in the same direction. The decrement of the tensile stress at the straight edge in the minimum section which has been reported in the case of isotropic material\(^{(1)}\), is also larger when the fiber is perpendicular to the straight edges.

In the study of this subject, the author has been greatly enlightened by the papers by R.C.J. Howland and M. Isida, and encouraged by Professor T. Ota of Osaka University. He expresses his sincere thanks to these people.

References

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On the Bending of an Orthogonally Aeolotropic Strip with a Circular Hole*

By Takuo HAYASHI**

In this paper, the bending of an orthotropic strip with a circular hole is treated in the same way as in the previous report\(^{(1)}\) in which the tension in an orthotropic strip with a hole has been studied.

Numerical examples are worked out for the plate of oak. They show that the existence of a circular hole in a strip decreases the stress at the straight edge of the minimum section, especially in the case of a strip bounded by lines orthogonal to the wood fiber.

1. Introduction

In the previous paper\(^{(1)}\), the author treated the stress distribution in a perforated strip of orthotropic material under tension, and presented a solution according to the Howland\(^{(2)}\)-Isida's\(^{(3)}\) methods in the isotropic material. After the previous study, he has worked out the stresses in an orthotropic strip with a circular hole under bending and the results will be reported here.

2. The expressions of the boundary conditions at the rim of the strip

Suppose an orthotropic plate to be in a state of the generalized plane stress and take the \(x, y\) co-ordinates parallel to its elastic axis in its plane, then the stresses in a plate may be written as\(^{(4)}\)

\[
\begin{align*}
\sigma_x &= -E_x \left[p_1 \gamma_1 + p_2 \gamma_2 \right] \\
\sigma_y &= -G_{xy} \left[p_1 \gamma_1 + p_2 \gamma_2 \right] \\
\tau_{xy} &= I_w \left[p_1 \gamma_1 + p_2 \gamma_2 \right]
\end{align*}
\]

\[
\begin{align*}
\sigma_z &= x + 2 \mu \sigma_y \\
\tau_{x\gamma} &= (E_x/G_{xy}) - 2\nu_s
\end{align*}
\]

\(E_x, E_y\) : Modulus of elasticity in the \(x, y\) directions respectively
\(G_{xy}\) : Modulus of rigidity of the plate
\(\nu_s\) : Poisson's ratio in the \(x\) direction

where \(\gamma_j\) is an analytic function of \(z_j\) and \(p_j\) is a real positive or conjugate complex number.

Consider now an orthotropic strip with a circular hole under bending bounded by lines parallel to the elastic axis of the material. For the sake of simplicity, the half breadth of the strip will be taken as unit length and the radius of the circular hole as \(\lambda\). \(x-y\) and \(r-\theta\) co-ordinates will also be chosen as in Fig. 1.

The boundary conditions in our case are as follows.

\[
\begin{align*}
at x = \infty, \quad y &= y, \quad \sigma_x = T y \\
at y = \pm 1, \quad \sigma_x &= y = 0 \\
at r = \lambda, \quad \sigma_r &= r = 0
\end{align*}
\]

Fig. 1

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where $T$ is the bending stress at the rim of the strip having no hole in it and has the relation with the bending moment $M$ as $T=3M/2$.

Considering the stress distribution to be symmetric with respect to $x$ and antisymmetric with respect to $y$, we put the fundamental stress function as

$$
\varphi_{2n+1,j}(z_j)=iA_{2n+1,j}z_j^{2n+1}+\int_0^\infty R_{2n+1,j}(z)\sin\alpha z\,d\alpha \quad \text{for} \quad n \geq 1, \quad j=1, 2 \tag{3}
$$

where $A_{2n+1,j}$ and $R_{2n+1,j}(z)$ take real or conjugate complex numbers according to $p_j$, and $R_{2n+1,j}$ will be determined so that the fundamental stress function (3) may satisfy the free edge condition at the rim of the strip.

In the same way as presented in the previous paper, we have

$$
R_{2n+1,j} = \left[ \frac{A_{2n+1,j}(p_j \sinh p\alpha + p_k \cosh p\alpha)e^{\gamma\alpha} + A_{2n+1,j}(p_k \sinh p\alpha - p_j \cosh p\alpha)}{p_j \cosh p\alpha \sinh p\alpha - p_k \cosh p\alpha \sinh p\alpha} \right]^{2n+1}(2n)! \tag{4}
$$

$$
\begin{align*}
&j=1, 2, \quad k=2, 1, \quad j \neq k
\end{align*}
$$

It is obvious that the stresses obtained from the fundamental stress functions (3) vanish at infinity of the strip.

Since the linear combination of those stress functions also satisfies the free edge condition at the rim of the strip and vanishes at infinity, we put the stress function in this case as

$$
\varphi_j(z_j) = T \left[ \frac{i z_j}{p_j(p_j^2 - p_k^2)} + \sum_{n=1}^\infty \varphi_{2n+1,j}(z_j) \right] \quad \text{for} \quad j=1, 2 \tag{5}
$$

or expanding every term of $\varphi_{2n+1,j}(z_j)$ in the form of the power series of $z_j$ around the origin, we have

$$
\varphi_j(z_j) = i T \left[ \frac{z_j}{p_j(p_j^2 - p_k^2)} + \sum_{n=1}^\infty A_{2n+1,j}z_j^{2n+1} + \sum_{n=0}^\infty B_{2n+1,j}z_j^{2n+1} \right] \tag{6}
$$

The first term of the stress function (5) or (6), represents the uniform bending of the strip without any hole satisfying also the free edge conditions at the rim of the strip, and so the boundary conditions at $y=\pm 1$ can be written from Eqs. (4) and (6) as

$$
B_{2n+1,j} = \sum_{r=1}^\infty \left[ 2z_j^{2n+1}\alpha_{2r+1,j} A_{2r+1,j}^2 + 2z_j^{2n+1}\beta_{2r+1,j} A_{2r+1,j} \right] \tag{7}
$$

where $2z_j^{2n+1}\alpha_{2r+1,j}$ and $2z_j^{2n+1}\beta_{2r+1,j}$ are constants defined as

$$
2z_j^{2n+1}\alpha_{2r+1,j} = \frac{(-1)^{n+r}}{(2n+1)!(2r)!}z_j^{2n+1+r}I_j, \quad 2z_j^{2n+1}\beta_{2r+1,j} = \frac{(-1)^{n+r}}{(2n+1)!(2r)!}z_j^{2n+1+r}J_j \tag{8}
$$

and the symbols $^*I_j$, $^*J_j$, have the meaning as shown below.

$$
^*I_j = \int_0^\infty \frac{\alpha^{n+r}e^{-p\alpha}(p_j \sinh p\alpha + p_k \cosh p\alpha)}{p_j \cosh p\alpha \sinh p\alpha - p_k \cosh p\alpha \sinh p\alpha} \, d\alpha \tag{9}
$$

$$
^*J_j = \int_0^\infty \frac{\alpha^{n+r-1}p_j \cosh p\alpha \sinh p\alpha - p_k \cosh p\alpha \sinh p\alpha}{p_k \cosh p\alpha \sinh p\alpha - p_j \cosh p\alpha \sinh p\alpha} \, d\alpha \tag{9}
$$

The definite integrals (9) can be rewritten, in the same way as presented in the previous paper, in the form of the infinite series as

$$
^*I_j = \frac{1}{2^{n-1}} \sum_{q=0}^\infty \frac{q!(n-1)!}{r!s!} \left( \frac{p_j + p_k}{p_j - p_k} \right)^{r+s+1} \left( \frac{p_j - p_k}{p_j + p_k} \right)^{r+s} \left[ \frac{1}{[p_j(1+s+t) + p_k(1+r+t)]^n} \right] \tag{10}
$$

$$
^*J_j = \frac{4p_k}{p_j - p_k} \sum_{q=0}^\infty \frac{q!(n-1)!}{r!s!} \left( \frac{p_j + p_k}{p_j - p_k} \right)^{r+s+1} \left( \frac{p_j - p_k}{p_j + p_k} \right)^{r+s} \left[ \frac{1}{[p_j(1+s+t) + p_k(1+r+t)]^n} \right] \tag{10}
$$

3. The expressions of the boundary conditions at the rim of the hole

According to the well-known mapping function:

$$
z_j = \frac{1}{2} \left( \frac{(1 + p_j)w_j + (1 - p_j)w_j^{-1}}{w_j + w_j^{-1}} \right), \quad w_j = r_j e^{i\theta}, \quad j=1, 2, \ldots \tag{11}
$$

a circle of radius $\lambda$ with its center at the origin of the physical plane will be transformed into a unit circle on the $w_j$ plane $r_j=1$, $\theta_j=\theta$ and the stress function $\varphi_j(z_j)$ may be written as a function of $w_j$ as follows.
\[ \varphi_j(x) = f_j(w_j) = T \sum_{n=-\infty}^{\infty} A_{2n+1,j} w_j^{2n+1} \]  

The boundary conditions at the rim of the hole require the following relation between the coefficients in the stress function (12):

\[ A_{-2n+1,j}' = \frac{1}{p_j} \left[ \frac{1}{p_j - p_k} \right] \sum_{r=0}^{\infty} B_{2r+1,j} \left( \frac{\lambda}{2} \right)^{2r+1} \left[ \frac{1}{r+1} \right] \left( 1 + p_j \right) \left( 1 - p_j \right) \]  

On the other hand, we have the relations between the coefficients in the stress function \( \varphi_j(x) \) and \( f_j(w_j) \), which are derived from the mapping function, as

\[ A_{2n+1,j} = \frac{1}{p_j} \left( \frac{\lambda}{2} \right)^{2n+1} \left[ \sum_{r=0}^{\infty} B_{2r+1,j} \left( \frac{\lambda}{2} \right)^{2r+1} \left( \frac{1}{r+1} \right) \left( 1 + p_j \right) \left( 1 - p_j \right) \right] \]  

Thus, substituting the Eq. (13) into Eqs. (14), we have, as the boundary condition at the rim of the hole, the next expression with respect to the coefficients in the stress function \( \varphi_j(x) \).

\[ A_{2n+1,j} = \frac{1}{p_j} \left( \frac{\lambda}{2} \right)^{2n+1} \left[ \sum_{r=0}^{\infty} B_{2r+1,j} \left( \frac{\lambda}{2} \right)^{2r+1} \left( \frac{1}{r+1} \right) \left( 1 + p_j \right) \left( 1 - p_j \right) \right] \]  

where \( 2^{n+1} \beta_{2r-1,j} \) and \( 2^{n+1} \beta_{2r-1,j} \) are the constants defined as below.

\[ 2^{n+1} \beta_{2r-1,j} = (1 - p_j)^{2r} \left[ \sum_{q=1}^{n} \left( 1 + p_j \right) \left( 1 - p_j \right) \right] \left( \frac{\lambda}{2} \right)^{2r} \left( \frac{1}{r+1} \right) \left( 1 + p_j \right) \left( 1 - p_j \right) \]  

4. Stresses in a strip

Two sets of linear Eqs. (7) and (15) should be solved to obtain the stress distribution in a strip. To do so, the perturbation method(9) will be used conveniently.

Provided that \( A_{2n+1,j} \) and \( B_{2n+1,j} \) can be represented in the power series of \( \lambda/2 \), they may be written as

\[ A_{2n+1,j} = \sum_{r=0}^{\infty} 2^{n+1} \beta_{2r-1,j} \left( \frac{\lambda}{2} \right)^{2r} \]  

Substituting Eqs. (7) into Eqs. (17) and (15), we have

\[ 2^{n+1} \beta_{2r-1,j} = \frac{2}{(1 + p_j)(1 - p_j)^{n+1}} \left( \frac{\lambda}{2} \right)^{2r} \left( \frac{1}{r+1} \right) \left( 1 + p_j \right) \left( 1 - p_j \right) \]  

and the unknown constants \( 2^{n+1} \beta_{2r-1,j} \) and \( 2^{n+1} \beta_{2r-1,j} \) can be determined one by one alternately using the above two equations.

Stresses in a strip can be found from Eqs. (1) and (6) or (12). After some calculations, the expressions become as follows.

1. **Stresses in any point in the strip**

\[ \sigma_{ij} = \frac{\lambda}{2} \sum_{r=0}^{\infty} \left[ \sum_{q=1}^{n} 2^{n+1} \beta_{2r-1,j} \right] \left[ \sum_{r=0}^{\infty} 2^{n+1} \beta_{2r-1,j} \right] \]  

where

\[ M_{2n+1,j} = p_j \left[ 2 \left( \frac{\lambda}{2} \right)^{2n+1} \left( \frac{1}{r+1} \right) \left( 1 + p_j \right) \left( 1 - p_j \right) \right] \]
or to improve the convergence of the infinite series in Eq. (20), it is more convenient to write $M_{2q+1,j}$, considering the first term of Eq. (10), as follows.

$$M_{2q+1,j} = p_j z_j r^{(2r+1)} + \sum_{r=1}^{\infty} \left\{ \frac{2r+1}{p_j - p_k} \frac{i}{2q+1} \left( \frac{i}{2p_j} \right)^{2(r+1)} \left( \frac{i}{p_j} \right)^{2(r+1)} \right\} p_j z_j r^{2r+1} + \sum_{r=1}^{\infty} \left\{ \frac{2r+1}{p_j - p_k} \frac{4p_j}{2q+1} \left( \frac{i}{p_j} \right)^{2(r+1)} \frac{i}{p_j} \right\} p_j z_j r^{2r+1} + \sum_{r=1}^{\infty} \left\{ \frac{2r+1}{p_j - p_k} \frac{i}{2q+1} \left( \frac{i}{p_j} \right)^{2(r+1)} \frac{i}{p_j} \right\} p_j z_j r^{2r+1}$$

$$\frac{\bar{\theta}_{\theta} T \lambda}{T \lambda} = \frac{1}{2} \left( 1 + p_j - p_k \right) + \sum_{r=1}^{\infty} \left\{ \frac{2q-p_j}{q} \left( \frac{1}{2} \right)^{2r-1} \right\} L_{2q-1,1} + \sum_{r=1}^{\infty} \left\{ \frac{1+q}{q} \left( \frac{1}{q} \right)^{2q-1} \right\}$$

5. Numerical examples

The numerical calculations are worked out for the strip of oak. The modulus of elasticity of the oak plate is

$$E_x = 58 \text{ 200 kg/cm}^2 \quad E_y = 21 \text{ 900 kg/cm}^2 \quad G_{xy} = 13 \text{ 200 kg/cm}^2$$

$$\nu_x = 0.33$$

and so $p_y = 5/3 \quad p_n = 49/51$

From Eqs. (19) and (22), the stress distribution along the minimum section of the strip are shown in Fig. 2 for $\lambda = 0.2$. The figures on the right side of the stress curve mean the stress values in the strip parallel to the oak fiber and those in parenthesis concern the stresses in the strip orthogonal to the fiber.

The stress concentration factor $\frac{\bar{\theta}_{\theta} T \lambda}{T \lambda}$ and the tensile stress at the rim of the strip in the minimum section are given as

(1) when the fiber of oak is parallel to the straight edges

$$\frac{\bar{\theta}_{\theta} T \lambda}{T \lambda} = 2.314 + 5.062 \lambda^4 + 0.826 \lambda^6 + 7.576 \lambda^8 - 7.795 \lambda^{10} + \cdots$$

$$\frac{\bar{\sigma}_{T \lambda}}{T \lambda} = 1 - 1.107 \lambda^4 + 1.955 \lambda^6 - 3.197 \lambda^8 + 7.704 \lambda^{10} + \cdots$$

(2) when it is perpendicular to them

$$\frac{\bar{\theta}_{\theta} T \lambda}{T \lambda} = 1.821 + 8.184 \lambda^4 - 7.751 \lambda^6 + 41.88 \lambda^8 - 188.58 \lambda^{10} + \cdots$$

$$\frac{\bar{\sigma}_{T \lambda}}{T \lambda} = 1 - 2.652 \lambda^4 + 12.231 \lambda^6 - 44.36 \lambda^8 + 183.27 \lambda^{10} + \cdots$$

6. Conclusion

The stress distribution in an orthotropic strip with a circular hole under uniform bending has been analyzed here, and the numerical examples are worked out for the strip of oak. The results show that the stress interference between the straight edges and the circular hole is less in the case of the bending than of the uniform tension and it may be so because the circular hole is in the less stressed region of the strip under bending than under tension, especially when $\lambda$ is small.

The maximum stress occurs now at the straight edge of the strip for small $\lambda$, and yet the stress value there is to decrease slightly by the existence of a circular hole, as is in the case of uniform tension. Accordingly it may be said that the small hole on the center line of the strip has little influence on the strength of the plate in the case of uniform bending.

As is announced in the previous paper\(^{15}\), the stress interference between both boundaries is larger in this case too, when the fiber is orthogonal to the straight edges than when it is parallel to them.

In this study, the author has referred much to the papers by R. C. J. Howland and M. Isida,
and has been encouraged by Professor T. Ota of Osaka University. He expresses his sincere thanks to these people.

References

(1) T. Hayashi: This Bulletin, p. 265.


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