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Influence of Intermediate Principal Stress on Plastic Flow*

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Let σ_1 , σ_2 and σ_3 ($\sigma_1 > \sigma_2 > \sigma_3$) be the principal stresses, and $\sigma_3 = 0$. Taking σ_1 and σ_2 as the axes, the criterion of plastic flow is not expressed by a smooth curve by the authors' theory, but a curve composed of two straight lines which intersect at $\sigma_1 = 4\sigma_2$ in the range between tension and pure shear.

Experiments were carried out. The results show that there is an abrupt bent as expected according to the authors' theory.

A new theory of plasticity was advanced by the authors, and it was explained in the previous paper⁽¹⁾⁽²⁾ that not only the stress relation between plastic tension and plastic pure shear, but also the hysteresis loops, the anisotropy due to plastic deformation, and the relation between the yield point of mild steel and the plastic deformation beyond that point can be accounted for by this theory. Moreover, the strain ratio relation in plastic deformation will also be accounted for.

In this paper the investigation on the criterion of plastic flow in the range between tension and pure shear will be reported.

The theory of plasticity

Let, $\sigma_x, \sigma_y, \sigma_z$ be the principal stresses,

τ_1, τ_2, τ_3 be the principal shearing stresses, or

$$\left\{ \begin{array}{l} \tau_1 = \frac{1}{2}(\sigma_y - \sigma_z) \\ \tau_2 = \frac{1}{2}(\sigma_z - \sigma_x) \\ \tau_3 = \frac{1}{2}(\sigma_x - \sigma_y) \end{array} \right.$$

S_1, S_2, S_3 be the shearing stresses such as shown in Fig. 1; or

$$\left\{ \begin{array}{l} \tau_1 = S_1 - \frac{1}{2}S_2 - \frac{1}{2}S_3 \\ \tau_2 = -\frac{1}{2}S_1 + S_2 - \frac{1}{2}S_3 \\ \tau_3 = -\frac{1}{2}S_1 - \frac{1}{2}S_2 + S_3 \end{array} \right\} \dots\dots\dots (1)$$

$\epsilon_x, \epsilon_y, \epsilon_z$ be the principal strains,

$\gamma_1, \gamma_2, \gamma_3$ be the shear strains, or

$$\left\{ \begin{array}{l} \gamma_1 = \epsilon_y - \epsilon_z \\ \gamma_2 = \epsilon_z - \epsilon_x \\ \gamma_3 = \epsilon_x - \epsilon_y \end{array} \right.$$

I, II, III be the directions of shears; the directions of shear strains γ_1, γ_2 and γ_3 are here to be called the directions I, II and III, then, the authors' theory of plasticity is as follows:—

1. The three shears in directions I, II and III are to be considered individually.

Then the shearing resistances in the three directions must also be considered individually, and let them be R_1, R_2 and R_3 . Now consider a deformation of an elemental sphere shown in Fig. 2; this is a pure shear in the direction II. It is generally thought that such a deformation is formed by a slip in II under the shearing stress τ_2 , and in such a case τ_2 must be

$$\tau_2 = \frac{1}{2}R_1 + R_2 + \frac{1}{2}R_3$$

R_1, R_2 and R_3 are the resistances to the increase of shear, and they must correspond to the shearing stresses, but as seen in the above equation they do not correspond directly

Fig. 1 Shearing stresses S_1, S_2 and S_3

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to τ_1 , τ_2 , and τ_3 , but to S_1 , S_2 , and S_3 .

2. When the strains are increasing monotonously, the magnitudes of R_1 , R_2 , and R_3 are equal, or

$$R_1 = R_2 = R_3 = R$$

provided that the directions of principal strains coincide with those of principal stresses.

3. When the direction of strain is reversed concerning each direction, the change is elastic, and the stress range of this elastic change is R .

Those explained in 1, 2, and 3 are the essence of the authors' theory of plasticity which is characterized by individual treatment of the shears in the three directions and so far as each direction is concerned, it represents a shearing stress theory.

Now the states of stress in thin hollow cylinders subjected to combined tension and internal pressure will be considered according to this theory, taking z -axis parallel to the axis of the cylinder, y -axis to the circumferential direction and x -axis to the radial direction.

In the case of pure shear shown in Fig. 2, the strain in II is of positive sign and those in I and III are of negative sign, so the state of stress is⁽¹⁾

$$\left. \begin{aligned} S_1 &= -R \\ S_2 &= R \\ S_3 &= -R \end{aligned} \right\} \dots\dots\dots (2)$$

Expressed in τ 's, (2) becomes

$$\left. \begin{aligned} \tau_1 &= -R \\ \tau_2 &= 2R \\ \tau_3 &= -R \end{aligned} \right\} \dots\dots\dots (2)'$$

In the case of tension, the deformation will be as shown in Fig. 3, and the state of stress is⁽¹⁾

$$\left. \begin{aligned} S_1 &= -R \\ S_2 &= R \\ S_3 &= \pm \frac{1}{2}R \end{aligned} \right\} \dots\dots\dots (3)$$

$$\left. \begin{aligned} \tau_1 &= -\frac{7}{4}R \\ \tau_2 &= \frac{7}{4}R \\ \tau_3 &= 0 \pm \frac{1}{2}R \end{aligned} \right\} \dots\dots\dots (3)'$$

It is considered that small slips in I and II take place alternately in tension. When a slip takes place in I the strain induced in III is of positive sign, but, when the slip is in II, it is of negative sign, and thus the shears of positive and negative sign are alternated in III. τ_1 and τ_2 must be equal in magnitude in tension, and as the stress range of elastic change is R , the value of S_3 should

Fig. 2 Pure shear

Fig. 3 Deformation under tension

Fig. 4 Diagrammatic expressions of stress states in the range between tension and pure shear

be $+R/2$ for the slip in I, and $-R/2$ for the slip in II. τ_3 is macroscopically zero, but there exists an internal stress of $\pm R/2$ in III.

Influence of intermediate principal stress

Let,

$$\left. \begin{aligned} \sigma_1 &= \sigma_z - \sigma_x \\ \sigma_2 &= \sigma_y - \sigma_x \\ \sigma_3 &= 0 \end{aligned} \right\} \dots\dots\dots (4)$$

Considering that the hydrostatic compression or tension has no effect on plastic deformation, σ_1 , σ_2 , and σ_3 can be taken as principal stresses without losing generality.

In tension $\sigma_2 = 0$, and in pure shear $\sigma_2 = \sigma_1/2$. Here the states of stress in the plastic flow in the range between tension and pure shear, or in the range $0 < \sigma_2 < \sigma_1/2$, will be investigated. And if the states in this range are known, we can get easily the states in all the range.

The state of stress of plastic tension expressed by (3), will be diagrammatically shown as in Fig. 4 (a), and that of pure shear expressed by (2) will be shown as in Fig. 4 (e). Comparing these two, the difference lies in the value of S_3 in III. The value of S_3 varies as the intermediate principal stress σ_2 increases, and reaches $-R$ as shown in Fig. 4 (e) at $\sigma_2 = \sigma_1/2$. Thus the states of stress in the range between tension and pure shear should be as shown in Fig. 4 (b), (c), and (d).

From Fig. 4 (a) to (c), the elastic stress range of $\pm R/2$ moves as σ_2 increases, and the state of

stress in this range can be given by

$$\left. \begin{aligned} S_1 &= -R \\ S_2 &= R \\ S_3 &= -\alpha R \pm \frac{1}{2}R \end{aligned} \right\} \dots\dots\dots (5)$$

where, $0 < \alpha < \frac{1}{2}$

Or,

$$\left. \begin{aligned} \tau_1 &= -\frac{7}{4}R + \frac{1}{2}\alpha R \\ \tau_2 &= \frac{7}{4}R + \frac{1}{2}\alpha R \\ \tau_3 &= -\alpha R \pm \frac{1}{2}R \end{aligned} \right\} \dots\dots\dots (5)'$$

$\pm R/2$ in τ_3 is an internal stress, and when the stress induced by the external load is considered, this term can be neglected. On the other hand,

$$\left. \begin{aligned} \tau_1 &= -\frac{1}{2}(\sigma_1 - \sigma_2) \\ \tau_2 &= \frac{1}{2}\sigma_2 \\ \tau_3 &= -\frac{1}{2}\sigma_2 \end{aligned} \right\} \dots\dots\dots (6)$$

From Eqs. (5)' and (6), we have

$$\left. \begin{aligned} \sigma_1 &= \frac{7}{2}R + \alpha R \\ \sigma_2 &= 2\alpha R \end{aligned} \right\} \dots\dots\dots (7)$$

Or,

$$\sigma_1 = \frac{7}{2}R + \frac{1}{2}\sigma_2 \dots\dots\dots (8)$$

Fig. 5 Criterion of plastic flow

This is the relation between σ_1 and σ_2 in the plastic flow. The straight line TQ in Fig. 5 shows this relation; the point T corresponds to tension or the stress state in Fig. 4 (a), and the point Q to the state in Fig. 4 (c).

In Fig. 4 (c) the value of S_3 for the slip in II is $-R$, and S_3 cannot decrease further. Meanwhile, the value of S_3 for the slip in I will continue to decrease, and in the state in Fig. 4 (e), or in pure shear, S_3 should become $-R$ for both slips. Thus, the states of stress between Fig. 4 (c) and (e) are

$$\left. \begin{aligned} S_1 &= -R \\ S_2 &= R \\ S_3 &= -\alpha R \pm (1-\alpha)R \end{aligned} \right\} \dots\dots\dots (9)$$

where, $\frac{1}{2} < \alpha < 1$.

Or,

$$\left. \begin{aligned} \tau_1 &= -2R + \alpha R \\ \tau_2 &= 2R \\ \tau_3 &= -\alpha R \pm (1-\alpha)R \end{aligned} \right\} \dots\dots\dots (9)'$$

From Eqs. (9)' and (6), we have

$$\left. \begin{aligned} \sigma_1 &= 4R \\ \sigma_2 &= 2\alpha R \end{aligned} \right\} \dots\dots\dots (10)$$

The horizontal line QP in Fig. 5 shows this relation.

Thus, the criterion of plastic flow is not expressed by a smooth curve by the authors' theory, but by a curve composed of two straight lines in the range between tension and pure shear.

By putting

$$\sigma_2/\sigma_1 = n$$

$n=0$ in tension, and in the range $0 < n < 1/4$ the criterion is expressed by a straight line of inclination equal to $1/2$, and in the range $1/4 < n < 1/2$ it is expressed by a horizontal line, and there is an abrupt bent at $n=1/4$.

Experiment 1

It was already ascertained that the ratio of plastic stresses under tension and under pure shear was $7/8$, comparing the states of stress where the values of R are equal⁽¹⁾. To investigate the relations in the range between tension and pure shear, experiments were carried out for various values of n .

Test pieces were hollow cylinders of 18 mm outer diameter and 16 mm inner diameter. The material was brass; the compositions were Cu 59.59%, Zn 38.36%, Pb 1.29%, Sn 0.47%, and Fe 0.29%. The experiments were carried out by applying various ratios of combined tension and internal pressure. The results are shown in Fig. 6; σ_1 is taken in the ordinate and longitudinal strain ϵ_z in the abscissa.

Fig. 6 Stress-strain relations for various values of n

In Fig. 5 the stresses are expressed in the ratios to R . R increases with strain, and is considered to be a function of the amount of total slips, and so, in the range between tension and pure shear, the values of R will be equal when the values of ε_z are equal.

σ_1 in tension is equal to $7R/2$. Taking the ratios of σ_1 of various curves in Fig. 6 to that of tension at equal longitudinal strain, the values of σ_1 become as shown in Fig. 7. Comparing with Fig. 5, the theory and experiments are found to conform with each other very closely.

Test pieces were cut out from the same bar, but the material could not be considered to be perfectly uniform. And this experiment, though it conforms with theory very closely, will not be sufficient as the evidence of the existence of an abrupt bent at $n=1/4$. So we carried out another experiment with one test piece.

Experiment 2

A test piece of hollow cylinder was at first deformed under tension, and then an internal pressure was applied step by step to examine whether there was an abrupt bent at $n=1/4$ or not. The material and dimensions of the test piece were the same as before.

Fig. 8 shows the results of such an experiment.

Fig. 7 Experimental results on the criterion of plastic flow

In this case the testing machine used was of self-balance type. At first tension was applied until ε_z became ε_{z0} , and after that an internal pressure was applied, the testing machine being left as it was. It can be seen in the Figure that there is an abrupt change at $n=1/4$, but the shape of the curve is quite different from that in Fig. 5.

This is perhaps due to the fact that the method of measurement was not suitable to the purpose of this experiment. The reasons are :

1. The state of stress under internal pressure can be considered to be composed of a hydrostatic tension and a shear of negative sign in III. Elastic extension in all the direction will be induced by the hydrostatic tension, and plastic longitudinal strain will be induced by the shear in III, as the change in II is plastic and that in I is elastic. Thus the test piece elongates a little under the internal pressure, and as the testing machine is of self balance type the tensile load decreases. Thus, what we measured in this experiment was the decrease of the tensile load due to the internal pressure.

2. As explained above, the change in II under the internal pressure is plastic, but that in I is elastic because the direction of shear is reversed in I, and the absolute value of strain in I decreases elastically. But, what we want to measure is the stresses in the state of plastic flow where the changes are plastic in all three directions.

To obtain the state of plastic flow where the change in I should be also plastic, it is necessary, after the application of internal pressure, to increase the tensile load until the beginning of plastic flow under tension.

Another experiment was carried out by this revised method. At first the internal pressure was increased a little, and then, the internal pressure being held constant, the tensile load was increased

until the beginning of longitudinal plastic flow, and after the load settled, the reading of the tensile load was taken. This procedure was repeated.

Figs. 9 and 10 show the results obtained in such a way. In the experiment shown in Fig. 9, the test piece was at first deformed under tension until the strain ϵ_z became ϵ_{z0} , and then the internal pressure and tension were applied in such a way as mentioned above. During the measurement of the points shown in the Figure the longitudinal strain increased about 2.1%; this strain is

Fig. 9 Experimental results of the criterion of plastic flow

Fig. 8 Stress change due to internal pressure

Fig. 10 Experimental results on the criterion of plastic flow

composed of the elongation due to the internal pressure, that due to the increase of tensile load and that due to the longitudinal plastic flow for measurements. Similarly in the case shown in Fig. 10 the increase of the longitudinal strain was 2.4%.

The results show clearly that there is an abrupt bent at $n=1/4$.

In Figs. 9 and 10, the ratio of σ_1 at $n=0$ to that at $n=1/4$ is not $7/8$, and the values of σ_1 in the range $1/4 < n < 1/2$ are not constant. But this is quite natural as the value of R varies in these experiments. It is in terms of σ_1/R that the ratio at $n=0$ to that at $n=1/4$ should be $7/8$, or constant in the range $1/4 < n < 1/2$. The purpose of these experiments is to examine whether there is an abrupt bent at $n=1/4$ or not.

Conclusions

The criterion of plastic flow by the authors' theory is not expressed by a smooth curve, but, in the range between tension and pure shear, it is composed of two straight lines intersecting at $n=1/4$. Experiments show that there is an abrupt bent as expected according to the authors' theory.

As the criterion in the range between tension and pure shear is known, we can get easily the criterion in the whole range. It is shown in Fig. 11.

Mises-Hencky's theory is generally adopted as the theory of plasticity. It is rather used for the convenience of computation than for the strict

Fig. 11 Criterion of plastic flow

correctness. Mises-Hencky's theory is also shown in Fig. 11, its point of tension being superposed upon that of the authors'. Authors consider of course that our theory is correct. Comparing with it, Mises-Hencky's theory is a very good approximation as shown in the Figure, and it can be applied to the plastic flow without much error if we pay attention to the condition of application.

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