The Out-plane Impulse Response of a Curved Bar with Varying Cross-sections*

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This paper is concerned with the dynamic response of a plane curved bar with varying cross-sections under a dynamic load working perpendicular to the plane comprising the center line. As numerical examples, time responses of displacement and bending moment are found for a symmetric semi-elliptic arc bar having varying cross-sections and clamped ends subjected to a step concentrated impact load. The behaviors of responses due to variations of the cross-sections and the load point are clarified.

Key Words: Vibration, Out-Plane Impulse Response, Curved Bar, Varying Cross-Section, Dynamic Load

1. Introduction

The study on vibration problems of a curved bar of which the center line is a plane curve (so-called "a plane curved bar") is of practical use in fields such as earthquake-proof designs of piping systems of chemical plants, atomic plants and the design of earthquake resistant structures. On the vibrations of plane curved bars, the authors reported in References (1)~(10) (free vibration), (11)~(12) (steady-state response) and (13) (in-plane impulse response). Irie and others^{(14),(15)}, studied the steady-state in-plane and out-plane responses of curved Timoshenko beams with internal damping (clamped-free) by using the transfer matrix approach. But enough investigation on the out-plane impulse response of plane curved bars has not been made.

The purpose of the present work is to present a method for analyzing the out-plane impulse response of a plane curved bar with varying cross-section under an impact load working perpendicular to the plane comprising the center line. Based on the classical theory, Lagrangian of a plane curved bar with varying cross-section subjected to a distributed external force along the center line is obtained. Equations of motion and boundary conditions are derived from stationary condition of the Lagrangian, and the equations of motion are solved by the eigenfunction expansion method.

As numerical examples time responses of displacement and bending moment are found for a symmetric semi-elliptic arc bar having varying cross-section and both ends clamped under a step concentrated impact load. The

behaviors of the responses due to the variations of the cross-section and the position of the impact load are clarified in discussion of numerical results.

2. Analysis

2.1 Lagrangian and equations of motion

Let l denote a representative length of a curved bar; s , the arc length measured along the center line; $1/\rho$, the curvature at any point of the center line; t , the time, $\bar{v}(s,t)$, $\bar{\phi}(s,t)$, the bending and the torsional displacements, respectively. Let $q(s,t)$ denote a distributed load working perpendicular to the plane comprising the center line. Let ρ_0 , S , I , EI , C denote the density, the cross-sectional area, the polar moment of inertia of area, the bending rigidity and the torsional rigidity, respectively.

Let us now neglect the effects of the shear deformation and the rotatory inertia of the bar. By considering the kinetic energies due to $\bar{v}$ and $\bar{\phi}$, the bending and the torsional strain energies and the work by the external load, the Lagrangian L of a plane curved bar with varying cross-section may be expressed as follows⁽²⁾ (in general, a functional without integral with respect to time is called Lagrangian).

$$L = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \left[\frac{\rho_0 S}{2} \left(\frac{\partial \bar{v}}{\partial t} \right)^2 + \frac{\rho_0 I \rho}{2} \left(\frac{\partial \bar{\phi}}{\partial t} \right)^2 - \frac{EI}{2} \bar{X}^2 - \frac{C}{2} \bar{Z}^2 + q \bar{v} \right] ds dt \quad \dots\dots\dots (1)$$

where s_1 and s_2 are the values of s at both ends of the bar, and $\bar{X}$ and $\bar{Z}$ are

$$\bar{X} = \frac{\partial^2 \bar{v}}{\partial s^2} - \frac{\bar{\phi}}{\rho}, \quad \bar{Z} = \frac{\partial \bar{\phi}}{\partial s} + \frac{1}{\rho} \frac{\partial \bar{v}}{\partial s} \quad \dots\dots\dots (2)$$

Let us express S , I , ρ , E and C as follows:

$$\left. \begin{aligned} S &= S_0 H_1(s), & I &= I_0 H_2(s) \\ I &= I_0 H_3(s), & C &= C_0 H_4(s) \end{aligned} \right\} \quad \dots\dots\dots (3)$$

where S_0 , I_0 , ρ_0 and C_0 are the cross-sectional area, the polar moment of inertia of area, the moment of inertia of area and the torsional rigidity at the origin of s on the center line, respectively, and $H_1(s) \sim H_4(s)$ are functions of s . Set the slenderness ratio $\bar{v}_0$ as follows:

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$$\bar{\nu}_0^2 = S_0 I^2 / I_{p_0} \dots \dots \dots (4)$$

Assuming ρ_0 and E as constants and rewriting Eq.(1) by using Eq.(3) yield the following Lagrangian:

$$L = \frac{EI_0}{2} \int_0^{\tau} \int_{s_1}^{s_2} \left[\frac{\rho_0 S_0 H_1}{EI_0} \left(\frac{\partial \bar{v}}{\partial t} \right)^2 + \frac{\rho_0 I_{p_0} H_2}{EI_0} \left(\frac{\partial \bar{\phi}}{\partial t} \right)^2 - H_3 \bar{X}^2 - \frac{C_0 H_1}{EI_0} \bar{Z}^2 + \frac{2q}{EI_0} \bar{v} \right] ds dt \dots \dots \dots (5)$$

Taking the first variation of L to obtain the stationary condition of the Lagrangian, $\delta L = 0$ and integrating by parts, one obtains

$$\begin{aligned} \frac{\delta L}{EI_0} = & \int_0^{\tau} \int_{s_1}^{s_2} \left[-\frac{\rho_0 S_0 H_1}{EI_0} \frac{\partial^2 \bar{v}}{\partial t^2} - \frac{\partial^2 (H_3 \bar{X})}{\partial s^2} \right. \\ & + \nu_0 \frac{\partial}{\partial s} \left(\frac{H_1 \bar{Z}}{\rho} \right) + \frac{q}{EI_0} \left. \right] \delta \bar{v} ds dt \\ & + \int_0^{\tau} \int_{s_1}^{s_2} \left[-\frac{\rho_0 I_{p_0} H_2}{EI_0} \frac{\partial^2 \bar{\phi}}{\partial t^2} + \frac{H_3 \bar{X}}{\rho} \right. \\ & + \nu_0 \frac{\partial}{\partial s} (H_3 \bar{Z}) \left. \right] \delta \bar{\phi} ds dt \\ & + \int_{s_1}^{s_2} \left[\frac{\rho_0 S_0 H_1}{EI_0} \left(\frac{\partial \bar{v}}{\partial t} \right) \delta \bar{v} \right. \\ & + \frac{\rho_0 I_{p_0} H_2}{EI_0} \left(\frac{\partial \bar{\phi}}{\partial t} \right) \delta \bar{\phi} \left. \right] \Big|_{t=0}^{t=\tau} ds \\ & - \int_0^{\tau} \left[-\left\{ \frac{\partial}{\partial s} (H_3 \bar{X}) - \nu_0 H_1 \frac{\bar{Z}}{\rho} \right\} \delta \bar{v} \right. \\ & + H_3 \bar{X} \delta \left(\frac{\partial \bar{v}}{\partial s} \right) + \nu_0 H_1 \bar{Z} \delta \bar{\phi} \left. \right] \Big|_{s=s_1}^{s=s_2} dt \dots \dots \dots (6) \end{aligned}$$

where

$$\nu_0 = \frac{C_0}{EI_0} \dots \dots \dots (7)$$

Setting the bracketed expressions of the first and the second integrands on the right side of Eq.(6) equal to zero, one obtains equations of motion. The third integrand vanishes because of the additional conditions of displacement at $t=0, \tau$,

$$\delta \bar{v}|_{t=0} = \delta \bar{\phi}|_{t=0} = \delta \bar{v}|_{t=\tau} = \delta \bar{\phi}|_{t=\tau} = 0$$

The fourth integrand denotes the displacement boundary conditions and the natural boundary conditions. The quantities of the fourth integrand

$$\left. \begin{aligned} & EI_0 H_3 \bar{X}, \quad C_0 H_1 \bar{Z} \\ & EI_0 \left\{ \frac{\partial}{\partial s} (H_3 \bar{X}) - \nu_0 H_1 \frac{\bar{Z}}{\rho} \right\} \end{aligned} \right\} \dots \dots \dots (8)$$

are the bending moment, the twisting moment and the shearing force, respectively.

It is convenient to use the following transformation of variable:

$$\frac{dx}{ds} = \frac{1}{\rho} = G \Psi(x) \dots \dots \dots (9)$$

where G is a constant with a dimension of a reciprocal of length, x is a variable that shows an angle between the tangent at the origin of s and the one at any point on the center line and $\Psi(x)$ is a function of x . As an example, consider an ellipse, for which the equation is denoted by the rectangular coordinates (ξ_1, ξ_2) with $\xi_1 = a \cos \theta$, $\xi_2 = b \sin \theta$, in which $2a$, $2b$ and θ are the major and minor axes and a parameter, respectively. Setting the representative length l and the ellipticity of the curve μ as

$$\left. \begin{aligned} & l = \sqrt{(a^2 + b^2)/2} \\ & \mu = (a^2 - b^2)/(a^2 + b^2) \end{aligned} \right\} \dots \dots \dots (10)$$

one obtains

$$\left. \begin{aligned} \tan x &= \sqrt{\frac{1+\mu}{1-\mu}} \tan \theta \\ \frac{1}{G l} &= 1 - \mu^2 \\ \Psi(x) &= (1 + \mu \cos 2x)^{1/2} \end{aligned} \right\} \dots \dots \dots (11)$$

One can denote $\bar{\Psi}(x)$ by a simple expression for a number of other curves as well (cf.(3)). Rewriting Eq.(6) by use of Eq.(9), one obtains

$$\begin{aligned} \frac{\delta L}{EI_0 G^2} = & \int_0^{\tau} \int_{x_1}^{x_2} \left[-\frac{\rho_0 S_0 H_1}{EI_0 G^2} \frac{\partial^2 \bar{v}}{\partial t^2} \right. \\ & - \Psi \frac{\partial}{\partial x} \left\{ \Psi \frac{\partial (H_3 \bar{U})}{\partial x} \right\} + \nu_0 \Psi \frac{\partial}{\partial x} (\Psi H_1 \bar{W}) \\ & + \frac{q}{EI_0 G^2} \left. \right] \delta \bar{v} dx dt \\ & + \int_0^{\tau} \int_{x_1}^{x_2} \left[-\frac{\rho_0 I_{p_0} H_2}{EI_0 G^2} \frac{\partial^2 \bar{\phi}}{\partial t^2} + H_3 \bar{U} \right. \\ & + \nu_0 \frac{\partial (H_1 \bar{W})}{\partial x} \left. \right] \Psi \delta \bar{\phi} dx dt \\ & + \int_{x_1}^{x_2} \left[\frac{\rho_0 S_0 H_1}{EI_0 G^2} \left(\frac{\partial \bar{v}}{\partial t} \right) \delta \bar{v} \right. \\ & + \frac{\rho_0 I_{p_0} H_2}{EI_0 G^2} \left(\frac{\partial \bar{\phi}}{\partial t} \right) \delta \bar{\phi} \left. \right] \Big|_{t=0}^{t=\tau} \frac{dx}{\Psi} \\ & - \int_0^{\tau} \Psi \left[H_3 \bar{U} \delta \left(\frac{\partial \bar{v}}{\partial x} \right) - \left\{ \frac{\partial (H_3 \bar{U})}{\partial x} \right\} \right. \\ & \left. - \nu_0 H_1 \bar{W} \right] \delta \bar{v} + \nu_0 H_1 \bar{W} \delta \bar{\phi} \Big|_{x=x_1}^{x=x_2} dt \dots \dots \dots (12) \end{aligned}$$

where x_1 and x_2 are the values of x corresponding to s_1 and s_2 and

$$\left. \begin{aligned} \bar{U} &= \Psi \frac{\partial}{\partial x} \left(\Psi \frac{\partial \bar{v}}{\partial x} \right) - \Psi^2 \bar{\phi} \\ \bar{W} &= \Psi \frac{\partial}{\partial x} (\Psi \bar{\phi}) + \Psi^2 \frac{\partial^2 \bar{v}}{\partial x^2} \\ \bar{\phi} &= G \Psi \bar{\phi} \end{aligned} \right\} \dots \dots \dots (13)$$

Accordingly one has the following equations of motion from $\delta L = 0$:

$$\left. \begin{aligned} & -\frac{\rho_0 S_0 H_1}{EI_0 G^2} \frac{\partial^2 \bar{v}}{\partial t^2} - \Psi \frac{\partial}{\partial x} \left\{ \Psi \frac{\partial (H_3 \bar{U})}{\partial x} \right\} \\ & + \nu_0 \Psi \frac{\partial}{\partial x} (\Psi H_1 \bar{W}) + \frac{q}{EI_0 G^2} = 0 \\ & -\frac{\rho_0 I_{p_0} H_2}{EI_0 G^2} \frac{\partial^2 \bar{\phi}}{\partial t^2} + H_3 \bar{U} + \nu_0 \frac{\partial (H_1 \bar{W})}{\partial x} = 0 \end{aligned} \right\} \dots \dots \dots (14)$$

The bending moment, the twisting moment and the shearing force are

$$\left. \begin{aligned} & EI_0 G^2 H_3 \bar{U}, \quad C_0 G^2 H_1 \bar{W} \\ & EI_0 G^2 \Psi \left[\frac{\partial}{\partial x} (H_3 \bar{U}) - \nu_0 H_1 \bar{W} \right] \end{aligned} \right\} \dots \dots \dots (15)$$

2.2 Solutions of the equations of motion

Let us employ the eigenfunction expansion method to solve the equations of motion (14). By using the solutions for out-plane free vibrations of a plane curved bar having varying cross-section which were given previously by the authors⁽¹⁰⁾, one may assume the solutions of Eq.(14) in the following form:

$$\left. \begin{aligned} \bar{v} &= \sum_{m=1}^{\infty} v_m(x) T_m(t) \\ \bar{\phi} &= \sum_{m=1}^{\infty} \phi_m(x) T_m(t) \end{aligned} \right\} \dots \dots \dots (16)$$

where $v_m(x)$ and $\phi_m(x)$ are eigenfunctions for m -th mode of vibration and $T_m(t)$ is a function of t .

By setting $q=0$, $\bar{v}=v_m(x)\sin\omega_m t$, $\bar{\Phi}=\Phi_m(x)$ $\times \sin\omega_m t$ in Eq.(14), the equations of vibration for out-plane free vibrations of a plane curved bar having varying cross-section can be written as follows:

$$\left. \begin{aligned} & \left(\frac{\alpha_{0m}}{Gl} \right)' H_1 v_m - \psi \frac{d}{dx} \left(\psi \frac{d(H_2 U_m)}{dx} \right) \\ & + \nu_0 \psi \frac{d}{dx} (\psi H_1 W_m) = 0 \\ & \nu_0 \left(\frac{\beta_{0m}}{Gl} \right)^2 H_2 \Phi_m + H_3 U_m + \nu_0 \frac{d(H_2 W_m)}{dx} = 0 \end{aligned} \right\} \dots\dots\dots (17)$$

where

$$\left. \begin{aligned} U_m &= \psi \frac{d}{dx} \left(\psi \frac{dv_m}{dx} \right) - \psi^2 \Phi_m \\ W_m &= \psi \frac{d}{dx} (\psi \Phi_m) + \psi^2 \frac{dv_m}{dx} \\ \alpha_{0m}^4 &= \frac{\rho_0 S_0 \omega_m^2 l^4}{EI_0}, \quad \beta_{0m}^2 = \frac{\rho_0 l \rho_0 \omega_m^2 l^2}{C_0} \end{aligned} \right\} \dots\dots\dots (18)$$

and where α_{0m} , β_{0m} and ω_m are nondimensional frequencies and a circular frequency for the m -th mode, respectively. If (v_m, Φ_m) and (v_n, Φ_n) are eigenfunctions for the m -th and the n -th modes respectively, one may readily find the following orthogonality property of the eigenfunctions (see Appendix).

$$\left. \begin{aligned} & \int_{x_1}^{x_2} \left[H_1 v_m v_n + \left(\frac{Gl}{\nu_0} \right)^2 H_2 \psi^2 \Phi_m \Phi_n \right] \frac{dx}{G \psi} = 0, \\ & \quad (m \neq n) \\ & \int_{x_1}^{x_2} \left[H_1 v_m^2 + \left(\frac{Gl}{\nu_0} \right)^2 H_2 \psi^2 \Phi_m^2 \right] \frac{dx}{G \psi} = M_n \\ & \quad (\neq 0) \end{aligned} \right\} \dots\dots\dots (20)$$

where

$$\begin{aligned} GM_n &= \left(\frac{Gl}{\nu_0} \right)^4 \psi \left| v_m \frac{d}{dx} \left(\frac{d}{dx} (H_3 U_m) \right) \right. \\ & \quad \left. - \nu_0 H_1 W_m \right\} - H_3 \frac{dv_m}{dx} \frac{dU_m}{d\alpha_{0m}} \\ & \quad - \frac{dv_m}{d\alpha_{0m}} \left\{ \frac{d}{dx} (H_3 U_m) - \nu_0 H_1 W_m \right\} \\ & \quad + H_3 U_m \frac{d}{d\alpha_{0m}} \left(\frac{dv_m}{dx} \right) - \nu_0 H_1 \Phi_m \frac{dW_m}{d\alpha_{0m}} \\ & \quad + \nu_0 H_1 W_m \frac{d\Phi_m}{d\alpha_{0m}} \Big|_{x_1}^{x_2} \dots\dots\dots (21) \end{aligned}$$

Substituting Eq.(16) into Eq.(14) and using the relationship in Eq.(17), one has

$$\begin{aligned} & \frac{\rho_0 S_0 H_1}{EI_0} \sum_{m=1}^{\infty} v_m \dot{T}_m + \sum_{m=1}^{\infty} \left(\frac{\alpha_{0m}}{l} \right)^4 H_1 v_m T_m = \frac{q}{EI_0} \\ & \dots\dots\dots (22) \\ & \frac{\rho_0 l \rho_0 H_2}{EI_0} \sum_{m=1}^{\infty} \Phi_m \dot{T}_m + \sum_{m=1}^{\infty} \nu_0 \left(\frac{\beta_{0m}}{l} \right)^2 H_2 \Phi_m T_m = 0 \\ & \dots\dots\dots (23) \end{aligned}$$

and from the relationship

$$\nu_0 \left(\frac{\beta_{0m}}{l} \right)^2 = \left(\frac{l}{\nu_0} \right)^2 \left(\frac{\alpha_{0m}}{l} \right)^4$$

Eq.(23) becomes

$$\left(\frac{l}{\nu_0} \right)^2 \left[\frac{\rho_0 S_0 H_2}{EI_0} \sum_{m=1}^{\infty} \Phi_m \dot{T}_m + \sum_{m=1}^{\infty} \left(\frac{\alpha_{0m}}{l} \right)^4 H_2 \Phi_m T_m \right] = 0 \dots\dots\dots (24)$$

Multiplying both sides of Eq.(22) by $v_n(x)/G\psi$ and those of Eq.(24) by $\Phi_n(x)/G\psi$ and adding both expressions, one obtains

$$\begin{aligned} & \frac{\rho_0 S_0}{EI_0} \sum_{m=1}^{\infty} \left\{ H_1 v_m v_n + \left(\frac{Gl}{\nu_0} \right)^2 H_2 \psi^2 \Phi_m \Phi_n \right\} \frac{\dot{T}_m}{G \psi} \\ & + \sum_{m=1}^{\infty} \left(\frac{\alpha_{0m}}{l} \right)^4 \left\{ H_1 v_m v_n + \left(\frac{Gl}{\nu_0} \right)^2 H_2 \psi^2 \Phi_m \Phi_n \right\} \\ & \times \frac{T_m}{G \psi} = \frac{q}{EI_0} \frac{v_n}{G \psi} \dots\dots\dots (25) \end{aligned}$$

Integrating Eq.(25) over the interval x_1 to x_2 and using the orthogonality relationship given by Eq.(20), the equations of motion reduce to the form:

$$\begin{aligned} & \frac{\rho_0 S_0}{EI_0} M_n \dot{T}_n + \left(\frac{\alpha_{0n}}{l} \right)^4 M_n T_n \\ & = \frac{1}{EI_0} \int_{x_1}^{x_2} q(x, t) v_n \frac{dx}{G \psi} \dots\dots\dots (26) \end{aligned}$$

from which one has

$$\dot{T}_n + \omega_n^2 T_n = \frac{1}{\rho_0 S_0} \frac{1}{GM_n} \int_{x_1}^{x_2} q(x, t) v_n(x) \frac{1}{\psi} dx \dots\dots\dots (27)$$

The general solution of Eq.(27) is

$$\begin{aligned} T_n &= A_n \cos \omega_n t + B_n \sin \omega_n t \\ & + \frac{1}{\omega_n} \int_0^t Q(\tau) \sin \omega_n (t - \tau) d\tau \dots\dots\dots (28) \end{aligned}$$

where A_n and B_n are arbitrary constants and

$$Q_n(t) = \frac{1}{\rho_0 S_0} \frac{1}{GM_n} \int_{x_1}^{x_2} q(x, t) v_n(x) \frac{1}{\psi} dx \dots\dots\dots (29)$$

If the initial conditions are taken to be $T_n|_{t=0}=0$, $\frac{dT_n}{dt}|_{t=0}=0$, one obtains $A_n=B_n=0$ and T_n

becomes

$$T_n = \frac{1}{\omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) d\tau \dots\dots\dots (30)$$

Now suppose the load $q(x, t)$ is a step concentrated impact load applied at the point of $x=x_f$. For this case, q is expressed as

$$q(x, t) = q_0 u(t) \delta(x - x_f) \dots\dots\dots (31)$$

where $\delta(x - x_f)$ is a Dirac delta function and $u(t)$ is a unit step function. Then the function T_n becomes

$$T_n = \frac{q_0 l^4}{EI_0 G} \frac{1}{\psi(x_f)} \frac{v_n(x_f)}{\alpha_{0n} M_n} (1 - \cos \omega_n t) \dots\dots\dots (32)$$

and the solutions $\bar{v}$ and $\bar{\Phi}$ are expressed as

$$\left. \begin{aligned} \bar{v} &= \frac{q_0 l^4}{EI_0 G} \frac{1}{\psi(x_f)} \sum_{n=1}^{\infty} v_n(x) \\ & \times \frac{v_n(x_f)}{\alpha_{0n} M_n} (1 - \cos \omega_n t) \\ \bar{\Phi} &= \frac{q_0 l^4}{EI_0 G} \frac{1}{\psi(x_f)} \sum_{n=1}^{\infty} \Phi_n(x) \\ & \times \frac{v_n(x_f)}{\alpha_{0n} M_n} (-\cos \omega_n t) \end{aligned} \right\} \dots\dots\dots (33)$$

3. Numerical Calculations

Numerical calculations are made to obtain time responses of displacements $\bar{v}$, $\bar{\Phi}$ and bending moment $M = EI_0 G^2 H_3 \bar{v}$ of a symmetric semi-elliptic arc bar having both ends clamped

$(-x_1=x_2=\pi/2, \mu=0.2, \bar{V}_0=100)$ subjected to a step concentrated impact load. The bar has its diameter varying with $d=d_0(1+kx^2)$ and then $H_1(x) \sim H_k(x)$ are taken as

$$\begin{aligned} H_1(x) &= (1+kx^2)^2, & H_2(x) &= (1+kx^2)^4 \\ H_3(x) &= (1+kx^2)^4, & H_4(x) &= (1+kx^2)^4 \end{aligned} \quad \dots\dots\dots (34)$$

where k is an arbitrary constant. In the calculations, three cases where $k=0, k=\pm 0.2$ were considered. For reference, the contour and the variation of d of the elliptic arc bar ($\xi_1=a\cos\theta, \xi_2=b\sin\theta$) are shown in Fig.1 in the case where $\bar{V}_0=100, \mu=0.2, k=\pm 0.2, d_0=6\text{mm}$.

By utilizing the analysis by Suzuki and others⁽¹⁰⁾ for out-plane free vibrations of a plane curved bar with varying cross-section, natural frequencies and eigenfunctions are calculated for the first to the eighth modes of vibration. The nondimensional frequencies α_{0n} are shown in Table 1 and the corresponding values of GM_n of Eq.(20) in Table 2. The responses of the displacements and the bending moment were calculated by using these eight natural frequencies, GM_n and eigenfunctions. The accuracies of the responses for the displacements are about three digits.

Fig.1 Elliptic arc bar with varying diameter

Table 1 Nondimensional natural frequencies α_{0n} of symmetric semi-elliptic arc bar with clamped ends ($\mu=0.2, \bar{V}_0=100$)

n	k=-0.2	k=0	k=0.2
1	0.830427	1.318324	1.711290
2	1.744750	2.259759	2.665032
3	2.721067	3.296151	3.732476
4	3.718359	4.337124	4.824129
5	4.658110	5.367627	5.921700
6	5.579842	6.390561	7.018254
7	6.490187	7.408489	8.114324
8	7.394965	8.423040	9.209416

The number $n=1,3,5,7$ correspond to symmetric vibrations and $n=2,4,6,8$ to antisymmetric vibrations

n	k=-0.2	k=0	k=0.2
1	1.18182	1.20445	1.20477
2	4.53977×10^1	4.01959×10^1	3.59955×10^1
3	2.07106	1.69947	1.50736
4	1.98319×10^1	1.45492×10^1	1.17601×10^1
5	1.95781	1.65987	1.49587
6	9.95351×10^2	7.08572×10^2	5.61410×10^1
7	1.91397	1.65134	1.50112
8	5.94261×10^2	4.17561×10^2	3.28364×10^3

Table 2 Values of $GM_n(\mu=0.2, \bar{V}_0=100)$

The convergence of the bending moments is poor compared with those of the displacements and in places, it is barely up to 1~2 digits. But the results obtained will be considered enough to show the tendency of time responses for the bending moment.

Figures 2 to 8 show the time responses of the displacements and the bending moment of symmetric semi-elliptic arc bars having both ends clamped subjected to a step concentrated impact load. The ordinates adopted are non-dimensional displacements $\bar{v}/(l^4 q_0/EI_0), \bar{\phi}/(l^4 q_0/EI_0)$ or a non-dimensional bending moment $M/(G^2 l^4 q_0)$ and the abscissas are non-dimensional times $T^*=(EI_0/(\rho_0 S_0 l^4))^{1/2}$. In Figures 2 to 8, full lines show the response curves for $k=0$, chain lines for $k=-0.2$ and broken lines for $k=0.2$, respectively. The responses for a static load, which were obtained approximately from the terms exclusive of $\cos \omega_n t$ in Eq.(33), are depicted by straight lines.

Figures 2 to 4 show the time responses of $\bar{v}$ and M at the load point and those of M at the clamped end under an impact load at the point of $x=0(x_f=0)$. With the values for a static load as a center, the bending

Fig.2 Time response of bending displacement at $x=0(x_f=0)$

Fig.3 Time response of bending moment at $x=0(x_f=0)$

Fig.4 Time response of bending moment at $x=\pi/2(x_f=0)$

displacements and the bending moment vary with a nearly constant period. The period of the response curves becomes shorter in the order of $k=-0.2, 0, 0.2$ and all of them almost agree with the periods of the first symmetric mode of free vibrations. The values of the bending displacements and the bending moment at the load point become smaller in the order of $k=-0.2, 0, 0.2$ and on the contrary, the values of the bending moment at the clamped end become larger.

Figures 5 to 8 show the time responses of $\bar{v}$, $\bar{\phi}$ and M at the load point and those of M at the clamped end in case of $x_f = \pi/4$. Similarly to the case of $x_f = 0$, the displacements and the bending moment vary with the static values as a center and also with a nearly constant period. Only the periods of the time responses of the bending moment at the load point almost agree with those of the first antisymmetric mode of free vibrations but the periods of other response curves do with those of the first symmetric mode. The values of the displacements and the bending moment at the load point become smaller in the order of $k=-0.2, 0, 0.2$ and on the contrary, the values of the bending moment at the clamped end become larger.

4. Conclusions

In this paper, the dynamic response of a plane curved bar with varying cross-section has been studied under a dynamic load working perpendicular to the plane comprising the center line. As numerical examples, time responses of displacements and bending moment were found for a symmetric semi-elliptic arc bar having varying cross-section and both ends clamped subjected to a step concentrated impact load, and the behaviors of responses due to the variations of cross-section and position of the impact load were studied.

Fig.5 Time response of bending displacement at $x = \pi/4$ ($x_f = \pi/4$)

Fig.6 Time response of torsional displacement at $x = \pi/4$ ($x_f = \pi/4$)

From the results, one can summarize as follows:

(1) With the static values as a center, almost all the displacements and the bending moment vary with a nearly constant period. The period of displacements and bending moment in case of $x_f = 0$ almost agree with that of the first symmetric mode of free vibrations, and in case of $x_f = \pi/4$, the period of displacements does with that of the first symmetric mode and the period of bending moment with that of the first antisymmetric mode.

(2) The values of displacements and bending moment at the load point become smaller as k (and hence the cross-section) becomes larger, but the values of bending moment at the clamped end become larger.

The numerical calculations in this paper were carried out by using ACOS System 1000 Computer of Tohoku University.

Finally, The authors wish to express their sincere gratitude to Mrs. Yamaguchi, Technical Official in Engineering Faculty, Yamagata University for her help throughout this work.

Appendix

Orthogonality properties of eigenfunctions

Setting $q=0$, $\bar{v} = v_m(s) \sin \omega_m t$, $\bar{\phi} = \phi_m(s) \sin \omega_m t$ in Eq.(6), one has the following equations of vibration for out-plane free vibrations of a plane curved bar having varying cross-section.

$$\left(\frac{\alpha_{0m}}{l}\right)' H_1 v_m = \frac{d^2}{ds^2} \left\{ \left(\frac{d^2 v_m}{ds^2} - \frac{\phi_m}{\rho} \right) H_2 \right\} - \nu_0 \frac{d}{ds} \left\{ \frac{H_1}{\rho} \left(\frac{d\phi_m}{ds} + \frac{1}{\rho} \frac{dv_m}{ds} \right) \right\} \quad (35)$$

$$\nu_0 \left(\frac{\beta_{0m}}{l} \right)^2 H_2 \phi_m = -\frac{1}{\rho} \left\{ \left(\frac{d^2 v_m}{ds^2} - \frac{\phi_m}{\rho} \right) H_2 \right\} - \nu_0 \frac{d}{ds} \left\{ H_1 \left(\frac{d\phi_m}{ds} + \frac{1}{\rho} \frac{dv_m}{ds} \right) \right\} \quad (36)$$

Fig.7 Time response of bending moment at $x = \pi/4$ ($x_f = \pi/4$)

Fig.8 Time response of bending moment at $x = \pi/2$ ($x_f = \pi/4$)

Now multiply both sides of Eq.(35) by $(-v_n)$ and the expression obtained replacing m with n in Eq.(35) by (v_m) . Integrating both expressions over the interval s_1 to s_2 and adding them, one obtains

$$\begin{aligned} & \frac{\alpha_{0n}^4 - \alpha_{0m}^4}{l^4} \int_{s_1}^{s_2} H_1 v_m v_n ds \\ &= \left[v_m \left\{ \frac{d}{ds} \left(\left(\frac{d^2 v_n}{ds^2} - \frac{\phi_n}{\rho} \right) H_3 \right) \right. \right. \\ & \quad \left. \left. - v_n \left\{ \frac{H_1}{\rho} \left(\frac{d\phi_n}{ds} + \frac{1}{\rho} \frac{dv_n}{ds} \right) \right\} \right\} \right. \\ & \quad \left. - \frac{dv_m}{ds} \left\{ \left(\frac{d^2 v_n}{ds^2} - \frac{\phi_n}{\rho} \right) H_3 \right\} \right. \\ & \quad \left. - v_n \left[\frac{d}{ds} \left\{ \left(\frac{d^2 v_m}{ds^2} - \frac{\phi_m}{\rho} \right) H_3 \right\} \right. \right. \\ & \quad \left. \left. - v_m \left\{ \frac{H_1}{\rho} \left(\frac{d\phi_m}{ds} + \frac{1}{\rho} \frac{dv_m}{ds} \right) \right\} \right\} \right. \\ & \quad \left. + \frac{dv_n}{ds} \left\{ \left(\frac{d^2 v_m}{ds^2} - \frac{\phi_m}{\rho} \right) H_3 \right\} \right]_{s_1}^{s_2} \\ & \quad + \int_{s_1}^{s_2} \left[\frac{dv_n}{ds} \frac{v_0}{\rho} H_1 \frac{d\phi_n}{ds} - \frac{d^2 v_m}{ds^2} \frac{\phi_n}{\rho} H_3 \right. \\ & \quad \left. - \frac{dv_n}{ds} \frac{v_0}{\rho} H_1 \frac{d\phi_m}{ds} + \frac{d^2 v_n}{ds^2} \frac{\phi_m}{\rho} H_3 \right] ds \dots (37) \end{aligned}$$

Next multiply both sides of Eq.(36) by $(-\phi_n)$ and the expression obtained replacing m with n in Eq.(36) by (ϕ_m) . Integrating both expressions over the interval s_1 to s_2 and adding them, one obtains

$$\begin{aligned} & \nu_0 \left(\frac{\beta_{0n}^2 - \beta_{0m}^2}{l^2} \right) \int_{s_1}^{s_2} H_3 \phi_m \phi_n ds \\ &= \left[\phi_n \nu_0 \left\{ H_1 \left(\frac{d\phi_m}{ds} + \frac{1}{\rho} \frac{dv_m}{ds} \right) \right\} \right. \\ & \quad \left. - \phi_m \nu_0 \left\{ H_1 \left(\frac{d\phi_n}{ds} + \frac{1}{\rho} \frac{dv_n}{ds} \right) \right\} \right]_{s_1}^{s_2} \\ & \quad + \int_{s_1}^{s_2} \left[\frac{d\phi_m}{ds} H_1 \frac{v_0}{\rho} \frac{dv_n}{ds} - \frac{\phi_m}{\rho} \frac{d^2 v_n}{ds^2} H_3 \right. \\ & \quad \left. - \frac{d\phi_n}{ds} H_1 \frac{v_0}{\rho} \frac{dv_m}{ds} + \frac{\phi_n}{\rho} \frac{d^2 v_m}{ds^2} H_3 \right] ds \dots (38) \end{aligned}$$

Rewriting β_0 in Eq.(38) by α_0 using the relationship of Eq.(19), and adding Eqs.(37) and (38) one has

$$\begin{aligned} & \frac{\alpha_{0n}^4 - \alpha_{0m}^4}{l^4} \int_{s_1}^{s_2} \left\{ H_1 v_m v_n + \left(\frac{l}{\nu_0} \right)^2 H_2 \phi_m \phi_n \right\} ds \\ &= \left[v_m \left\{ \frac{d}{ds} \left(\left(\frac{d^2 v_n}{ds^2} - \frac{\phi_n}{\rho} \right) H_3 \right) \right. \right. \\ & \quad \left. \left. - v_n \left\{ \frac{H_1}{\rho} \left(\frac{d\phi_n}{ds} + \frac{1}{\rho} \frac{dv_n}{ds} \right) \right\} \right\} \right. \\ & \quad \left. - \frac{dv_m}{ds} \left\{ \left(\frac{d^2 v_n}{ds^2} - \frac{\phi_n}{\rho} \right) H_3 \right\} \right. \\ & \quad \left. - v_n \left[\frac{d}{ds} \left\{ \left(\frac{d^2 v_m}{ds^2} - \frac{\phi_m}{\rho} \right) H_3 \right\} \right. \right. \\ & \quad \left. \left. - v_m \left\{ \frac{H_1}{\rho} \left(\frac{d\phi_m}{ds} + \frac{1}{\rho} \frac{dv_m}{ds} \right) \right\} \right\} \right. \\ & \quad \left. + \frac{dv_n}{ds} \left\{ \left(\frac{d^2 v_m}{ds^2} - \frac{\phi_m}{\rho} \right) H_3 \right\} \right. \\ & \quad \left. + \phi_n \nu_0 \left\{ H_1 \left(\frac{d\phi_m}{ds} + \frac{1}{\rho} \frac{dv_m}{ds} \right) \right\} \right. \\ & \quad \left. - \phi_m \nu_0 \left\{ H_1 \left(\frac{d\phi_n}{ds} + \frac{1}{\rho} \frac{dv_n}{ds} \right) \right\} \right]_{s_1}^{s_2} \dots (39) \end{aligned}$$

Under usual boundary conditions, the right side of Eq.(39) is zero. Therefore, when $m \neq n$, the following relationship must be true.

$$\int_{s_1}^{s_2} \left\{ H_1 v_m v_n + \left(\frac{l}{\nu_0} \right)^2 H_2 \phi_m \phi_n \right\} ds = 0 \dots (40)$$

For $m=n$, the right side of Eq.(39) may be constant. Now put as follows:

$$\left. \begin{aligned} \alpha_{0n} &= \alpha_{0m} + \delta \alpha_{0m} \\ v_n &= v_m + \frac{dv_m}{d\alpha_{0m}} \delta \alpha_{0m} \\ \phi_n &= \phi_m + \frac{d\phi_m}{d\alpha_{0m}} \delta \alpha_{0m} \end{aligned} \right\} \dots (41)$$

where $\delta \alpha_{0m}$ is an infinitely small quantity. Substituting Eq.(41) into Eq.(39) and neglecting small quantities of higher order, one obtains

$$\int_{s_1}^{s_2} \left\{ H_1 v_m^2 + \left(\frac{l}{\nu_0} \right)^2 H_2 \phi_m^2 \right\} ds = M_n \dots (42)$$

Denoting Eqs.(40) and (42) with the variable x , one may obtain Eq.(20).

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