A New Approach to Bending Problems of Circular Plates with Mixed Boundary Conditions*

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This paper describes a new approach to bending problems of circular plates with uniform or a concentrated load under mixed boundary conditions. The cases are considered when circular plates have boundary portions of two kinds, i.e. clamped portions and simply supported portions. A method proposed in this study is an iterative method, and by comparing the results obtained by this method with ones by other investigators its validity is proved.

Key Words : Elasticity, Numerical Analysis, Circular Plate, Bending, Mixed Boundary Condition, Iterative Method

1. Introduction

The bending problem of a thin elastic circular plate, some boundary portions of which are simply supported and the others are clamped, was formerly studied by H.D. Conway et al.⁽¹⁾, A.W. Leissa et al.⁽²⁾ and B. Stahl et al.⁽³⁾, but their calculated results do not coincide in some degree with each other.

In this study a new approach is proposed to solve the above-mentioned problem. It is an iterative method, which was adopted by one of the authors and a coauthor⁽⁴⁾ to solve bending problems of circular plates with plural circular holes.

In this study, another method is applied to the above-mentioned problem of circular plates with mixed boundary conditions, i.e. a method in which the general solution to bending problems of elastic plates expressed in the polar coordinates is used, a system of simultaneous equations with very many unknowns are established by describing the boundary conditions by integrating the above-mentioned general solution, and those simultaneous equations are solved. The results obtained by this method are compared with those by the iterative method and by other investigators.

2. Iterative Method

It is assumed that a thin elastic circular plate is simply supported in some boundary portions and clamped in the others, and that it is subjected to a uniformly distributed load q or to a concentrated load P at the center of the plate, O , as shown in Fig.1. Symbols indicated in Fig.1 and Fig.2 are used in this study. Also the following non-dimensional quantities are defined ;

$$\rho = \frac{r}{a}, \quad \bar{w} = \frac{w}{qa^4/D} \quad \left(\text{or} \quad \frac{w}{Pa^2/D} \right)$$

$$\bar{M}_r = \frac{M_r}{qa^2} \quad \left(\text{or} \quad \frac{M_r}{P} \right)$$

$$\bar{M}_t = \frac{M_t}{qa^2} \quad \left(\text{or} \quad \frac{M_t}{P} \right) \dots\dots\dots (1)$$

where w is the deflection, M_r and M_t are the moments, and D is the flexural rigidity.

First, the closed-form solution to the bending problem of a circular plate with a simply supported edge is obtained for a given load. Next, keeping always zero deflection at the edge, the above solution is modified so that $\partial \bar{w} / \partial \rho$ may vanish at the clamped portion of the edge. Next, the solution is modified so that $\bar{M}_r$ may vanish at the simply supported portion. In this way, the modification is repeated until the solution satisfies all boundary conditions. By the theory of bending of thin plates, the governing differential equation is obtained as follows ;

$$\nabla^4 \bar{w} = 1$$

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} \dots\dots\dots (2)$$

In each step of modification of the solution, the well-known following homogeneous solution⁽⁵⁾ for Eq.(1) is used :

$$\bar{w} = \sum_{m=0}^{\infty} (A_m \rho^m + C_m \rho^{m+2}) \cos m\theta \dots\dots\dots (3)$$

The above equation was simplified by considering the symmetricity of the deflection about the x -axis and the finiteness

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of the deflection and the moments at the point O.

From Eq.(3) the moments $\bar{M}_r$ and $\bar{M}_t$ as shown in Fig.2 are expressed as follows ;

$$\left. \begin{aligned} \bar{M}_r &= -2(1+\nu)C_0 - 2(3+\nu)C_1\rho \cos \theta \\ &\quad - \sum_{m=2}^{\infty} [m(m-1)(1-\nu)A_m\rho^{m-2} + (m+1) \\ &\quad \times \{m(1-\nu) + 2(1+\nu)\}C_m\rho^m] \cos m\theta \\ \bar{M}_t &= -2(1+\nu)C_0 - 2(1+3\nu)C_1\rho \cos \theta \\ &\quad + \sum_{m=2}^{\infty} [m(m-1)(1-\nu)A_m\rho^{m-2} + (m+1) \\ &\quad \times \{m(1-\nu) + 2(1+\nu)\}C_m\rho^m] \cos m\theta \end{aligned} \right\} \dots\dots\dots(4)$$

where ν is Poisson's ratio. Also from Eq.(3) we obtain

$$\frac{\partial \bar{w}}{\partial \rho} = 2C_0\rho + \sum_{m=1}^{\infty} \{mA_m\rho^{m-1} + (m+2)C_m\rho^{m+1}\} \cos m\theta \dots\dots\dots(5)$$

Now each step of the modification of solution is described in the following. The numerical results as shown in Fig.3 and Fig.4 are those for the case of Fig.1 (a) where $\alpha=45^\circ$ and also for the case of a uniformly distributed load acting and Poisson's ratio $\nu=0.3$.

Step (I) : The deflection of a circular plate with a simply supported edge is given by ⁽⁶⁾

$$\bar{w}_0 = \frac{1}{64(1+\nu)} \{2(3+\nu)(1-\rho^2) - (1+\nu)(1-\rho^4)\} \dots\dots\dots(6)$$

From this equation, the distribution of $\partial \bar{w}_0 / \partial \rho$ at the edge of the plate is obtained as shown by the solid line in Fig.3.

Fig.1 Circular plate with some portions of its boundary clamped and others simply supported

Fig.2 Definition of $\bar{M}_r$ and $\bar{M}_t$

Step (II) : The condition $\partial \bar{w}_0 / \partial \rho = 0$ for the clamped portions of the edge is satisfied by adding to the solution the quantity $p(\theta)$ indicated by the dotted line in Fig.3 keeping zero deflection at the edge. In this case, considering that it is symmetric about both the x - and y -axes, $p(\theta)$ is expressed as follows ;

$$p(\theta) = \sum_{m=0}^{\infty} a_m \cos m\theta \quad (m=0, 2, 4, \dots) \dots\dots\dots(7)$$

By comparing Eq.(7) with the equation obtained by substituting $\rho=1$ into Eq.(5), and by substituting $\rho=1; \bar{w}=0$ into Eq.(3), the undetermined coefficients A_m and C_m are estimated. Thus from Eqs.(3)~(5) the modified value for each variable is obtained. This value is added to that obtained from Eq.(6).

In the above-mentioned procedures, $p(\theta)$ is a step function and it is difficult to obtain accurate values of the coefficients a_m in Eq.(7). Therefore, as shown in Fig.3, a narrow transient range of β is arranged and $p(\theta)$ is assumed to be variable linearly in this range.

The distribution of $\bar{M}_r$ modified by the step (II) for $\beta=10^\circ$ is indicated in Fig.4 by the solid line.

Step (III) : In order to satisfy the condition at the simply supported edge, the quantity as shown by the dotted line in Fig.4 is obtained in the same way as in the step (II), and it is added to the solution obtained by the steps (I) and (II).

Step (IV) : By the modification of the step (III) the boundary condition $\partial \bar{w}_0 / \partial \rho \neq 0$ at the clamped portions ceases to be satisfied. Therefore, the step (II) is repeated.

In this way, the modifications of the steps (II) and (III) are repeated until the boundary conditions at the clamped and simply supported portions of the edge come to be numerically satisfied and then the solution to our problem is obtained.

But the above solution is merely a solution for some specified value of β as shown in Fig.3. Therefore, for various values of β the above-mentioned procedures are carried out, and by the extrapolation method the solution for $\beta=0$ is estimated.

For the case of Fig.1 (b), the terms for odd number of m should be added to the

Fig.3 Distribution of $\partial \bar{w}_0 / \partial \rho$ at boundary of circular plate

series of Eq.(7). And in the case of the concentrated load at the center of the circular plate, the solution for the concentrated load⁽⁷⁾ should be used instead of Eq.(6).

3. Method by Simultaneous Equations

Formerly one of the authors and his coauthor⁽⁶⁾ proposed an energy method to solve various problems of rectangular plates with mixed boundary conditions. The method to treat the boundary conditions in that study is adopted here for the case of a circular plate.

Here the case of Fig.1 (a) and of the uniformly distributed load is considered. The general solution is expressed by superposing $\bar{w}_0$ in Eq.(6) with $\bar{w}$ in Eq.(3) as follows ;

$$w = \bar{w}_0 + \sum_{m=0}^{\infty} (A_m \rho^m + C_m \rho^{m+2}) \cos m\theta$$

$$(m=0, 2, 4, \dots) \dots \dots \dots (8)$$

From symmetry of this problem, the region $0 \leq \theta \leq \pi/2$ is considered. One of the boundary conditions is $\bar{w}=0$ at $\rho=1$ for the whole region of θ . From this condition we obtain $A_m + C_m = 0$. Next, the condition that $(\partial \bar{w}_0 / \partial \rho) = 0$ for $\alpha \leq \theta \leq \pi/2$ and $\rho=1$ leads to the following equation ;

$$\int_{\alpha}^{\pi/2} \left(\frac{\partial \bar{w}}{\partial \rho} \right) \cos n\theta d\theta = 0 \quad (n=0, 2, 4, \dots)$$

$$\dots \dots \dots (9)$$

Also, the condition that $\bar{M}_r = 0$ for $0 \leq \theta \leq \alpha$ and $\rho=1$ leads to

$$\int_0^{\alpha} \bar{M}_r \cos n\theta d\theta = 0, \quad (n=0, 2, 4, \dots) \dots (10)$$

By substituting Eq.(8) and the equations of $\bar{M}_r$ obtained from Eq.(8) into Eq.(9) and Eq.(10), respectively, simultaneous linear

Fig.4 Distribution of $\bar{M}_r$ at boundary of circular plate in step (I) of modification

equations of A_m are obtained. These equations are solved by Gaussian elimination method and the solution to our problem is obtained.

4. Numerical Calculations and Considerations

By the iterative method stated in Chapter 2, the problems for a uniformly distributed load and a concentrated load were solved. The boundary conditions of two kinds as shown in Fig.1 (a) and (b) were considered. Giving various values to α , the solutions of $\bar{w}$ and $\bar{M}_r$, $\bar{M}_t$ were obtained. Poisson's ratio was invariably assumed as $\nu=0.3$.

As for the value of β stated in the step (II), various values between $0.8^\circ \sim 10^\circ$ were given for the case of the uniformly distributed load with the boundary conditions of Fig.1 (a) when $\alpha=45^\circ$, and the results as shown in Fig.5 were obtained. For the other cases, almost the same tendency as in Fig.5 was recognized. Thus, the solutions for the case of $\beta=0^\circ$ were able to be estimated by extrapolation.

Next, by the method stated in Chapter 3, the cases of the uniformly distributed load with the boundary conditions of Fig.1 (a) were solved. It was found that solutions accurate enough can be obtained by solving the simultaneous equations with 80 ~ 100 unknowns. Further it is recognized that their results almost coincide with those by the iterative method stated in Chapter 2.

Formerly three studies were published for the case of the uniformly distributed load. The calculated results in these studies are compared with our results in Tables 1~3. As for the present solutions, the results outside the parentheses are those obtained by the iterative method, and the results inside the parentheses are those by the method stated in Chapter 3. From these tables it is found that the two results calculated by our methods coincide well with each other, and that from our viewpoint the results by Stahl et al. are most accurate among those presented in the former three studies. Table 4 indicates the results of the maximum deflection and the maximum moment in the interior domain of the circular plate for the case of Fig. 1 (b) with the uniformly distributed load.

Fig.5 Relation between $\bar{w}_{\max}$ and β

Table 5 or Table 6 indicates the maximum deflection for the case of the concentrated load with the boundary conditions of Fig.1 (a) or Fig.1 (b), respectively. In Table 1~Table 6, figures with underlines may vary slightly. In Table 2~Table 4, $(\bar{M}_r)_{\max}$ and $(\bar{M}_t)_{\max}$ are the maximum moments

Table 1 $\bar{w}_{\max}$ in case of uniformly distributed load and both sides clamped

α deg.	Present sol.	Stahl	Conway	Leissa
	$\bar{w}_{\max}$	$\bar{w}_{\max}$	$\bar{w}_{\max}$	$\bar{w}_{\max}$
0	0.01563	0.0156	0.0156	0.0156
11.25		0.0162	0.0162	
15.00	0.01657 (0.01659)	0.0166		0.0166
22.50		0.0176	0.0178	
30.00	0.01907 (0.01909)	0.0191		0.0192
33.75		0.0199	0.0200	
45.00	0.02292 (0.02294)	0.0229	0.0230	0.0235
56.25		0.0266	0.0268	
60.00	0.02817 (0.02821)	0.0281		0.0288
67.50		0.0313	0.0316	
75.00	0.03554 (0.03555)	0.0353		0.0366
78.75		0.0376	0.0381	
85.00	0.04339			
89.00		0.0509	0.0530	
90.00	0.06370	0.0637	0.0637	0.0637

Table 2 $(\bar{M}_r)_{\max}$ in case of uniformly distributed load and both sides clamped

α deg.	Present sol.	Stahl	Leissa
	$(\bar{M}_r)_{\max}$	$(\bar{M}_r)_{\max}$	$(\bar{M}_r)_{\max}$
0	0.08125	0.0812	0.0812
11.25		0.0812	
15.00	0.08116 (0.08115)	0.0812	0.0810
22.50		0.0813	
30.00	0.08185 (0.08185)	0.0818	0.0816
33.75		0.0824	
45.00	0.08566 (0.08564)	0.0856	0.0858
56.25		0.0921	
60.00	0.09532 (0.09540)	0.0953	0.0958
67.50		0.1033	
75.00	0.1150 (0.1150)	0.1145	0.1177
78.75		0.1217	
85.00	0.1403		
89.00		0.1648	
90.00	0.2063		

Table 3 $(\bar{M}_t)_{\max}$ in case of uniformly distributed load and both sides clamped

α deg.	Present sol.	Stahl	Leissa
	$(\bar{M}_t)_{\max}$	$(\bar{M}_t)_{\max}$	$(\bar{M}_t)_{\max}$
0	0.08125	0.0812	0.0812
11.25		0.0842	
15.00	0.08627 (0.08634)	0.0863	0.0864
22.50		0.0917	
30.00	0.09857 (0.09864)	0.0986	0.0995
33.75		0.1024	
45.00	0.1148 (0.1149)	0.1147	0.1175
56.25		0.1277	
60.00	0.1325 (0.1325)	0.1321	0.1351
67.50		0.1409	
75.00	0.1511 (0.1511)	0.1502	0.1534
78.75		0.1552	
85.00	0.1677		
89.00		0.1809	
90.00	0.2063		

in the interior domain of the circular plate except the boundary and its neighbourhood, since at the joining point of the clamped portion and the simply supported portion of the boundary the moments become infinitively large.

In adopting the iterative method, the relaxation factor was used in the iterative procedures in order to promote the convergence of the solution. The values of the relaxation factors were 0.1~0.5 when $\beta \leq 4^\circ$ and 0.5~1 when $\beta > 4^\circ$. These values were decided by the trial and error method. By using these relaxation factors, the solution was revised 8~15 times. The number of terms of the Fourier series was 200, and the width of the divided range in numerical integration to obtain the Fourier coefficients was 0.1° . These two values were also decided by the trial and error method.

5. Conclusions

In this study two methods were proposed to solve the bending problem of a circular plate, some portions of which are clamped and the others are simply supported. For the case of the uniformly distributed load, these two methods were applied and their results were compared with

Table 4 $\bar{w}_{\max}$, $(\bar{M}_r)_{\max}$ and $(\bar{M}_t)_{\max}$ in case of uniformly distributed load and one side clamped

α deg.	Present sol.	Conway	Present sol.	
	$\bar{w}_{\max}$	$\bar{w}_{\max}$	$(\bar{M}_r)_{\max}$	$(\bar{M}_t)_{\max}$
0	0.01563		0.0813	0.0813
22.40	0.01664		0.0813	0.0870
22.50		0.02661		
45.00	0.01940	0.01956	0.0847	0.1008
67.40	0.02338		0.1013	0.1116
67.50		0.02361		
90.00	0.02803	0.02827	0.1239	0.1193
112.40	0.03314		0.1424	0.1270
112.50		0.03327		
135.00	0.03830	0.03872	0.1578	0.1373
157.40	0.04516		0.1738	0.1546
157.50		0.04514		
173.00	0.05142		0.1855	0.1717
179.00		0.05638		
180.00	0.06370		0.2063	0.2063

Table 5 $\bar{w}_{\max}$ in case of concentrated load and both sides clamped

α deg.	Present sol.
	$\bar{w}_{\max}$
0	0.01989
15.00	0.02050
30.00	0.02209
45.00	0.02456
60.00	0.02788
75.00	0.03247
85.00	0.03741
90.00	0.05050

Table 6 $\bar{w}_{\max}$ in case of concentrated load and one side clamped

α deg.	Present sol.
	$\bar{w}_{\max}$
0	0.01989
22.40	0.02052
45.00	0.02209
67.40	0.02432
90.00	0.02706
112.40	0.03021
135.00	0.03384
157.40	0.03822
173.00	0.04267
180.00	0.05050

each other and also with the results of three studies by other investigators. Also the case of the concentrated load was solved by our iterative method. Among our two methods, the iterative method is considered to be applicable to many kinds of the boundary condition of the circular plate. The authors wish to apply this method in the future to many problems of circular plates which have not yet treated.

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