Error Estimate of Numerical Integration in Boundary Element Method Analysis

by Takao SAWADA** and Masafumi IMANARI***

Errors included in solutions obtained by the boundary element method analysis are generally larger than those by the finite element method analysis in the case that the number of discretized elements is small. One of the reasons is supposed to be attributable to the error which will be produced in the numerical integration of the singular functions of the type \(1/r^p\) in a two dimensional elastic problem. In this report the distance \(r\) between a load point and an observation point on a boundary element was represented graphically by two parameters. Then the functions composing the fundamental solutions could be classified into seven basic ones and their tendencies along an element were characterized. The following results were obtained: (1) Errors of the numerical integration of the functions are small enough in the region \(r/L > 3\) even if four points are chosen in the Gaussian quadrature formula. (2) The higher the exponent, the more increase the errors in the integration of the functions \(\gamma(r)^{p'}/r\).

Then, the methods to reduce computing time and to decrease errors of the numerical integrations are proposed.

Key words : Elasticity, Boundary Element Method, Numerical Integration, Gaussian Quadrature, Computing Time

1. Introduction

The boundary element method is expected to take advantage of the finite element method in three dimensional analysis such as contact problems and so on, since only the boundary of the object is divided into a finite number of elements. Applying this method to numerical analysis, many kinds of approaches have been made individually or combinedly to obtain more accurate solutions and to reduce the computation time; (1) determination of the number of the discretized elements and the range of the independent variables included in the integrand, (2) the choice of the points for the Gaussian quadrature formula, (3) the error estimates of the numerical integration, and (4) the numerical operations at the corners of the object. Thus many trials have been in the light of experience. As an example of the error in the BEM solution, it is reported that the boundary displacement and the interior stress of a square block subjected to tension are far from their exact solutions. This may be primarily caused by the numerical integration of the function \(\gamma(r)^{p'}/r\) which presents singularities.

In this report the states of the error in the numerical integration were clarified in the following way. That is, the fundamental solutions in two dimensional elasticity are separated into simple functions, and the basic function \(1/r\) is presented by two parameters to know the property of the integration in advance. In addition, other functions are presented in a similar manner and the errors in their numerical integrations are characterized. Furthermore, two methods are presented to improve accuracy in the numerical integrations and to correct the resulting errors in a numerical way, and attempt is also made to reduce the computing time.

2. Integrands with Singularity

2.1 Classification of the integrands

The fundamental solutions related to the two dimensional elasticity are presented as follows:

\[
u_I = \frac{1}{8\pi(1-\nu)}[\frac{1}{r}(1+\frac{\partial}{\partial x}+\frac{\partial}{\partial y})] \tag{1}
\]

\[
\rho_I = -\frac{1}{4\pi(1-\nu)}[\frac{\partial}{\partial x}(1-2\nu)\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\frac{\partial}{\partial y}] \tag{2}
\]

\[
D_{\alpha\beta} = \frac{1}{2\pi(1-\nu)}[\frac{\partial}{\partial x}(1-2\nu)\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\frac{\partial}{\partial y}] \tag{3}
\]

\[
S_{\alpha\beta} = \frac{\mu}{2\pi(1-\nu)}\frac{\partial}{\partial x}[\frac{1}{r}(1-2\nu)\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\frac{\partial}{\partial y}] \tag{4}
\]

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where $i,j,k,l = x,y$. These equations are separated into the individual functions $1/r^2, r/r^3, r^2/r^3$, which are shown in Tables 1 and 2 with the number of the repetitions of the numerical integration for each function. For the sake of simplicity, the functions multiplied by $\partial r/\partial \eta$ are expressed only by $1/r$. The tables show that the fundamental solutions $D_{ij}$ and $S_{ij}$ are composed of some of the same integrands, which can be classified into 20 kinds of functions. Therefore, it is sufficient in the case of the discretization of the object with the constant elements to investigate those simple functions to estimate the error in the numerical integration. In the case of the linear elements, 40 kinds of functions including 20 kinds of functions multiplied by a nondimensional co-ordinate $\xi$ on the boundary element have to be investigated since the shape function is presented by a linear equation. In the case of a quadratic element, 60 kinds of function including those multiplied by $\xi^2$ should be examined similarly. The estimation of errors caused by those numerical integrations becomes very complicated as the number of the functions is increased. In this study, the integrands are converted into simple formulae, and the method to estimate the integration error is described.

2.2 Graphical representation of the integrands The functions of the distance between the loading point $P$ and the point $Q$ are integrated along the discretized boundary element $\Gamma$ in the boundary element method. The distance $r$ is derived from Fig.1 as follows:

\begin{align}
    r &= x_1 - x_P \cos \theta + y_1 - y_P \sin \theta \\
    r &= (x_1 - x_P)^2/L^2 + (y_1 - y_P)^2/L^2 \\
    r^2 &= r_0^2 + r_0^4/L^2 \\
    &= L'(a a' + b b' + c) \\
    \Gamma &= (1 + \xi)|L|/2 \\
    a &= 1/4
\end{align}

Table 1 Functions composing fundamental solution and number of repetition of integrations.

<table>
<thead>
<tr>
<th>Num. of</th>
<th>$\log_{10}r$</th>
<th>$r^2$</th>
<th>$r^3$</th>
<th>$r^5$</th>
<th>$r^7$</th>
<th>$r^{11}$</th>
<th>$r^{13}$</th>
<th>$r^{15}$</th>
<th>$r^{17}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cal.</td>
<td>2 1 1 1 1 1 1 1 1 1 1 2 2 2 2 1 1 1 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2 Functions $D_{ij}$ and $S_{ij}$ and number of repetition of integrations.

<table>
<thead>
<tr>
<th>Num. of</th>
<th>$\log_{10}r$</th>
<th>$r^2$</th>
<th>$r^3$</th>
<th>$r^5$</th>
<th>$r^7$</th>
<th>$r^{11}$</th>
<th>$r^{13}$</th>
<th>$r^{15}$</th>
<th>$r^{17}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cal.</td>
<td>3 3 3 1 1 1 2 2 6 3 3 1 1 1 2 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig.1 Geometrical relationship between points $P$ and $Q$ on boundary element.

Fig.2 Representation of function $r$ by two parameters $r_0$ and $s(\theta = x)$.
the influence of which is considered later. In this study the case \( \phi = \pi \) is considered to predict easily the property of the integrand including \( r_x \) and \( r_y \). (Fig.2). The directional cosines of the line normal to the boundary element \( \Gamma \) are \( n_x = 0 \) and \( n_y = 1 \). Since \( r_s \) is constant, \( r_s = gy/gy = r_s/r_s \) is derived and its property can be the same as that of the function \( 1/r \). These operations reduce 20 kinds of functions to 12 kinds, that is, \( \log(1/r), 1/r, r_x/r, r_y/r, r_x^2/r, r_y^2/r, r_x/r^2, r_y/r^2, r_x/r^3, r_y/r^3 \).

2.3 Property of integrand
Four kinds of the basic functions \( \log(1/r), 1/r, r_x, \) and \( r_y/r \) are represented when \( \phi = \pi \). The following properties are obtained from Fig. 2:

\[
\begin{align*}
 r_s &= L(\xi/2 - s/L) \quad \text{(16)}
 r^2 &= r_x^2 + r_y^2 \\
 &= L^2(\xi/2 - s/L) + s^2/L + (r_x/r_y) \\
 &= L^2(\xi(2 - 4s + 2s^2)) + (r_x/r_y) \quad \text{(17)}
\end{align*}
\]

Therefore, since \( r_x = -r_y \) is derived, we obtain

\[
\begin{align*}
\int r_x dr_x &= \frac{1}{2} \int (1/2L) \xi d\xi \quad \text{(18)}
\int r_y dr_y &= \frac{1}{2} \int (1/2L) \xi d\xi \quad \text{(19)}
\int r_x r_y dr_x dr_y &= \frac{1}{2} \int (1/2L) (\xi^2) d\xi \quad \text{(20)}
\int \log(1/r) dr &= -\frac{L}{2} \int (\log L + \log A(\xi)) d\xi \quad \text{(21)}
\end{align*}
\]

As shown in Eq. (17) the function \( r^2 \) is a quadratic expression of \( \xi \), and the four kinds of the functions are shown graphically in Fig. 3. Representing the distance \( r \) by \( r_x \) and \( s \) on the \( \xi - \eta \) coordinate in Eq. (17), the qualitative properties of these functions of \( r \) can be evaluated.

Now, we integrate the above functions with the Gaussian quadrature formulas. In this integration the function values at the prescribed points in the formula are multiplied by their own weight and then added. So, its principle is mathematical based on adding the mean values of the area of the function near a given point.

Accordingly, a considerable error between the true value and that of a function numerically calculated by the Gaussian quadrature may occur if the integrand is unsymmetric with respect to the point of the formula or if it changes very steeply except for the point. Therefore, the properties of the integration of the functions are found from the above viewpoint as follows:

(1) The maximum value of the function \( (1/r)^n \) is \( (L/r_s)^n \) as shown in Fig. 3(a), so that it becomes larger as the \( m \) takes a larger integer when the value \( r_s/L \) is smaller than unit. The error in numerical integration becomes larger especially in the case the maximum point falls in the range of \( |\xi| = 2s/L \leq 1 \).

(2) The function \( r_x^2/r^4 \) changes its sign from plus to minus and takes zero value when \( \xi = 2s/L \) in the case \( m \) is odd. Its radius of curvature is large when \( n = 0 \) (Fig. 3(b)) and the function shows a concave-convex one when \( n > 0 \), the center of which is \( \xi = 2s/L \) (Fig. 3(c)). A convex-convex curve is shown in the case the value \( m \) is even. The maximum value of the function increases with a decrease in the value \( r_s/L \) and the error in the numerical integration becomes large.

(3) The above description shows that the error in the numerical integration decreases in the case the maximum point lies in the range \( |\xi| > 1 \) (\( s/L > 0.5 \)). Fig. 4 shows the errors in the numerical integration of the functions \( 1/r^2 \) and \( (r_x/r)^4 \) based on the analytical integrations, where the value \( r_s/L \) is 0.2 and the four point Gaussian quadrature is employed. The errors damp with the increase of \( s/L \), which is true for the other functions. From the above discussion it is sufficient to consider only the parameter \( r_s/L \) when \( s/L < 0.5 \) in order to investigate.

![Fig. 3 Characteristics of basic integrand.](image)

![Fig. 4 Errors in numerical integration and analytical one.](image)
the integration errors quantitatively. It is shown in the following that twelve kinds of functions are classified into even and odd ones and the influence of the shape function on the integration is easily discussed.

Letting $s/L = 0$, the four kinds of functions Eqs. (18)-(21) become

\[ \int_0^r \frac{1}{r} \, dl = \frac{1}{2} \int_0^1 \frac{1}{\sqrt{\xi^2 + (r/L)^2}} \, d\xi \quad \cdots \cdots \(22\) \]

\[ \int_0^r \frac{1}{r} \, dl = \frac{1}{2} \int_0^1 \frac{-\xi}{\sqrt{\xi^2 + (r/L)^2}} \, d\xi \quad \cdots \cdots \(23\) \]

\[ \int_0^r \frac{1}{r} \, dl = \frac{1}{2} \int_0^1 \frac{1}{\sqrt{\xi^2 + (r/L)^2}} \, d\xi \quad \cdots \cdots \(24\) \]

\[ \int \log \frac{1}{r} \, dl = -\frac{L}{2} \left( \int_0^1 \log L \, d\xi \right) \]

\[ + \int \log \left( \frac{\xi \sqrt{\xi^2 + r/L}}{L} \right) \, d\xi \quad \cdots \cdots \(25\) \]

The functions which show the even property with respect to $\xi$ as described in section 2.2 are represented as $r_1(r/L)$, $\log r_1(r/L)$, such as Eqs. (22) and (25), and $r_2/r_1$, $(m = 1, 2, 3)$ whose numerator consists of the multiplication by $\xi^m$. The function $r_3/r_2$ is multiplied by $\xi$. The function consists of $\xi^{-1}$ such as Eqs. (23) and (24) is an odd one. It is not necessary to discuss their integration errors since its integrated values are zero. Therefore, only seven kinds of even functions are required to discuss under the discretization by means of the constant boundary element, that is, $\log(1/r_1), 1/r_1, r_2, r_3/r_1, r_3/r_2, r_4/r_3, r_5/r_4, r_6/r_5, r_7/r_6$. In the case of linear element discretization, twelve kinds of functions are multiplied by the shape functions $N_1 = (-\xi/2)$ and $N_2 = (1 + \xi/2)$, and they are added. Hence, we have to examine 24 kinds of functions. The functions multiplied by $\xi$ change from even to odd ones and vice versa. Consequently, nine kinds of functions including $r_3/r_2$ and $r_3/r_1$ are required to discuss. Similarly, 11 kinds of functions are required in the case of the quadratic element.

3. Errors in Numerical Integration and its Improvement

3.1 Error of the integration with the Gaussian quadrature formula. In this section we discuss the errors in the numerical integration of the integrands explained in the previous section by the Gaussian quadrature formula. First we consider the influence of the number of Gaussian points on the integration errors. The errors in the integration of $1/r^3$ with the four and eight point Gaussian quadrature

<table>
<thead>
<tr>
<th>$r/L$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>1.73</td>
<td>5.96</td>
<td>3.44</td>
<td>2.24</td>
<td>1.57</td>
<td>0.46</td>
</tr>
<tr>
<td>4 points</td>
<td>0.61</td>
<td>1.2</td>
<td>1.79</td>
<td>0.47</td>
<td>0.14</td>
<td>1.6</td>
</tr>
<tr>
<td>8 points</td>
<td>0.12</td>
<td>0.15</td>
<td>2x10^-2</td>
<td>1.6</td>
<td>3x10^-2</td>
<td></td>
</tr>
</tbody>
</table>

Table 3 Analytical integration of $1/r^3$ and errors for 4 and 8 point Gaussian quadratures.

The errors for $r_1/r^3$ and $r_2/r^3$ are represented in Fig. 6 as an example. Although the angle $\theta$ varies, the tendency of the errors is similar except for $\theta = \pi$ and the values are almost the same. Accordingly, it may be concluded that the errors in the numerical integration is small in the range $r/L > 0.5$.

3.2 Error of the internal stress. The above errors of the numerical integration can not be ignored, when the interior stresses and displacements near the boundary of the object are calculated. The errors in the internal stresses are obtained for the square block under plane strain and uniaxial tension. That is, the

\[ r_2 = r_3 \cos \phi + r_4 \sin \phi \quad \cdots \cdots \(26\) \]

\[ r_4 = r_3 \sin \phi - r_4 \cos \phi \quad \cdots \cdots \(27\) \]

where the values $r_3$ and $r_4$ are consistent with $r_7$ and $r_5$, respectively, when $\phi = \pi$.

The error for $r_3/r_4$ is represented in Fig. 6 as an example. Although the angle $\theta$ varies, the tendency of the errors is similar except for $\theta = \pi$ and the values are almost the same. According to the calculations made in this paper, the errors in the numerical integration is small in the range $r/L > 0.5$. The errors in the internal stresses are obtained for the square block under plane strain and uniaxial tension. That is, the
length $4L$ of the boundary of the cross section of the block is divided into 16 elements, and the origin of the coordinate system is taken at the center of the block. The internal stresses are obtained by the exact solutions of the boundary displacements and tractions, the error of which is summarized in Table 4. The error in the internal stress in the case of the linear element is larger than that for the constant element. Some of the former integrands represent higher singularities. The errors in the stress are large at the center near the boundary element and small near the ends in both cases. It is caused by the fact that the error is large when $a/L=0$ and small for a large value of $a/L$, as shown in Fig.4. The errors on the side of the loading are smaller than those on the free side, the reason for which is not clear.

3.4 Improvement of internal stress errors

In the above section we know that the error in the numerical integration near the boundary is large. It is the best way to integrate analytically each function along the boundary element near the given

Fig.5 Errors in numerical integration.

Fig.6 Errors in numerical integration for several values of inclined angle of element.
Table 4 Error in internal stress values near boundary.

(a) Loaded surface

<table>
<thead>
<tr>
<th>y/ε</th>
<th>Element</th>
<th>x/ε</th>
<th>0</th>
<th>1/2</th>
<th>1/3</th>
<th>2/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8</td>
<td>Constant</td>
<td>-6.3</td>
<td>-22</td>
<td>-5.0</td>
<td>-7.6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>-6.5</td>
<td>-5.4</td>
<td>-7.6</td>
<td>-10.0</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>Constant</td>
<td>-6.3</td>
<td>-22</td>
<td>-5.0</td>
<td>-7.6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>-6.5</td>
<td>-5.4</td>
<td>-7.6</td>
<td>-10.0</td>
<td></td>
</tr>
</tbody>
</table>

(b) Free surface

<table>
<thead>
<tr>
<th>y/ε</th>
<th>Element</th>
<th>x/ε</th>
<th>0</th>
<th>1/2</th>
<th>1/3</th>
<th>2/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8</td>
<td>Constant</td>
<td>-5.1</td>
<td>-4.6</td>
<td>-9.0</td>
<td>-12.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>-6.5</td>
<td>-5.4</td>
<td>-7.6</td>
<td>-10.0</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>Constant</td>
<td>-5.1</td>
<td>-4.6</td>
<td>-9.0</td>
<td>-12.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>-6.5</td>
<td>-5.4</td>
<td>-7.6</td>
<td>-10.0</td>
<td></td>
</tr>
</tbody>
</table>

Table 5 Error curves applied to integrands for improvement of internal stress values.

<table>
<thead>
<tr>
<th>Error Curve</th>
<th>Integrand for Correction</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.08 e-10 log z</td>
<td>( \frac{1}{r_z} ) ( \frac{1}{r} ) ( \frac{1}{r_y} )</td>
</tr>
<tr>
<td>0.196(10z) 1/3</td>
<td>( r_x ) ( r_{x y} ) ( r_{x y} )</td>
</tr>
<tr>
<td>0.77(0.33(1-0.055)</td>
<td>( r_x ) ( r_{x y} ) ( r_{x y} )</td>
</tr>
<tr>
<td>0.32 e-10 log 62</td>
<td>( r_x^3 ) ( r_{x y} ) ( r_{x y} )</td>
</tr>
<tr>
<td>0.175(7.7(1-0.196)</td>
<td>( r_x^3 ) ( r_{x y} ) ( r_{x y} )</td>
</tr>
<tr>
<td>0.19 e-15.68z</td>
<td>( r_x ) ( r_{x y} ) ( r_{x y} )</td>
</tr>
<tr>
<td>0.28(12.5(1-0.20)</td>
<td>( r_x ) ( r_{x y} ) ( r_{x y} )</td>
</tr>
<tr>
<td>0.32 e-10 log 62</td>
<td>( r_x^3 ) ( r_{x y} ) ( r_{x y} )</td>
</tr>
</tbody>
</table>

The errors in internal stress values are obtained by some integration methods.

Fig. 7 Errors in internal stress values obtained by some integration methods.

4. Trial for Reduction of Computing Time

The computing time required for the boundary element method is shown in Fig. 8. The rate of time required for both the solution of the simultaneous equation and the assembly of the coefficient matrix including the integration is almost the same. The Gauss-Jordan method is adopted here for the solution of equations. Only the method to reduce the computing time is tried in this study for the latter case. It is only necessary to integrate 10 kinds of functions in Table 7 in the case where the boundary solution is analyzed by the use of the constant element. However, there are five kinds of functions which are repeated to integrate two times. The number of integrations decreases from 15 to 10 times without this repetition. The rate of time to assemble the matrix without the repetition of integration decreases 44% regardless of the number of the elements. The rate can be decreased more for the odd integrands and also in the case the boundary elements lie in a straight line, where the integrands multiplied by \( r_z \) become zero due to \( r_{z}=0 \). It can be also decreased in the case the interior stresses should be calculated at many points.
5. Application to Three-dimensional Elasticity

It is necessary to integrate fundamental solutions over the surface of the object in three dimensional analysis by the boundary element method. The line normal to the quadrilateral element is chosen as \( y \) axis, and the \( \xi - \zeta \) axes are chosen as \( x \) and \( z \) ones, so that the plane \( \xi \) is defined as in Fig. 9. The distance \( r \) is represented by \( \xi \)-coordinate in the same way as in section 2.2 as

\[
\begin{align*}
  r^2 &= L^2 \left( \frac{\xi^2}{4} - \frac{(s/L)\xi}{2} + \frac{r_d}{L} \right)^2 \\
        &+ \left( \frac{h}{L} \right)^2 + \left( \frac{h}{L} \right)^2 \\
  \end{align*}
\]

(28)

where the coordinates of the point \( P \) are \((r_x, s, l)\). Since the \( \xi \) and \( \zeta \) axes are interchangeable with respect to the point \( P \), the property of the integration along the \( \xi \) axis is the same as that along the \( \zeta \) axis. The functions \((1/r)^2\) and \((1/r)^4\) are integrated over the \( \xi \) plane by the four point Gaussian quadrature, where a point \( P \) is chosen as \((g, 0)\) and \((1, 0)\) on the \( \xi - \zeta \) plane. The error is shown in Fig. 10. Integrating along the \( \xi \) axis, it shows a similar tendency to that of the two dimensional case because \( r_1 \) is constant even if the point \( Q \) varies. The error for \((1/r)^2\) is almost the same, where the integrated value by the 11 point Gaussian quadrature is chosen as the standard value for the error calculation. The values \( r_1, r_2 \), and \( r_3 \) in the case where the plane inclines arbitrarily against the \( x, y \) and \( z \) axes are obtained by the coordinate transformation into \( \xi, \zeta, \xi \) and \( \eta \). The error may be presumed to present a similar tendency to that in the two-dimensional case.

![Fig. 9 Schematic representation of the surface integration of \( r \).](image)

![Fig. 10 Sample results for errors in surface integration.](image)

6. Concluding Remarks

The functions \( r_s, \xi, \xi, r' \) included in the fundamental solutions were classified into simple forms, the tendency of which was estimated by graphical representation. The properties of the numerical integration by the Gaussian quadrature have been found and can be summarized as follows:

1. The errors in the numerical integration of 20 kinds of integrands in the case of constant element by the four point Gaussian quadrature are small enough in the range \( r_d/L > 0.5 \). The tendency of the error in three dimensions is the same as that in two dimensions, where the error becomes large with an increment in the order of the integrand.

2. The error in the interior stress is large near the center of an element \((s/L=0)\), but small at the ends \((s/L=0.5)\).

Some methods to decrease the error and to reduce the computing time were attempted and desirable results were obtained. In this study the error of the
numerical integration was reported, which is one of the factors influencing the accuracy of the solution by the boundary element method. The other factors may be studied as the future work. The authors wish to thank Mr. Saito (post graduate student) for his numerical calculation.

References