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Theoretical Solutions of Heat Transfer in the Hydrodynamic Entrance Length of Double Pipes*

By Katsuhisa MURAKAWA**

In the industrial heat-exchangers, there are many cases of unsteady heat transfer in the hydrodynamic entry length with temperature distributions of pipe-walls along the axial direction of double pipes and the pressure-drop from the inlet to the outlet, and in the atomic reactors (LMFR), the heat generates from the inside of the fluid.

Therefore, in these cases, the research of unsteady heat transfer generating heat from the inside of the fluid is necessary.

In this paper, theoretical solutions of the unsteady heat transfer in the hydrodynamic entrance length of vertical double pipes are made. In these solutions, such cases of unsteady heat transfer in the hydrodynamic entry length of vertical double pipes are treated that the heat generates from the inside of the fluid, free convection and laminar flow coexist at the same time, the pressure drops from the inlet to the outlet, and arbitrary temperature distributions of pipe-walls vary in the axial direction.

In the solutions, the finite Hankel transform⁽⁴⁾, the finite Fourier transform⁽⁴⁾, the Laplace transform⁽²⁾ and Bessel functions⁽¹⁾⁽³⁾ are used. But, there is not a case of such an experiment. Theoretical solutions are reduced to the steady heat transfer and compared with experiments⁽³⁾.

1. Introduction

In the industrial heat-exchangers, there are many cases that heat-exchange begins at the inlet of pipes instantly e.g., the cases of heat transfer in the hydrodynamic entry length. Moreover, the temperature is distributed in the axial direction of pipes and we encounter many cases of unsteady heat transfer accompanied by the pressure drop from the inlet to the outlet, and in recent years, there is an atomic reactor (LMFR) in which the small particles of uranium suspended in bismuth are used as heat source.

In these cases, the research of the unsteady heat transfer generating the heat in the inside of fluid is very important. Therefore, as a part of research of this case, the author accomplished the theoretical solutions of unsteady heat transfer in the hydrodynamic entrance length of vertical double pipes with annular space (annulus) in which the heat is generated from the inside of fluid, and

free convection and laminar flow coexist at the same time, and there are a pressure drop and arbitrary temperature distributions of pipe walls along the axial direction.

But, few experiments have been carried out on these cases, therefore, in the following, the author attempts the theoretical solutions of the case without heat generation of the steady heat transfer of the laminar flow only in the hydrodynamic entrance length of the vertical double pipes and investigates indirectly the theoretical results of the unsteady heat transfer accompanied by heat generation, as compared with the experimental results.

2. Theory of unsteady heat transfer in the hydrodynamic entrance length of vertical double pipes accompanied by heat generation in the fluid and having arbitrary temperature distributions of the pipe walls and the pressure drop, with free con- vection and laminar flow coexisting at the same time

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** Professor of Mechanical Engineering, Faculty of Engineering, University of Yamaguchi, Ube, Yamaguchi.

Equations of motion in hydrodynamics :

$$\frac{\partial C_r}{\partial t} + C_r \frac{\partial C_r}{\partial r} + C_z \frac{\partial C_r}{\partial Z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\nabla^2 C_r - \frac{C_r}{r^2} \right) \quad (1)$$

$$\frac{\partial C_z}{\partial t} + C_r \frac{\partial C_z}{\partial r} + C_z \frac{\partial C_z}{\partial Z} = -\frac{1}{\rho} \frac{\partial p}{\partial Z} + \nu \nabla^2 C_z + g\beta(T - T_0) \quad (2)$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial Z^2} \quad (3)$$

Equation of continuity :

$$\frac{1}{r} \frac{\partial}{\partial r}(r \cdot C_r) + \frac{\partial C_z}{\partial Z} = 0 \quad (4)$$

Energy equation :

$$\frac{\partial T}{\partial t} + C_r \frac{\partial T}{\partial r} + C_z \frac{\partial T}{\partial Z} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial Z^2} \right) + \frac{Q(r, Z, t)}{c \cdot \gamma} \quad (5)$$

Boundary conditions :

$$(C_z)_{Z=0} = \phi_0(t, r) \quad (6)$$

$$(C_z)_{r=r_1} = 0 \quad (7)$$

$$(C_z)_{r=r_2} = 0 \quad (8)$$

$$(C_z)_{t=0} = 0 \quad (9)$$

$$(C_z)_{Z=L_V} = \phi_1(t, r) \quad (10)$$

$$(C_r)_{r=r_1} = 0 \quad (11)$$

$$(C_r)_{t=0} = 0 \quad (12)$$

$$(T)_{t=0} = T_1 \quad (13)$$

$$(T)_{r=r_1} = F_1(t, Z) \quad (14)$$

$$(T)_{r=r_2} = F_2(t, Z) \quad (15)$$

$$(T)_{Z=0} = T_0 \quad (16)$$

where t : time sec, C_r : radial velocity cm/sec, C_z : axial velocity cm/sec, (r, Z) : cylindrical coordinates, ρ : density kg/cm³, p : pressure kg/cm², ν : kinematic viscosity cm²/sec, r_1 : outside radius of inner pipe cm, r_2 : inside radius of outer pipe cm, g : acceleration of gravity cm/sec², β : coefficient of expansion of volume 1/°C, T_0 : fluid temperature at the entrance °C, T : fluid temperature °C, a : thermal diffusivity cm²/sec, Q : heat generated from the inside of fluid kcal/cm³sec, c : specific heat kcal/kg°C, γ : specific gravity kg/cm³, $\phi_0(t, r)$: axial velocity distribution at the entrance cm/sec, L_V : length of hydrodynamic entrance length cm, $\phi_1(t, r)$: axial velocity distribution at the end of the hydrodynamic entrance length cm/sec, T_1 : initial temperature °C at the time $t=0$, $F_1(t, Z)$: outer surface-temperature of inner pipe °C, $F_2(t, Z)$: inner surface-temperature of outer pipe °C.

To vanish the terms of p , operate

$$\left[\frac{\partial}{\partial r}(2) - \frac{\partial}{\partial Z}(1) \right]$$

To write in dimensionless forms, put

$$C_z = \frac{1}{r} \frac{\partial \Psi}{\partial r} \quad (17)$$

$$C_r = -\frac{1}{r} \frac{\partial \Psi}{\partial Z} \quad (18)$$

stream function

$$\Psi = \bar{w} \cdot (r_2 - r_1)^2 \cdot \phi \quad (19)$$

$$x = (r - r_1) / (r_2 - r_1) \quad (20)$$

$$z = Z / L \quad (21)$$

$$L = L_V, T_1 = T_0,$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{x + \frac{r_1}{r_2 - r_1}} \frac{\partial}{\partial x} + \left(\frac{r_2 - r_1}{L} \right)^2 \frac{\partial^2}{\partial z^2} \quad (22)$$

$$Gr = \frac{g\beta(r_2 - r_1)^3(T_{w1m} - T_{w2m})}{\nu^2} \quad (23)$$

$$Re = \frac{\bar{w} \cdot 2(r_2 - r_1)}{\nu} = \frac{\bar{w} \cdot (D_2 - D_1)}{\nu} \quad (24)$$

$$\varepsilon = \frac{Gr}{Re^2} \quad (25)$$

$$\phi = \phi_0 + \varepsilon \cdot \phi_1 + \varepsilon^2 \cdot \phi_2 + \dots, \quad (26)$$

$$\frac{T - T_0}{T_{w1m} - T_{w2m}} = \theta = \frac{1}{Re} (\theta_0 + \varepsilon \cdot \theta_1 + \varepsilon^2 \cdot \theta_2 + \dots), \quad (27)$$

where Ψ : stream function, $\bar{w}$: mean velocity of axial direction cm/sec, ϕ : dimensionless stream function, L : heating length cm, T_{w1m} : mean temperature of the surface of inner pipe wall °C, T_{w2m} : mean temperature of the surface of outer pipe wall °C, $D_2 = 2r_2$: inner diameter of outer pipe cm, $D_1 = 2r_1$: outer diameter of inner pipe cm, Gr : Grashof number, Re : Reynolds number, θ : dimensionless temperature.

As written in dimensionless forms, compare with coefficients of ε and we obtain the equations of stream functions in dimensionless forms. Consequently we can obtain the temperature distribution, therefore we can obtain the Nusselt Number N_{u1} of the inside pipe.

$$N_{u1} = \frac{2 \cdot \alpha_1 (r_2 - r_1)}{\lambda_1} = -\frac{2 \left(\frac{\partial \theta}{\partial x} \right)_{x=0}}{f_0(t, z)} = a \text{ function}$$

$$\text{of } \left[\text{Graetz Number } G_{z2} = \frac{\pi}{4} \cdot Re \cdot Pr \cdot \frac{2(r_2 - r_1)}{L} \right. \\ = \frac{\pi}{4} \sigma_2, \text{ or } \left(Re \cdot Pr \cdot \frac{(D_2 - D_1)}{L} = \sigma_2 \right), \left(Gr \cdot Pr \right. \\ \left. \times \frac{D_2 - D_1}{L} \right), \text{ the dimensionless temperature}$$

$$\text{distribution of the outer surface of inner pipe } f_0(t, z), \text{ the dimensionless temperature distribution of the inner surface of outer pipe } f_1(t, z), \text{ heat generated from the inside of fluid } Q, \text{ and time } t. \quad (28)$$

where α_1 : heat transfer coefficient of the inside

pipe kcal/m²hr°C, λ_1 : thermal conductivity of the inside pipe kcal/mhr°C, $Pr = \nu/\alpha$: Prandtl number.

Thereby we obtain the theoretical Nusselt Number N_{u1} of unsteady heat transfer for arbitrary Graetz Number, arbitrary G_r , R_s , P_r , f_0 , f_1 , Q and arbitrary dimensions on such occasions in the hydrodynamic entrance length of vertical double pipes with annular space (annulus) in which the heat is generated from the inside of fluid, and free convection and laminar flow coexist at the same time, and there are a pressure drop and arbitrary temperature distributions of pipe walls along the axial direction.

But, there is not an experiment to compare with this solution, therefore, in the following, the author induces the theoretical solutions for the case without heat generation of the steady heat transfer of the laminar flow only in the hydrodynamic entrance length of the vertical double pipes to investigate indirectly the theoretical results of the unsteady heat transfer accompanied by heat generation, as compared with the experimental results.

3. Theory of steady heat transfer of laminar flow only without heat generation from the inside of fluid in the hydrodynamic entrance length of vertical double pipes with annular space (Annulus)

We obtain the next equation as the first approximation of $C_z/\bar{w}$ which satisfies the boundary conditions completely in the place close to the wall of inner pipe.

$$\frac{C_z}{\bar{w}} = \frac{(1+z)}{\left(x + \frac{r_1}{r_2 - r_1}\right)} \sum_{s=1}^{\infty} \frac{2U_0(x, s)}{\sinh k_s \frac{L}{r_2 - r_1} \left[\left(\frac{r_2}{r_2 - r_1}\right)^2 \cdot \left\{ U_0' \left(k_s \frac{r_2}{r_2 - r_1} \right) \right\}^2 - \left(\frac{r_1}{r_2 - r_1}\right)^2 \cdot \left\{ U_0' \left(k_s \frac{r_1}{r_2 - r_1} \right) \right\}^2 \right]} \times \left[\sinh k_s \frac{L}{r_2 - r_1} (1-z) \cdot \int_0^1 f_0(t) \cdot U_0(t, s) \cdot t \cdot dt + \frac{1}{2} \sinh k_s \frac{L}{r_2 - r_1} z \int_0^1 f_1(t) \cdot U_0(t, s) \cdot t \cdot dt \right] \dots (29)$$

$$U_0(x, s) = \frac{J_0 \left\{ k_s \left(x + \frac{r_1}{r_2 - r_1} \right) \right\}}{J_0 \left\{ k_s \frac{r_1}{r_2 - r_1} \right\}} - \frac{Y_0 \left\{ k_s \left(x + \frac{r_1}{r_2 - r_1} \right) \right\}}{Y_0 \left\{ k_s \frac{r_1}{r_2 - r_1} \right\}} \dots (30)$$

k_s is a root of the next equation.

$$J_0 \left\{ k \frac{r_2}{r_2 - r_1} \right\} \cdot Y_0 \left\{ k \frac{r_1}{r_2 - r_1} \right\} = Y_0 \left\{ k \frac{r_2}{r_2 - r_1} \right\} \cdot J_0 \left\{ k \frac{r_1}{r_2 - r_1} \right\} \dots (31)$$

J_0 : Bessel function, Y_0 : Neumann function.

In the numerical calculation of velocity distribution in the hydrodynamic entry length from Eq. (29), we must give $f_0(t)$, inlet velocity distribution and $f_1(t)$, velocity distribution of steady flow in the downstream passed over the hydrodynamic entry length and fully developed.

The inlet velocity distribution is generally $F_0(r)$, the function of radius and considered to be the distribution of a rectangle with roundish corners. When we represent $f_0(t)$ in this curve, we must use the elliptic functions of the first and second kind, and therefore, here, to simplify, $f_0(t)$ is treated as the uniform distribution, that is to say,

$$(C_z)_{z=0} = \bar{w} \dots (32)$$

$$\therefore f_0(t) = \left(t + \frac{r_1}{r_2 - r_1} \right) \dots (33)$$

Next, the velocity distribution of steady flow in the downstream fully developed is shown by the next equation.

$$(C_z/\bar{w})_{z=L} = 2 \cdot w_0(x) \dots (34)$$

$$\therefore f_1(t) = 2 \cdot w_0(t) \times \left(t + \frac{r_1}{r_2 - r_1} \right) \dots (35)$$

$$f_1(t) = 2 \cdot \frac{[-\{(r_2/r_1 - 1)t + 1\}^2 + 1 + \{(r_2/r_1)^2 - 1\} / \log_e r_2/r_1 \times \log_e \{(r_2/r_1 - 1)t + 1\}]}{(r_2/r_1)^2 + 1 - \{(r_2/r_1)^2 - 1\} / \log_e r_2/r_1} \cdot \left(t + \frac{r_1}{r_2 - r_1} \right) \dots (36)$$

To simplify Eq. (29), take the first term of sinh and substitute $f_0(t)$ and $f_1(t)$ mentioned above, and we obtain the next equation as the approximate equation for the numerical calculation.

$$C_z/\bar{w} = (1 - z^2) + (z + z^2) \cdot w_0(x) \dots (37)$$

This equation is the case when only the first term of sinh is taken and considered simply like (32), and therefore, this equation does not satisfy the boundary conditions at $x=0$ and $x=1$, but it is found to be

$(Cz/\bar{w})_{x=0, x=1}=0$ from Eq. (29) which is a more accurate solution than Eq. (37) and so, Eq. (37) becomes an approximate equation which is very convenient for the calculation of velocity distribution at the arbitrary points except for $x=0$ and $x=1$ only, i.e., the velocity ($=0$) on the surface of inside and outside pipe-walls.

Energy equation and boundary conditions are shown by the next relations.

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{1}{x + \frac{r_1}{r_2 - r_1}} \cdot \frac{\partial \theta}{\partial x} + \left(\frac{r_2 - r_1}{L} \right)^2 \cdot \frac{\partial^2 \theta}{\partial z^2} = \frac{\sigma_2}{4} \cdot w(x, z) \cdot \frac{\partial \theta}{\partial z} \quad (38)$$

$$(\theta)_{x=0} = f_0(z) \quad (39)$$

$$(\theta)_{x=1} = f_1(z) \quad (40)$$

$$(\theta)_{z=0} = 0 \quad (41)$$

where $\sigma_2 = R_e \cdot P_r \cdot \frac{D_2 - D_1}{L} = \frac{4}{\pi} \cdot G_{z2}$, G_{z2} = Graetz number $= \frac{\pi}{4} \cdot R_e \cdot P_r \cdot \frac{2(r_2 - r_1)}{L}$, $w(x, z) = Cz/\bar{w} = \text{Eq. (37)}$,

$f_0(z)$ = dimensionless axial temperature distribution of outside surface of inner pipe, $f_1(z)$ = dimensionless axial temperature distribution of inside surface of outer pipe.

To solve these equations, put

$$\theta = u + [\{f_1(z) - f_0(z)\} \cdot x + f_0(z)] \cdot R_e \quad (42)$$

$$F_1(x, z) = \frac{f_0(z) - f_1(z)}{x + \frac{r_1}{r_2 - r_1}} \cdot R_e - \left(\frac{r_2 - r_1}{L} \right)^2 \cdot \left[x \left\{ \frac{d^2}{dz^2} f_1(z) - \frac{d^2}{dz^2} f_0(z) \right\} + \frac{d^2}{dz^2} f_0(z) \right] \cdot R_e \\ + \frac{\sigma_2}{4} w(x, z) \left[x \left\{ \frac{d}{dz} f_1(z) - \frac{d}{dz} f_0(z) \right\} + \frac{d}{dz} f_0(z) \right] \cdot R_e \quad (43)$$

and we obtain u_0 the first approximation of u from the next relations.

$$\frac{\partial^2 u_0}{\partial x^2} + \frac{1}{x + \frac{r_1}{r_2 - r_1}} \cdot \frac{\partial u_0}{\partial x} + \left(\frac{r_2 - r_1}{L} \right)^2 \cdot \frac{\partial^2 u_0}{\partial z^2} = F_1(x, z) \quad (44)$$

$$(u_0)_{x=0} = 0 \quad (45)$$

$$(u_0)_{x=1} = 0 \quad (46)$$

$$(u_0)_{z=0} = 0 \quad (47)$$

Introducing the finite Hankel transform⁽⁴⁾ at (45) and (46), and Laplace transform⁽²⁾ at (47), we obtain the next solution from Eq. (42).

$$\theta_0 = [\{f_1(z) - f_0(z)\} x + f_0(z)] \cdot R_e + \sum_{s=1}^{\infty} \frac{2 \cdot k_s^2 \cdot J_0^2 \left(k_s \frac{r_1}{r_2 - r_1} \right)}{J_0^2 \left(k_s \frac{r_2}{r_2 - r_1} \right) - J_0^2 \left(k_s \frac{r_1}{r_2 - r_1} \right)} \left[J_0 \left\{ k_s \left(x + \frac{r_1}{r_2 - r_1} \right) \right\} \cdot Y_0 \left(k_s \frac{r_2}{r_2 - r_1} \right) \right. \\ \left. - J_0 \left(k_s \frac{r_2}{r_2 - r_1} \right) \cdot Y_0 \left\{ k_s \left(x + \frac{r_1}{r_2 - r_1} \right) \right\} \right] \times \frac{L}{r_2 - r_1} \cdot \frac{1}{k_s} \\ \times \int_0^z \left[\int_0^1 \left(x + \frac{r_1}{r_2 - r_1} \right) F_1(x, \zeta) \cdot \left[J_0 \left\{ k_s \left(x + \frac{r_1}{r_2 - r_1} \right) \right\} \cdot Y_0 \left(k_s \frac{r_2}{r_2 - r_1} \right) \right. \right. \\ \left. \left. - J_0 \left(k_s \frac{r_2}{r_2 - r_1} \right) \cdot Y_0 \left\{ k_s \left(x + \frac{r_1}{r_2 - r_1} \right) \right\} \right] \cdot dx \right] \times \sinh k_s \frac{L}{r_2 - r_1} (z - \zeta) \cdot d\zeta \quad (48)$$

When we find $u_1, u_2, \dots$, we only solve the equations of the same form by the same method and only the values of $F_1(x, \zeta)$ in the solutions change. Consequently we can obtain the temperature distribution and the mean Nusselt number N_{u1} of the inside pipe.

$$N_{u1} = \frac{2\alpha_1(r_2 - r_1)}{\lambda_1} = - \frac{2 \int_0^1 \left(\frac{\partial \theta}{\partial x} \right)_{x=0} \cdot dz}{\int_0^1 f_0(z) \cdot dz} \quad (49)$$

For the numerical calculation from Eq. (49), put $r_2/r_1 = D_2/D_1 = 55 \text{ mm}/35 \text{ mm} = 1.833 \text{ } 333$, $L = 1 \text{ } 400 \text{ mm}$, i.e., the dimensions of experimental equipment, and $f_0(z) = -0.716 \text{ } 7z^2 + 1.268z + 0.045 \text{ } 0$, $f_1(z) = 0.010 \text{ } 77z + 0.246 \text{ } 2$ from the experimental results⁽³⁾, and

$$w(x, z) \doteq 1 - z^2 + (z + z^2) \cdot w_0(x) \dots (37)'$$

where, from Eq. (36),

$$w_0(x) = \frac{[-\{(r_2/r_1 - 1)x + 1\}^2 + 1 + \{(r_2/r_1)^2 - 1\} / \log_e r_2/r_1 \times \log_e \{(r_2/r_1 - 1)x + 1\}]}{(r_2/r_1)^2 + 1 - \{(r_2/r_1)^2 - 1\} / \log_e r_2/r_1} \dots (50)$$

and after very much numerical calculation by the Table of Bessel functions⁽¹⁾⁽⁵⁾, we obtain the first approximation (calculated to $s=2$) in the next equation.

$$N_{u1} \doteq 0.8676 + \sigma_2 \times 0.106 = 0.8676 + R_e \cdot P_r \cdot (D_2 - D_1) / L \times 0.106 = 0.8676 + 4/\pi \times G_{z2} \times 0.106 \dots (51)$$

In Fig. 1, the relation of N_{u1} and σ_2 is shown, the marks $\bigcirc$ being the experimental results of laminar heat transfer of water heated from the inside pipe and the solid straight line showing (51).

4. Approximate solution of steady heat transfer of laminar flow through the double pipes without heat generation

Energy equation and boundary conditions in dimensionless forms :

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{1}{x + \frac{r_1}{r_2 - r_1}} \cdot \frac{\partial \theta}{\partial x} = \frac{\sigma_2}{4} \cdot w(x, z) \cdot \frac{\partial \theta}{\partial z} \dots (52)$$

$$(\theta)_{x=0} = f_0(z) \dots (53)$$

$$(\theta)_{x=1} = f_1(z) \dots (54)$$

$$w(x, z) \doteq 1 - z^2 + (z + z^2) \cdot w_0(x) \dots (55)$$

$$\text{Put } \theta = u + \{f_1(z) - f_0(z)\} \cdot x + f_0(z) \dots (56)$$

The Nusselt number is given by the next equation.

$$N_u = -2 \int_0^1 \left(\frac{\partial \theta}{\partial x} \right)_{x=0} \cdot dz / \int_0^1 f_0(z) \cdot dz \dots (57)$$

In the next, from the author's experimental results⁽³⁾ of water heated from the inside pipe in the hydrodynamic entry length of the vertical double pipes, put $f_0(z) = -0.7167z^2 + 1.268z + 0.0450$ (the temperature distribution of the wall of inside pipe), $f_1(z) = 0.01077z + 0.2461$ (the temperature distribution of the wall of outside pipe), where the dimensions of experimental apparatus are as follows :

$$2r_2/2r_1 = 55 \text{ mm}/30 \text{ mm}, \quad L = 1400 \text{ mm},$$

$$d_2 = 55 \text{ mm (the steel pipe),}$$

$$d_1 = 30 \text{ mm (the copper pipe)}$$

The values of k_s in the above mentioned dimensions calculated by the author, are as follows :

$$k_1 = 3.141592654, \quad k_2 = 6.283185308,$$

$$k_3 = 9.424777962, \dots$$

The first approximation of Eq. (57) in this case is given by the next equation.

$$N_{u1} = 0.7701 + 0.0201 \times \sigma_2 + 0.000596 \times \sigma_2^2 \dots (58)$$

where, $s=1$, i.e., the initial term only was calculated and Simpsons' 1/3 rule was applied to the method of numerical integration and the limits of application of σ_2 in Eq. (58) were $100 < \sigma_2 < 180$. If σ_2 is smaller than 100, free convection may influence and if is larger than 180, turbulent flow may occur, and therefore these cases are beyond the limits of application of σ_2 in Eq. (58). The approximate solution (58) is shown by the dotted line in Fig. 1.

5. The examples of numerical calculations

After very much numerical calculation by the Table of Bessel functions⁽¹⁾⁽⁵⁾, we obtain Eq. (51) for Eq. (49) and Eq. (58) for Eq. (57).

In the approximate calculations of Eqs. (51), (37) was used simply, but, in the exact calculations, the correct equation of $w(x, z)$, i.e., Eq. (29) has to be used, and so, we will obtain more correct results for N_{u1} . These results of calculations are shown by Fig. 1. In Fig. 1, the solid straight line represents the correct solution (51), the dotted line the approximate solution (58), the marks $\bigcirc$ being the experimental results of laminar heat transfer of the water heated from the inside pipe, with the influence of free convection eliminated. This experiment was accomplished by the author⁽³⁾.

The experimental points of Fig. 1 show the case of the laminar heat transfer only, without

Fig. 1 Comparison of theoretical and experimental results

the influence of free convection, and so, we can compare with the results of theoretical solutions of (49), (51) and (57), (58) directly.

There is no experiment to compare directly with the theoretical results of unsteady heat transfer in the hydrodynamic entrance length of vertical double pipes accompanied by heat generation in the inside of fluid and having arbitrary temperature distributions of the pipe walls and a pressure drop, with free convection and laminar flow coexisting at the same time, and therefore to investigate indirectly, simplifying these theoretical equations, the methods of solutions of (49), (51), (57) and (58) were obtained. Comparing with experimental results⁽³⁾, the theoretical results were found in good agreement.

6. Conclusion

The research of unsteady heat transfer with heat generated from the inside of fluid and temperature distributions of the surface of pipe-walls of heat exchangers becomes the necessary research item in future in the fields of the atomic reactors, the chemical factories etc., and the author attempted the theoretical solutions mentioned above, as a part of the research in this field. From the results of general theory, when free convection and laminar flow coexist at the same time, as the Nusselt number N_u is found in the form of $N_u = [\text{laminar heat transfer } N_{ul}] + [\text{free convection } N_{uf}]$, putting $N_{uf} = 0$ in the case of laminar heat transfer only, we obtain N_{ul} ; and putting $N_{ul} = 0$ in the case of free convection only, we obtain N_{uf} .

By this method of general solution, we can show the results with a wide scope of application, including the case of coexistence of free convection and laminar flow, the case of free convection only,

also laminar flow only and all the cases. From this results, it is found that the Graetz number G_{z2} is in the form of $R_e \cdot P_r \cdot (D_2 - D_1)/L$, the term of free convection is in the form of $G_r \cdot P_r \cdot (D_2 - D_1)/L$, and the mean Nusselt number N_{u1} in the direction of pipe axis in the case of unsteady heat transfer accompanied by heat generated in the inside of fluid, is a function of [Graetz number or $R_e \cdot P_r \cdot (D_2 - D_1)/L$, $G_r \cdot P_r \cdot (D_2 - D_1)/L$, the temperature distribution of inside pipe f_0 , the temperature distribution of outside pipe f_1 , the term of heat generated from the inside of fluid Q , time t]. There is no experiment to compare directly with the theoretical results of unsteady heat transfer accompanied by the heat generated from the inside of fluid, so to investigate indirectly, simplifying these theoretical equations, the methods of solutions of (49), (51), (57) and (58) were obtained. Compared with the experimental results⁽³⁾, the theoretical results were found in good agreement. The approximate solutions of (57) and (58) are not yet sufficient. It is found that we can obtain the results with greater accuracy, by the general method adopted in this theory.

Moreover it is very convenient that the Table of Bessel functions⁽¹⁾⁽⁵⁾ are available for numerical calculation of this theory.

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