sigma_0, \quad a/c < |\xi| < b/c \quad (17 \cdot a)$$

$$(v^I)_{\eta=0} = 0, \quad |\xi| \leq a/c, \quad |\xi| \geq b/c \quad (17 \cdot b)$$

$$(\tau_{xy}^I)_{\eta=0} = 0 \quad (17 \cdot c)$$

where we confine ourselves to the perturbation problem as in the case of one crack. By taking account of the symmetry of stress state and the conditions at infinity, stress functions may be taken as follows for each region:

$$\left. \begin{aligned} c_{66}\varphi_1^I &= \int_0^\infty \{f_1(\lambda)e^{\lambda\eta/\sqrt{\mu_1}} + f_2(\lambda)e^{-\lambda\eta/\sqrt{\mu_1}}\} \cos \lambda\xi d\lambda \\ c_{66}\varphi_2^I &= \int_0^\infty \{f_3(\lambda)e^{\lambda\eta/\sqrt{\mu_2}} + f_4(\lambda)e^{-\lambda\eta/\sqrt{\mu_2}}\} \cos \lambda\xi d\lambda \end{aligned} \right\} \quad (18)$$

$$\left. \begin{aligned} c'_{66}\varphi_1^{II} &= \int_0^\infty g_1(\lambda)e^{-\lambda\eta/\sqrt{\mu_1}} \cos \lambda\xi d\lambda \\ c'_{66}\varphi_2^{II} &= \int_0^\infty g_2(\lambda)e^{-\lambda\eta/\sqrt{\mu_2}} \cos \lambda\xi d\lambda \end{aligned} \right\} \quad (19)$$

where $\eta' = \eta - \gamma$ and c_{66} , μ_1 , c'_{66} , μ_1' are elastic constants and constants expressed with the elastic constants for the corresponding regions, and further $\text{Re}\sqrt{\mu_1} > 0$, $\text{Re}\sqrt{\mu_1'} > 0$ are assumed. Also $f_1(\lambda) \sim f_4(\lambda)$, $g_1(\lambda)$, $g_2(\lambda)$ are unknown functions which are to be determined from the boundary conditions. By substituting the stresses and displacements corresponding to Eqs. (18) and (19) into the boundary conditions Eqs. (16), (17.c) and using the method of Fourier transforms, we have a system of linear algebraic equations for the unknown functions, from which five of six unknown functions are expressed with the remaining one. Taking account of these relations, we have the following normal stress and displacement on the line where the crack lie:

$$\frac{(\sigma_y^I)_{\eta=0}}{\sigma_0} = \int_0^\infty \lambda\psi(\lambda)S(\lambda)\cos \lambda\xi d\lambda \quad (20)$$

$$\frac{c_{66}(v^I)_{\eta=0}}{\sigma_0 c} = -\frac{(k_1 - k_2)}{(1 + k_1)(1 + k_2)(\sqrt{\mu_1} - \sqrt{\mu_2})} \times \int_0^\infty S(\lambda)\cos \lambda\xi d\lambda \quad (21)$$

where $\psi(\lambda)$ is a known function and $S(\lambda)$ is expressed with the remaining function as mentioned above. Substitutions of Eqs. (20) and (21) into the boundary conditions Eqs. (17.a) and (17.b) lead to the following dual integral equations:

$$\int_0^\infty \lambda\{1 + \bar{\psi}(\lambda)\}S(\lambda)\cos \lambda\xi d\lambda = -1, \quad a/c < \xi < b/c \quad (22 \cdot a)$$

$$\int_0^\infty S(\lambda)\cos \lambda\xi d\lambda = 0, \quad 0 \leq \xi \leq a/c, \quad \xi \geq b/c \quad (22 \cdot b)$$

where, for analytical convenience, we put $\psi(\lambda) = 1 + \bar{\psi}(\lambda)$ by taking account of $\lim_{\lambda \rightarrow \infty} \psi(\lambda) = 1$

In order to transform the above equations into a singular integral equation, we define a new unknown function by

$$\begin{aligned} h(\xi) &= \frac{\partial}{\partial \xi} \int_0^\infty S(\lambda)\cos \lambda\xi d\lambda \\ &= -\int_0^\infty \lambda S(\lambda)\sin \lambda\xi d\lambda \end{aligned} \quad (23)$$

as in the aforementioned case. Application of Fourier transforms to Eq. (23), taking account of Eq. (22.b), leads to

$$\lambda S(\lambda) = -\frac{2}{\pi} \int_{a/c}^{b/c} h(t)\sin \lambda t dt \quad (24)$$

and substituting this into Eq. (22.a) we have the following singular integral equation after some manipulation:

$$\frac{1}{\pi} \int_{a/c}^{b/c} \frac{h(t)}{t - \xi} dt + \frac{1}{\pi} \int_{a/c}^{b/c} \left\{ \frac{1}{t + \xi} + k_1(t, \xi) + k_2(t, \xi) \right\} h(t) dt = 1, \quad a/c < \xi < b/c \quad (25)$$

where the integral kernels are given by

$$\left. \begin{aligned} k_1(t, \xi) &= \int_0^\infty \bar{\psi}(\lambda) \sin \lambda(t - \xi) d\lambda \\ k_2(t, \xi) &= \int_0^\infty \bar{\psi}(\lambda) \sin \lambda(t + \xi) d\lambda \end{aligned} \right\} \dots\dots\dots (26)$$

and also the single valued condition for the displacement yields

$$\int_{a/c}^{b/c} h(t) dt = 0 \dots\dots\dots (27)$$

For numerical convenience, we set $\beta = a/c$ and make a change of variables such as

$$\left. \begin{aligned} s &= t - \frac{b+a}{b-a} = t - 1 - \beta \\ \zeta &= \xi - \frac{b+a}{b-a} = \xi - 1 - \beta \end{aligned} \right\} \dots\dots\dots (28)$$

Eqs. (25) and (27) reduce to

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 \frac{H(s)}{s - \zeta} ds + \frac{1}{\pi} \int_{-1}^1 \left\{ \frac{1}{(s + \zeta + 2 + 2\beta)} \right. \\ \left. + K_1(s, \zeta) + K_2(s, \zeta) \right\} H(s) ds = 1, \\ |\zeta| < 1 \dots\dots\dots (29) \end{aligned}$$

$$\int_{-1}^1 H(s) ds = 0 \dots\dots\dots (30)$$

where $H(s) = h(s + 1 + \beta)$ and $K_1(s, \zeta)$, $K_2(s, \zeta)$ are given by

$$\left. \begin{aligned} K_1(s, \zeta) &= \int_0^\infty \bar{\psi}(\lambda) \sin \lambda(s - \zeta) d\lambda \\ K_2(s, \zeta) &= \int_0^\infty \bar{\psi}(\lambda) \sin \lambda(s + \zeta + 2 + 2\beta) d\lambda \end{aligned} \right\} \dots\dots\dots (31)$$

The equation (29), when the stress on the line where the cracks lie, $-(\sigma_y^I)_{\eta=0}/\sigma_0$, is replaced with unity in the right side, holds good for $|\zeta| \geq 1$ as well as $|\zeta| < 1$ and particularly near at the crack-tips, it is approximated as follows since the first term is dominant:

$$\frac{(\sigma_y^I)_{\eta=0}}{\sigma_0} = -\frac{1}{\pi} \int_{-1}^1 \frac{H(s)}{s - \zeta} ds, \quad |\zeta| \rightarrow 1 \dots\dots\dots (32)$$

Close examination of singular behaviour of the unknown function $H(s)$ gives

$$H(s) = G(s)/\sqrt{1-s^2} \dots\dots\dots (33)$$

and, substituting this into Eq. (32), we have

$$\begin{aligned} \frac{(\sigma_y^I)_{\eta=0}}{\sigma_0} &= -\frac{1}{\pi} \int_{-1}^1 \frac{G(s)}{(s - \zeta)\sqrt{1-s^2}} ds \\ &= \begin{cases} \frac{G(1)}{\sqrt{\zeta^2-1}} + R_1(\zeta), & \zeta \rightarrow +1 \quad (\zeta > 1) \\ -\frac{G(-1)}{\sqrt{\zeta^2-1}} + R_2(\zeta), & \zeta \rightarrow -1 \quad (\zeta < -1) \end{cases} \\ &\dots\dots\dots (34) \end{aligned}$$

where the functions $G(s)$ and $R_1(\zeta)$, $R_2(\zeta)$ are bounded at $s = \pm 1$ and $\zeta = \pm 1$, respectively

Therefore, we denote the stress intensity factors at the points A and B by $K_{I,A}$, $K_{I,B}$ and define them by

$$\left. \begin{aligned} K_{I,A} &= \lim_{\zeta \rightarrow -1} \sqrt{-2\pi c(\zeta + 1)} \cdot (\sigma_y^I)_{\eta=0} \\ K_{I,B} &= \lim_{\zeta \rightarrow +1} \sqrt{2\pi c(\zeta - 1)} \cdot (\sigma_y^I)_{\eta=0} \end{aligned} \right\} \dots\dots\dots (35)$$

, then substitution of Eq. (34) into Eq. (35) leads to

$$\frac{K_{I,A}}{\sigma_0 \sqrt{\pi c}} = -G(-1), \quad \frac{K_{I,B}}{\sigma_0 \sqrt{\pi c}} = G(1) \dots\dots\dots (36)$$

Lastly we consider two parallel cracks with equal lengths as shown in Fig. 3.

The region having two cracks is an orthotropic strip which is divided into two, i.e., regions (I) and (II) for analytical convenience. The outer region (III) is also orthotropic half plane. Although material directions of the two regions coincide, the elastic constants of region (III) generally differ from those of regions (I) and (II) which are the same. We call it a macromechanical model when the elastic constants of regions (I), (II) and (III) are the same and coincide with those of composite material; and, when the elastic constants of regions (I) and (II) coincide with those of matrix resin and those of region (III) with the elastic constants of composite material we call it a micromechanical one. The boundary conditions in this case can be stated as follows. That is, on the interface between regions (II) and (III), we have

$$(u^II)_{\eta=\gamma} = (u^III)_{\eta=\gamma} \dots\dots\dots (37 \cdot a)$$

$$(v^II)_{\eta=\gamma} = (v^III)_{\eta=\gamma} \dots\dots\dots (37 \cdot b)$$

$$(\sigma_y^II)_{\eta=\gamma} = (\sigma_y^III)_{\eta=\gamma} \dots\dots\dots (37 \cdot c)$$

$$(\tau_{xy}^II)_{\eta=\gamma} = (\tau_{xy}^III)_{\eta=\gamma} \dots\dots\dots (37 \cdot d)$$

and also on the line where the crack lies, we have

$$(\sigma_y^I)_{\eta=\beta} = (\sigma_y^II)_{\eta=\beta} = -\sigma_0, \quad |\xi| < 1 \dots\dots\dots (38 \cdot a)$$

$$(\tau_{xy}^I)_{\eta=\beta} = (\tau_{xy}^II)_{\eta=\beta} = 0, \quad |\xi| < 1 \dots\dots\dots (38 \cdot b)$$

$$(u^I)_{\eta=\beta} - (u^II)_{\eta=\beta} = 0, \quad |\xi| \geq 1 \dots\dots\dots (38 \cdot c)$$

$$(v^I)_{\eta=\beta} - (v^II)_{\eta=\beta} = 0, \quad |\xi| \geq 1 \dots\dots\dots (38 \cdot d)$$

$$(\sigma_y^I)_{\eta=\beta} = (\sigma_y^II)_{\eta=\beta} \dots\dots\dots (38 \cdot e)$$

$$(\tau_{xy}^I)_{\eta=\beta} = (\tau_{xy}^II)_{\eta=\beta} \dots\dots\dots (38 \cdot f)$$

Fig. 3 Two parallel cracks with equal lengths

where we confine ourselves to the perturbation problem as before. It should be noted here that Eqs.(38.a) and (38.b) are not exactly correct for the micromechanical model and they are approximate boundary conditions. In Eqs.(37) and (38), $\gamma=h/c$, $\beta=a/c$ and the superscripts "I", "II" and "III" refer to the corresponding regions. By taking account of the symmetry of stress state and the conditions at infinity, we take the stress functions in the following forms for each region:

$$\begin{aligned} c_{66}\varphi_1^I &= \int_0^\infty g_1(\lambda) \cosh(\lambda\eta/\sqrt{\mu_1}) \cos \lambda\xi d\lambda \\ c_{66}\varphi_2^I &= \int_0^\infty g_2(\lambda) \cosh(\lambda\eta/\sqrt{\mu_2}) \cos \lambda\xi d\lambda \end{aligned} \quad \dots\dots\dots (39)$$

$$\begin{aligned} c_{66}\varphi_1^{II} &= \int_0^\infty \{f_1(\lambda) e^{\lambda\eta'/\sqrt{\mu_1}} + f_2(\lambda) e^{-\lambda\eta'/\sqrt{\mu_1}}\} \cos \lambda\xi d\lambda \\ c_{66}\varphi_2^{II} &= \int_0^\infty \{f_3(\lambda) e^{\lambda\eta'/\sqrt{\mu_2}} + f_4(\lambda) e^{-\lambda\eta'/\sqrt{\mu_2}}\} \cos \lambda\xi d\lambda \end{aligned} \quad \dots\dots\dots (40)$$

$$\begin{aligned} c'_{66}\varphi_1^{III} &= \int_0^\infty g_3(\lambda) e^{-\lambda\eta'/\sqrt{\mu_1}} \cos \lambda\xi d\lambda \\ c'_{66}\varphi_2^{III} &= \int_0^\infty g_4(\lambda) e^{-\lambda\eta'/\sqrt{\mu_2}} \cos \lambda\xi d\lambda \end{aligned} \quad \dots\dots\dots (41)$$

where $\eta'=\eta-\gamma$, $\eta''=\eta-\beta$ and c_{66} , μ_1 , c'_{66} , μ_2 are elastic constants and constants expressed with the elastic constants of each region, respectively, and also $\operatorname{Re} \sqrt{\mu_1} > 0$, $\operatorname{Re} \sqrt{\mu_2} > 0$ are assumed. In addition, $f_1(\lambda) \sim f_4(\lambda)$, $g_1(\lambda) \sim g_4(\lambda)$ are unknown functions which are to be determined from the boundary conditions. Substitutions of the stresses and displacements corresponding to these stress functions into the boundary conditions Eqs.(37), (38.e) and (38.f) and application of the method of Fourier transforms to them lead to six relations between the eight unknown functions and, by taking account of them, we have the following stresses and displacements on the line where the crack lies:

$$\begin{aligned} \frac{(\sigma_y^I)_{\eta=\beta}}{\sigma_0} &= \frac{(\sigma_y^{II})_{\eta=\beta}}{\sigma_0} \\ &= \int_0^\infty \lambda \{ \psi_{11}(\lambda) S_1(\lambda) + \psi_{12}(\lambda) S_2(\lambda) \} \cos \lambda\xi d\lambda \\ \frac{(\tau_{xy}^I)_{\eta=\beta}}{\sigma_0} &= \frac{(\tau_{xy}^{II})_{\eta=\beta}}{\sigma_0} \\ &= \int_0^\infty \lambda \{ \psi_{21}(\lambda) S_1(\lambda) + \psi_{22}(\lambda) S_2(\lambda) \} \sin \lambda\xi d\lambda \\ \frac{c_{66}\{(u^I)_{\eta=\beta} - (u^{II})_{\eta=\beta}\}}{\sigma_0 C} &= \int_0^\infty S_2(\lambda) \sin \lambda\xi d\lambda \\ \frac{c_{66}\{(v^I)_{\eta=\beta} - (v^{II})_{\eta=\beta}\}}{\sigma_0 C} &= \int_0^\infty S_1(\lambda) \cos \lambda\xi d\lambda \end{aligned} \quad \dots\dots\dots (42)$$

where $\psi_{ij}(\lambda)$'s ($i, j=1, 2$) are known functions and $S_1(\lambda)$, $S_2(\lambda)$ unknown ones expressed with original unknown functions $f_i(\lambda)$ and $g_i(\lambda)$ ($i=1\sim 4$). Substitutions of Eq.(42) into the remaining boundary conditions Eqs.(38.a)~(38.d) lead to the following simultaneous dual integral equations:

$$\begin{aligned} \int_0^\infty \lambda \{ \alpha_{11} S_1(\lambda) + \bar{\psi}_{11}(\lambda) S_1(\lambda) + \bar{\psi}_{12}(\lambda) S_2(\lambda) \} \cos \lambda\xi d\lambda &= -1, \\ 0 \leq \xi < 1 \quad \dots\dots\dots (43\cdot a) \end{aligned}$$

$$\begin{aligned} \int_0^\infty \lambda \{ \alpha_{22} S_2(\lambda) + \bar{\psi}_{22}(\lambda) S_1(\lambda) + \bar{\psi}_{21}(\lambda) S_2(\lambda) \} \sin \lambda\xi d\lambda &= 0, \\ 0 \leq \xi < 1 \quad \dots\dots\dots (43\cdot b) \end{aligned}$$

$$\int_0^\infty S_1(\lambda) \cos \lambda\xi d\lambda = 0, \quad \xi \geq 1 \quad \dots\dots\dots (44\cdot a)$$

$$\int_0^\infty S_2(\lambda) \sin \lambda\xi d\lambda = 0, \quad \xi \geq 1 \quad \dots\dots\dots (44\cdot b)$$

where, for analytical convenience, noting $\lim_{\lambda \rightarrow \infty} \{\psi_{11}(\lambda), \psi_{12}(\lambda), \psi_{22}(\lambda)\} = \{\alpha_{11}, 0, \alpha_{22}\}$ we put

$$\left. \begin{aligned} \psi_{11}(\lambda) &= \alpha_{11} + \bar{\psi}_{11}(\lambda) \\ \psi_{22}(\lambda) &= \alpha_{22} + \bar{\psi}_{22}(\lambda) \\ \psi_{12}(\lambda) &= \psi_{21}(\lambda) = \bar{\psi}_{12}(\lambda) \end{aligned} \right\} \quad \dots\dots\dots (45)$$

in which α_{11} and α_{22} are constants expressed with the elastic constants and they are given by

$$\alpha_{11} = \frac{(1+k_1)(1+k_2)(\sqrt{\mu_1}-\sqrt{\mu_2})}{2(k_1-k_2)}, \quad \alpha_{22} = \frac{\alpha_{11}}{\sqrt{\mu_1\mu_2}} \quad \dots\dots\dots (46)$$

In order to transform Eqs.(43) and (44) into singular integral equations, we set

$$\left. \begin{aligned} h_1(\xi) &= \frac{\partial}{\partial \xi} \int_0^\infty S_1(\lambda) \cos \lambda\xi d\lambda \\ &= - \int_0^\infty \lambda S_1(\lambda) \sin \lambda\xi d\lambda \\ h_2(\xi) &= \frac{\partial}{\partial \xi} \int_0^\infty S_2(\lambda) \sin \lambda\xi d\lambda \\ &= \int_0^\infty \lambda S_2(\lambda) \cos \lambda\xi d\lambda \end{aligned} \right\} \quad \dots\dots\dots (47)$$

and apply the method of Fourier inverse transforms to them by taking account of Eq.(44), yielding

$$\left. \begin{aligned} \lambda S_1(\lambda) &= - \frac{2}{\pi} \int_0^1 h_1(t) \sin \lambda t dt \\ \lambda S_2(\lambda) &= \frac{2}{\pi} \int_0^1 h_2(t) \cos \lambda t dt \end{aligned} \right\} \quad \dots\dots\dots (48)$$

Therefore, substituting Eq.(48) into Eq.(43), we have the following simultaneous singular integral equations:

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 \frac{H_1(t)}{t-\xi} dt + \frac{1}{\pi} \int_{-1}^1 \{K_{11}(t, \xi) H_1(t) - K_{12}(t, \xi) H_2(t)\} dt &= 1, \quad |\xi| < 1 \dots\dots\dots (49) \\ \frac{1}{\pi} \int_{-1}^1 \frac{H_2(t)}{t-\xi} dt + \frac{1}{\pi} \int_{-1}^1 \{K_{21}(t, \xi) H_1(t) + K_{22}(t, \xi) H_2(t)\} dt &= 0, \quad |\xi| < 1 \dots\dots\dots (50) \end{aligned}$$

where $H_1(t) \equiv \alpha_{11} h_1(t)$, $H_2(t) = \alpha_{22} h_2(t)$ and the integral kernels $K_{ij}(t, \xi)$ are given by

$$\left. \begin{aligned} K_{ii}(t, \xi) &= \frac{1}{\alpha_{ii}} \int_0^\infty \bar{\phi}_{ii}(\lambda) \sin \lambda(t-\xi) d\lambda, \\ (i=1, 2) \\ K_{ij}(t, \xi) &= \frac{1}{\alpha_{ij}} \int_0^\infty \bar{\phi}_{ij}(\lambda) \cos \lambda(t-\xi) d\lambda, \\ (i \neq j, i, j=1, 2) \end{aligned} \right\} \dots\dots\dots (51)$$

In addition, single valued conditions for the displacements require

$$\int_{-1}^1 H_1(t) dt = \int_{-1}^1 H_2(t) dt = 0 \dots\dots\dots (52)$$

The same consideration as in Eq.(29) shows that the normal stress $(\sigma_y)_{\eta=\beta}$ and shear stress $(\tau_{xy})_{\eta=\beta}$ near the crack-tips may be approximated by the first terms of Eqs. (49) and (50), that is

$$\frac{(\sigma_y)_{\eta=\beta}}{\sigma_0} = -\frac{1}{\pi} \int_{-1}^1 \frac{H_1(t)}{t-\xi} dt, \quad |\xi| \rightarrow 1 \dots\dots (53)$$

$$\frac{(\tau_{xy})_{\eta=\beta}}{\sigma_0} = -\frac{1}{\pi} \int_{-1}^1 \frac{H_2(t)}{t-\xi} dt, \quad |\xi| \rightarrow 1 \dots\dots (54)$$

in which $(\sigma_y)_{\eta=\beta} = (\sigma_y^I)_{\eta=\beta} = (\sigma_y^{II})_{\eta=\beta}$, $(\tau_{xy})_{\eta=\beta} = (\tau_{xy}^I)_{\eta=\beta} = (\tau_{xy}^{II})_{\eta=\beta}$

Taking account of the singular behaviour of functions $H_1(t)$ and $H_2(t)$, we put

$$H_i(t) = G_i(t) / \sqrt{1-t^2}, \quad (i=1, 2) \dots\dots\dots (55)$$

which, substituted into Eqs.(53) and (54), reduces to

$$\frac{(\sigma_y)_{\eta=\beta}}{\sigma_0} = \frac{G_1(1)}{\sqrt{\xi^2-1}} + R_3(\xi), \quad \xi \rightarrow +1 \quad (\xi > 1) \dots\dots\dots (56)$$

$$\frac{(\tau_{xy})_{\eta=\beta}}{\sigma_0} = \frac{G_2(1)}{\sqrt{\xi^2-1}} + R_4(\xi), \quad \xi \rightarrow +1 \quad (\xi > 1) \dots\dots\dots (57)$$

where the functions $G_i(t)$ and $R_3(\xi)$, $R_4(\xi)$ are bounded at $t=\pm 1$ and $\xi=\pm 1$, respectively.

Defining the mode I and II stress intensity factors by

$$K_I = \lim_{\xi \rightarrow +1} \sqrt{2\pi C(\xi-1)} \cdot (\sigma_y)_{\eta=\beta} \dots\dots\dots (58)$$

$$K_{II} = \lim_{\xi \rightarrow +1} \sqrt{2\pi C(\xi-1)} \cdot (\tau_{xy})_{\eta=\beta} \dots\dots\dots (59)$$

and substituting Eqs.(56) and (57) into the above equations, we have

$$\frac{K_I}{\sigma_0 \sqrt{\pi C}} = G_1(1), \quad \frac{K_{II}}{\sigma_0 \sqrt{\pi C}} = G_2(1) \dots\dots\dots (60)$$

3. Numerical Calculations

In order to evaluate the stress intensity factors, $G(1)$, $G(-1)$, $G_1(1)$ and $G_2(-1)$ are needed as seen in Eq.(36) or (60).

Now, substitutions of Eq.(33) into Eqs.(29) and (30) or Eq.(55) into Eqs. (49), (50) and (52) lead to

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 \frac{G(s) ds}{(s-\xi) \sqrt{1-s^2}} + \frac{1}{\pi} \int_{-1}^1 \left\{ \frac{1}{(s+\xi+2+2\beta)} + K_1(s, \xi) + K_2(s, \xi) \right\} \frac{G(s)}{\sqrt{1-s^2}} ds &= 1, \quad |\xi| < 1 \\ \dots\dots\dots (61 \cdot a) \end{aligned}$$

$$\int_{-1}^1 \frac{G(s)}{\sqrt{1-s^2}} ds = 0 \dots\dots\dots (61 \cdot b)$$

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 \frac{G_1(t) dt}{(t-\xi) \sqrt{1-t^2}} + \frac{1}{\pi} \int_{-1}^1 \{K_{11}(t, \xi) G_1(t) - K_{12}(t, \xi) G_2(t)\} \frac{dt}{\sqrt{1-t^2}} &= 1, \quad |\xi| < 1 \\ \dots\dots\dots (62 \cdot a) \end{aligned}$$

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 \frac{G_2(t) dt}{(t-\xi) \sqrt{1-t^2}} + \frac{1}{\pi} \int_{-1}^1 \{K_{21}(t, \xi) G_1(t) + K_{22}(t, \xi) G_2(t)\} \frac{dt}{\sqrt{1-t^2}} &= 0, \quad |\xi| < 1 \\ \dots\dots\dots (63 \cdot a) \end{aligned}$$

$$\int_{-1}^1 \frac{G_1(t)}{\sqrt{1-t^2}} dt = 0 \dots\dots\dots (62 \cdot b)$$

$$\int_{-1}^1 \frac{G_2(t)}{\sqrt{1-t^2}} dt = 0 \dots\dots\dots (63 \cdot b)$$

Since it seems difficult to solve the above equations analytically, we try to solve them numerically by use of the Lobatto-Tchebysheff numerical integration rule⁽¹⁴⁾. That is, application of this rule to Eqs.(61) ~ (63) yields the discrete forms as follows:

$$\left. \begin{aligned} \frac{1}{n-1} \sum_{k=1}^n \lambda_k \left\{ \frac{1}{s_k - \xi_r} + \frac{1}{s_k + \xi_r + 2 + 2\beta} + K_1(s_k, \xi_r) + K_2(s_k, \xi_r) \right\} G(s_k) &= 1 \\ (r=1, 2, \dots, n-1) \\ \frac{\pi}{n-1} \sum_{k=1}^n \lambda_k G(s_k) &= 0 \end{aligned} \right\} \dots\dots\dots (64)$$

$$\left. \begin{aligned} \frac{1}{n-1} \sum_{k=1}^n \lambda_k \left\{ \frac{G_1(t_k)}{t_k - \xi_r} + K_{11}(t_k, \xi_r) G_1(t_k) - K_{12}(t_k, \xi_r) G_2(t_k) \right\} &= 1, \\ (r=1, 2, \dots, n-1) \end{aligned} \right\}$$

$$\left. \begin{aligned} & \frac{1}{n-1} \sum_{k=1}^n \lambda_k \left\{ \frac{G_2(t_k)}{t_k - \xi_r} + K_{21}(t_k, \xi_r) G_1(t_k) \right. \\ & \quad \left. + K_{22}(t_k, \xi_r) G_2(t_k) \right\} = 0, \\ & \quad (r=1, 2, \dots, n-1) \\ & \frac{\pi}{n-1} \sum_{k=1}^n \lambda_k G_1(t_k) = 0 \\ & \frac{\pi}{n-1} \sum_{k=1}^n \lambda_k G_2(t_k) = 0 \end{aligned} \right\} \dots\dots\dots (65)$$

where collocation points are given by

$$\left. \begin{aligned} s_k = t_k = \cos \frac{(k-1)\pi}{(n-1)}, \quad (k=1, 2, \dots, n) \\ \xi_r = \xi_r = \cos \frac{(2r-1)\pi}{2(n-1)}, \quad (r=1, 2, \dots, n-1) \end{aligned} \right\} \dots\dots\dots (66)$$

and $\lambda_k = 1/2$ ($k=1, n$), $\lambda_k = 1$ ($k=2, 3, \dots, n-1$). Equation (64) forms n linear algebraic equations for n unknown functions $G(s_1), \dots, G(s_n)$, and also Eq.(65) $2n$ linear equations for $2n$ unknown functions $G_1(t_1), \dots, G_1(t_n), G_2(t_1), \dots, G_2(t_n)$, and so these can be solved, from which $G(1), G(-1), G_1(1)$ and $G_1(-1)$ are determined completely.

4. Numerical Results

We are concerned with a unidirectional carbon fiber or glass fiber reinforced resin composite material and numerical calculations are carried out for three kinds of macro and micromechanical models. Table 1 shows the elastic constants of these models, i.e., No.1 and No.2 represent the elastic constants of carbon fiber

Table 1 Elastic constants

	(GPa)			
	E_x	E_y	G_{xy}	ν_{xy}
No.1	84.67	7.35	4.12	0.30
No.2	157.0	10.80	7.10	0.30
No.3	43.61	12.64	4.80	0.24
No.4	3.975	3.975	1.59	0.25

reinforced resin composite, No.3 those of glass fiber reinforced composite and No.4 those of epoxy resin which is a matrix of these composite.

The elastic constants of No.1~No.3 are used for macromechanical models, CFRP(1), CFRP(2) and GFRP, respectively, whereas, in micromechanical models C1, C2 and E, the elastic constants of No.4 are used for their strip regions and those of No.1~No.3 for outer half plane regions. In solving a system of linear equations (64) and (65), we need to evaluate the integral kernels given by Eqs.(31) and (51) in advance. Noting that the integrands $\bar{\psi}(\lambda)$ and $\bar{\psi}_{ij}(\lambda)$ decrease exponentially with an increased value of λ we evaluate these integrals numerically by use of Simpson's integration rule with upper limits $\lambda=15 \sim 50$ and increments $\Delta\lambda=0.5 \sim 1$. We solve these linear equations by truncating after $n=50$ terms and find that the stress intensity factors are thus obtained with good convergency and precision.

(a) Macromechanical model

(b) Micromechanical model

Fig.4 Stress intensity factor as a function of distance between two cracks (collinear cracks)

Fig.4 shows the stress intensity factor $K_I/\sigma_0\sqrt{\pi c}$ for two collinear cracks as a function of distance between them, i.e., their proximity. In macromechanical models, the stress intensity factors at points A and B are independent of elastic constants and coincide with those of an isotropic body, where the former increases rapidly with a reduced distance between two cracks and the increasing rate of the latter is not so remarkable. On the other hand, in micromechanical models, the stress intensity factors depend on the elastic constants and that of C1 model is the largest followed by E and C2 models. The stress intensity factors at points A and B for macromechanical models are larger than those of micromechanical ones. Fig.5 shows the stress intensity factors $K_I/\sigma_0\sqrt{\pi c}$ for two parallel cracks as a function of their proximity. Mode I stress intensity factors for macromechanical models decrease rapidly when two cracks are brought into proximity and they approach the limit value of $K_I/\sigma_0\sqrt{\pi c}=0.5$ which is independent of elastic constants. On the contrary, those of micromechanical ones decrease gradually with a reduced distance between two cracks and their values are generally smaller than those of macromechanical models. It is also confirmed that Mode II stress intensity factors ($-K_{II}/\sigma_0\sqrt{\pi c}$) have a tendency to increase when two cracks are brought into close proximity, which is contrary to Mode I stress intensity factors, and their values are much smaller than those of Mode I.

Fig.6 Influence of elastic constants on stress intensity factor (parallel cracks, macromechanical model)

Fig.6 shows the stress intensity factor for the macromechanical model containing two parallel cracks with $\beta=a/c=1$ as a function of elastic constants. Poisson's ratio is fixed here at $\nu_{xy}=0.25$ and E_y/E_x is varied from unity to 20 for different values of G_{xy}/E_x , i.e., 0.05, 0.1 and 0.5. It is seen from this figure that Mode I stress intensity factor increases with a decreased value of E_y/E_x and it becomes larger for a larger value of G_{xy}/E_x .

(a) Macromechanical model

(b) Micromechanical model

Fig.5 Stress intensity factor as a function of distance between two cracks (parallel cracks)

Finally, although the present analysis is concerned with an orthotropic plane stress problem, solution of the corresponding transversely isotropic plane strain problem can be easily obtained by taking account of the relations between their elastic constants as mentioned before. Comparisons of both results then show that there is no remarkable difference between them except that plane strain solution is slightly smaller than plane stress one in general.

5. Concluding Remarks

Crack/crack interaction effect, when two collinear or parallel cracks are embedded into an orthotropic elastic body under tension, is considered from macro and micromechanical point of view under plane stress and plane strain conditions. In the analysis, the problem is formulated as a singular integral equation of the first kind by using a stress function approach and a method of Fourier transforms. The results thus obtained are summarized as follows:

- (1) In the case of two collinear cracks with equal lengths the stress intensity factors K_I for macromechanical models are independent of the elastic constants and those of both ends increase when the cracks are brought into proximity and its increasing rate is so remarkable at the end point nearer to the interface. On the other hand, the corresponding stress intensity factors for micromechanical models have the same tendency as those of macromechanical ones, but their values are smaller in general.
- (2) In the case of two parallel cracks with equal lengths, Mode II deformation as well as Mode I one occurs as the cracks approach each other, and for both macro and micromechanical models K_I decreases and K_{II} increases

with a reduced distance between two cracks. The value is larger for macromechanical model than for micromechanical one.

- (3) When two parallel cracks with equal lengths are located at a crack-length distance between them, the stress intensity factor K_I for macromechanical model becomes large with a reduced value of E_y/E_x and an increased value of G_{xy}/E_x .

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