$$\left(\begin{aligned} &\left(k + \frac{F_0}{B}\right)\alpha_0 - \left(k + K + \frac{F_0}{B}\right)\beta_0 + \frac{4F_0}{3\pi B}\alpha_1 - \frac{4F_0}{3\pi B}\beta_1 = \frac{F_0}{2} \\ &\frac{8F_0}{3\pi B}\alpha_0 - \frac{8F_0}{3\pi B}\beta_0 - \left(k + \frac{3F_0}{2B}\right)\alpha_1 + \left(M\omega^2 - k - K - \frac{3F_0}{2B}\right)\beta_1 = \frac{4F_0}{3\pi} \end{aligned} \right)$$

Numerical method is again convenient to solve the above equations.

Fig. 12 shows both the experimental and the analytical results.

7. Conclusion

Vibratory machineries for practical use are not simple in their mechanism and performance, but can be considered to be one- or two-degrees-of-freedom vibrating systems from the dynamical point of view, and the fundamental motions of these machineries are investigated.

In the analytical method in this paper the

damping factor of the system has not been taken into consideration but the adequacy of the treatment is verified by the experimental results. The frequency response of the system with electro-magnetic vibrator is made clear.

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Problem of Large Deflection of Coiled Springs*

By Masao MIZUNO**

The deflection curve of the center line of coiled springs is discussed, expanding the Bernoulli-Euler equation for the elastica of thin rods.

Numerical results and their comparison to the experimental values are presented.

Introduction

When the elastic deformation in bending of bars, plates etc. is discussed from the view-point of the strength of materials and applied elasticity, it is a customary procedure to treat it under the following fundamental assumptions, not speaking of those concerning elasticity:

(1) The equilibrium of stresses between themselves and with external forces needs only to hold under the form and conditions of the undeformed states,

(2) The squares of deformation angles are negligible compared to unit.

As these fundamental assumptions are only approximately valid in small deformations, the theory grounded on them is called the theory of infinitesimal deformations. In this theory all the strains and stresses caused by external forces are linear in them and uniqueness of

the deformed state corresponding to definite external forces is demonstrated and a stable equilibrium will be established. Therefore (1) of the fundamental assumptions must be regarded as not valid in the theory of elastic stability of bars, such as the one of buckling of columns. Such a theory is called one of finite deformations. In larger deformations (2) of the fundamental assumptions must be treated as not true. Then we have a problems of large deflection. The large deflection problem of bars of uniform section is called the problem of elastica and was discussed by D. Bernoulli and L. Euler for the first time, so we must regard it as a classical problem⁽¹⁾. But the theory of infinitesimal deformations is simple and the test results are pretty close to the real deformations and often taken as a sufficient approximation for practical use except for exceptionally large deformations. So, as a practical computation method, it has been used until now more widely than the problem of elastica. Majority of the ordinary elastic theories such as those of

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elasticity, strength of materials, structures etc. belong to this theory of infinitesimal deformations⁽²⁾. Number of papers published recently concerning the large deflection problem is much less than the number of those founded on the theory of infinitesimal deformations. We can only name the work of Suhara, discussing the elastica of bars of gradually varied section⁽¹⁾, the work of E. J. Scott et al, the solution in series being obtained by Reversion method⁽³⁾, the work of H. D. Conway etc, discussing the large deflection of bars curved in circular arc⁽⁴⁾, the work of Hamada, treating the large deflection of plate⁽⁵⁾ and a few others, all of them treating bending deflection. Generally speaking, the spring is a machine element characterised by a large deformation in the whole even when its elements may be regarded to have infinitesimal strains in the elastic range. Therefore in order to get an accurate result, the computation of its strains and stresses must be carried out as problem of large deflection, but this is scarcely ever performed. There is only one work by Watari on laminated springs⁽⁶⁾. The author treats in this paper the deflection of a coiled spring in a state with very large deformations. Theoretically sound results are obtained only by the treatment of large deflection problem. However, such treatment will scarcely have practical importance except in case of these springs⁽⁷⁾.

1. Fundamental equations for the problem of large deflection of coiled springs

In order to determine the deflection curve of the central line of a buckled coiled spring, we treat the central line as a bar with a certain rigidity, then the spring will be subjected to a direct force depending on the position of each point on its central line, and so its pitch varies from point to point, i.e. we get a case of a bar with gradually varying rigidity. The elastica of a bar with gradually varying cross section has been treated by Suhara⁽¹⁾ and the author⁽⁸⁾, but the spring in the present case is more complicated. We must take into consideration the curvature of its central line caused by a varying shear which is comparable with the curvature caused by bending and cannot be neglected and also the gradually varying rigidity which is brought about by the dependence of the total length on its load. We assume that the coiled spring with uniform pitches and straight central line in its undeformed state is buckled under pressure R acting on both hinged ends and the deformed curve of the central line is $APOB$, as shown in Fig. 1. The deformed curve

Fig. 1

has a parallel tangent to the force R in its central point O . As in Fig. 1, let this tangent be ordinate and O be origin, then the deformed curve will be symmetrical relatively to the abscissa. As in elastica, the curve of the central line of a coiled spring will be easily determined from the view-point of large deflection problem by the coordinates of any point P on the deformed curve, as long as this is in a plane.

We shall denote as follows:

- θ : angle of the tangent to the deflection curve at any point P and the direction of load R , $\pi \geq \theta \geq 0$, being $\theta = \alpha$ at A and $\theta = 0$ at O
- s : length OP along the deflection curve of the central line of the spring
- s_0 : length s under no load
- l : total length of the spring under load R
- l_0 : total length of the spring under no load
- h : pitch at P under load R
- h_0 : pitch at P under no load
- E, G, m : Young's modulus, modulus of rigidity, and Poisson's number of the material of spring respectively
- I : moment of inertia of the cross section of wire
- r : mean radius of the coil

For simplicity, let the cross section of wire be circular, then we get the polar moment of inertia of area $J = 2I$.

Denoting by A_0, B_0, C_0 the rigidities for axial force, flexure and shear of the spring under no load respectively, we get after Biezeno and Grammel⁽⁹⁾⁽¹⁰⁾

$$A_0 = \frac{GIh_0}{\pi r^3}, \quad B_0 = \frac{2EGIh_0}{\pi r(E+2G)}, \quad C_0 = \frac{EIh_0}{\pi r^3} \dots (1.a)$$

Cases of wire with cross section of wire other than circular e.g. rectangular may be treated by the method described in the literature mentioned above⁽⁹⁾.

Now, let T, Q, M denote axial force, shearing force and bending moment respectively at point P

on the central line of the spring and let tension be counted positive, then we get

$$T = -R \cos \theta, \quad Q = R \sin \theta, \quad M = R \int_s^{l/2} \sin \theta ds \quad (1.b)$$

and

$$\frac{T}{A_0} = \frac{ds - ds_0}{ds_0} = \frac{h - h_0}{h_0}$$

so

$$\frac{h}{h_0} = 1 + \frac{T}{A_0} = 1 - \frac{R}{A_0} \cos \theta \quad (1.c)$$

Therefore, if we denote by A, B, C the rigidities at point P under load R , we get

$$\left. \begin{aligned} A &= A_0 \frac{h}{h_0} = A_0 \left(1 - \frac{R}{A_0} \cos \theta \right) \\ B &= B_0 \frac{h}{h_0} = B_0 \left(1 - \frac{R}{A_0} \cos \theta \right) \\ C &= C_0 \frac{h}{h_0} = C_0 \left(1 - \frac{R}{A_0} \cos \theta \right) \end{aligned} \right\} \quad (1.d)$$

On the other hand we find the curvature of deflection curve of the central line of the spring as follows⁽¹⁰⁾:

$$\frac{d\theta}{ds} = \frac{M}{B} + \frac{1}{C} \frac{dQ}{ds} \quad (1.1)$$

This is the fundamental equation of deflection curve. If we put Eqs. (1.b) and (1.d) into Eq. (1.1), we get

$$\frac{d\theta}{ds} = \frac{R \int_s^{l/2} \sin \theta ds}{B_0 \left(1 - \frac{R}{A_0} \cos \theta \right)} + \frac{R \cos \theta}{C_0 \left(1 - \frac{R}{A_0} \cos \theta \right)} \cdot \frac{d\theta}{ds}$$

In order to make the quantities non-dimensional we put

$$\left. \begin{aligned} s &= l\sigma, \quad R/A_0 = \lambda, \quad l^2 \frac{R}{B_0} = \mu, \quad R/C_0 = \nu \\ \lambda + \nu &= R \left(3 + \frac{2}{m} \right) \frac{\pi r^3}{EI h_0} = \tau \end{aligned} \right\} \quad (1.e)$$

then

$$(1 - \tau \cos \theta) \frac{d\theta}{ds} = \mu \int_\sigma^{1/2} \sin \theta d\sigma \quad (1.2)$$

$$x_l = \sqrt{f} \cdot r \int_0^\alpha \frac{(1 - \tau \cos \theta) \sin \theta}{\sqrt{F(\theta)}} d\theta = \sqrt{f} \cdot r \frac{1}{\tau} \left| -\sqrt{F(\theta)} \right|_0^\alpha = \sqrt{f} \cdot r \frac{1}{\tau} \sqrt{(1 - \tau \cos \alpha)^2 - (1 - \tau)^2} \quad (1.B)$$

$$y_l = \sqrt{f} \cdot r \int_0^\alpha \frac{(1 - \tau \cos \theta) \cos \theta}{\sqrt{F(\theta)}} d\theta \quad (1.C)$$

As $ds/ds_0 = 1 - \lambda \cos \theta$ by Eq. (1.c) and $\tau = \lambda + \nu$ by Eq. (1.e), we get

$$\begin{aligned} \frac{l_0}{2} &= \int_0^s \frac{ds}{1 - \lambda \cos \theta} = \sqrt{f} \cdot r \left[\left(1 + \frac{\nu}{\lambda} \right) \int_0^\alpha \frac{d\theta}{\sqrt{F(\theta)}} - \frac{\nu}{\lambda} \int_0^\alpha \frac{d\theta}{(1 - \lambda \cos \theta) \sqrt{F(\theta)}} \right] \\ &= \sqrt{f} \cdot r \int_0^\alpha \frac{(1 - \tau \cos \theta) d\theta}{(1 - \lambda \cos \theta) \sqrt{F(\theta)}} \quad (1.D) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{d\sigma} \left\{ (1 - \tau \cos \theta) \frac{d\theta}{d\sigma} \right\} &= -\mu \sin \theta \\ \left\{ (1 - \tau \cos \theta) \frac{d\theta}{d\sigma} \right\} \frac{d}{d\sigma} \left\{ (1 - \tau \cos \theta) \frac{d\theta}{d\sigma} \right\} d\sigma &= -\mu (1 - \tau \cos \theta) \sin \theta d\theta \\ (1 - \tau \cos \theta)^2 \left(\frac{d\theta}{d\sigma} \right)^2 &= \mu (2 \cos \theta - \tau \cos^2 \theta + H) \end{aligned} \quad (1.3)$$

where H is an integration constant. At point A i.e. at $\theta = \alpha$, we have

$$\frac{d\theta}{d\sigma} = l \frac{d\theta}{ds} = 0 \quad (1.f)$$

so we get

$$\begin{aligned} H &= -(2 \cos \alpha - \tau \cos^2 \alpha) \\ &= \frac{1}{\tau} \{ (1 - \tau \cos \alpha)^2 - 1 \} \end{aligned} \quad (1.4)$$

If we put Eq. (1.4) into Eq. (1.3),

$$\begin{aligned} (1 - \tau \cos \theta)^2 \left(\frac{d\theta}{d\sigma} \right)^2 &= \frac{\mu}{\tau} \{ (1 - \tau \cos \alpha)^2 - (1 - \tau \cos \theta)^2 \} \end{aligned}$$

Noting Eq. (1.e) and putting

$$\begin{aligned} \sqrt{\frac{\tau}{\mu}} &= \frac{1}{l} \sqrt{\frac{1/A_0 + 1/C_0}{1/B_0}} = \frac{r}{l} \sqrt{\frac{3m+2}{2m+1}} \\ &= \frac{r}{l} \sqrt{f} \end{aligned} \quad (1.g)$$

we get

$$d\sigma = \sqrt{f} \frac{r}{l} \cdot \frac{(1 - \tau \cos \theta) d\theta}{\sqrt{(1 - \tau \cos \alpha)^2 - (1 - \tau \cos \theta)^2}} \quad (1.5)$$

where we assumed $1 \geq \tau$ for the sake of simplicity and took note that

$$1 - \tau \cos \theta \geq 0, \quad 1 - \tau \cos \alpha \geq 0 \quad \text{and} \quad \frac{d\theta}{d\sigma} \geq 0$$

As $ds = l d\sigma$ by Eq. (1.e), putting

$$F(\theta) = (1 - \tau \cos \alpha)^2 - (1 - \tau \cos \theta)^2$$

we get by Eq. (1.5):

$$ds = \sqrt{f} \cdot \frac{r}{l} \cdot l \cdot \frac{(1 - \tau \cos \theta)}{\sqrt{F(\theta)}} d\theta \quad (1.5)'$$

and

$$\frac{l}{2} = \sqrt{f} \cdot r \int_0^\alpha \frac{(1 - \tau \cos \theta)}{\sqrt{F(\theta)}} d\theta \quad (1.A)$$

The coordinates of point A are as follows:

The Eqs. (1·A, B, C, D) denote the integrations along OA in Fig. 1 and so express the values at point A , but if we take θ as the upper limit of integration, we get formulae for s , x , y , s_0 at an arbitrary point P in general:

$$s = \sqrt{f} \cdot r \int_0^\theta \frac{(1 - \tau \cos \theta)}{\sqrt{F(\theta)}} d\theta \dots\dots\dots (1\cdot A)'$$

$$x = \sqrt{f} \cdot r \frac{1}{\tau} \sqrt{F(\theta)} \dots\dots\dots (1\cdot B)'$$

$$y = \sqrt{f} \cdot r \int_0^\theta \frac{(1 - \tau \cos \theta) \cos \theta}{\sqrt{F(\theta)}} d\theta \dots\dots\dots (1\cdot C)'$$

$$s_0 = \sqrt{f} \cdot r \left[\left(1 + \frac{\nu}{\lambda} \right) \int_0^\theta \frac{d\theta}{\sqrt{F(\theta)}} - \frac{\nu}{\lambda} \int_0^\theta \frac{d\theta}{(1 - \lambda \cos \theta) \sqrt{F(\theta)}} \right] \dots\dots\dots (1\cdot D)'$$

where

$$f = \frac{3m+2}{2m+1}, \quad \frac{\nu}{\lambda} = \frac{A_0}{C_0} = \frac{G}{E} = \frac{m}{2(m+1)}$$

$$\tau = R \left(3 + \frac{2}{m} \right) \frac{\pi r^3}{E I h_0}$$

$$F(\theta) = (1 - \tau \cos \alpha)^2 - (1 - \tau \cos \theta)^2$$

f , ν/λ being constants depending on the dimensions of the spring under no load and elastic modulus of its material. Generally, the integrals on the right of Eqs. (1·A, B, C, D) are reduced to elliptic integrals as in the case of elastica⁽¹⁾.

The fact that the deflection curve of the central line of a buckled coiled spring is expressible in elliptic integrals same as elastica may be of any significance to applied mechanics.

N.B.—the above calculations refer to the deformation in which the coils of the spring do not touch each other.

2. Numerical results and relations with usual data

Now we shall calculate the numerical values of the solutions obtained in section 1. Put Eq. (1·g) into Eq. (1·A), then we get

$$\frac{l}{2} = l \sqrt{\frac{\tau}{\mu}} \int_0^\alpha \frac{(1 - \tau \cos \theta) d\theta}{\sqrt{(1 - \tau \cos \alpha)^2 - (1 - \tau \cos \theta)^2}} \dots\dots\dots (2\cdot 1)$$

If we transform the variable θ into φ by the definition

$$\sin^2 \frac{\theta}{2} = \sin^2 \frac{\alpha}{2} \sin^2 \varphi \dots\dots\dots (2\cdot a)$$

we have

$$\varphi = 0 \text{ at } \theta = 0, \quad \varphi = \frac{\pi}{2} \text{ at } \theta = \alpha$$

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \frac{\alpha}{2} \sin^2 \varphi$$

$$d\theta = \frac{2 \sin \frac{\alpha}{2} \cos \varphi}{\sqrt{1 - \sin^2 \frac{\alpha}{2} \sin^2 \varphi}} d\varphi$$

$$\sqrt{(1 - \tau \cos \alpha)^2 - (1 - \tau \cos \theta)^2} = 2 \sqrt{\tau} \sin \frac{\alpha}{2} \cos \varphi \sqrt{1 - \tau + \tau(1 + \sin^2 \varphi) \sin^2 \frac{\alpha}{2}}$$

and

$$\sqrt{\mu} = 2 \int_0^{\pi/2} \frac{\left(1 - \tau + 2\tau \sin^2 \frac{\alpha}{2} \sin^2 \varphi \right) d\varphi}{\sqrt{1 - \sin^2 \frac{\alpha}{2} \sin^2 \varphi} \sqrt{1 - \tau + \tau(1 + \sin^2 \varphi) \sin^2 \frac{\alpha}{2}}} \dots\dots\dots (2\cdot 1)'$$

or expanding into series

$$\begin{aligned}\sqrt{\mu} &= 2 \int_0^{\pi/2} \left\{ (1-\tau) + 2\tau \sin^2 \varphi \sin^2 \frac{\alpha}{2} \right\} \left(1 + \frac{1}{2} \sin^2 \varphi \sin^2 \frac{\alpha}{2} + \frac{3}{8} \sin^4 \varphi \sin^4 \frac{\alpha}{2} + \dots \right) \\ &\quad \times (1-\tau)^{-1/2} \left\{ 1 - \frac{1}{2} \frac{\tau(1+\sin^2 \varphi)}{1-\tau} \sin^2 \frac{\alpha}{2} + \frac{3}{8} \frac{\tau^2(1+\sin^2 \varphi)^2}{(1-\tau)^2} \sin^4 \frac{\alpha}{2} - \dots \right\} d\varphi \\ &= 2(1-\tau)^{-1/2} \int_0^{\pi/2} \left[(1-\tau) + \frac{1}{2} \{-\tau + (1+2\tau) \sin^2 \varphi\} \sin^2 \frac{\alpha}{2} \right. \\ &\quad \left. + \frac{1}{8} \left\{ (3+5\tau) \sin^4 \varphi + \frac{3\tau^2 - 2\tau \sin^2 \varphi - (2\tau + 3\tau^2) \sin^4 \varphi}{1-\tau} \right\} \sin^4 \frac{\alpha}{2} + \dots \right] d\varphi\end{aligned}$$

As

$$\int_0^{\pi/2} \sin^2 \varphi d\varphi = \frac{1}{2} \cdot \frac{\pi}{2}, \quad \int_0^{\pi/2} \sin^4 \varphi d\varphi = \frac{3}{8} \cdot \frac{\pi}{2}, \dots, \dots \quad (2.b)$$

we get finally

$$\left. \begin{aligned}\sqrt{\mu} &= \pi(1-\tau)^{-1/2} S_1, \\ \text{where} \quad S_1 &= (1-\tau) + \frac{1}{4} \sin^2 \frac{\alpha}{2} + \frac{1}{64} \cdot \frac{9-8\tau}{1-\tau} \sin^4 \frac{\alpha}{2} + \dots\end{aligned} \right\} \dots \quad (2.2)$$

When $\alpha=0$, i.e. under buckling load

$$S_1 = 1 - \tau_{cr}$$

so we have

$$\sqrt{\mu_{cr}} = \pi (1 - \tau_{cr})^{1/2} \dots \quad (2.3)$$

Considering $\tau_{cr} = \lambda_{cr} + \nu_{cr}$ [cf. Eq. (1.e)], we can transform Eq. (2.3) as follows:

$$\frac{1}{\frac{\mu_{cr}}{\pi^2} + \nu_{cr}} = \frac{1}{1 - \lambda_{cr}} = \frac{1}{\lambda_{cr}} \left(\frac{1}{1 - \lambda_{cr}} - 1 \right)$$

Again from Eq. (1.e)

$$\lambda_{cr} = \frac{R_{cr}}{A_0}, \quad \mu_{cr} = \frac{R_{cr}}{B_0} l_{cr}^2, \quad \nu_{cr} = \frac{R_{cr}}{C_0}$$

and as the central line is straight under buckling load, there holds:

$$l_{cr} = (1 - \lambda_{cr}) l_0, \quad \frac{R_{cr}}{A_0} = \lambda_{cr} = \frac{l_0 - l_{cr}}{l_0} \dots \quad (2.c)$$

Therefore we get

$$\frac{1}{\frac{l_{cr}^2}{\pi^2} \frac{R_{cr}}{B_0} + \frac{R_{cr}}{C_0}} = \frac{A_0}{R_{cr}} \frac{l_0 - l_{cr}}{l_{cr}}$$

or

$$R_{cr} = \frac{A_0 (l_0 - l_{cr})}{l_0} = \frac{\pi^2 B_0}{l_{cr} l_0} \frac{1}{1 + \frac{\pi^2}{l_{cr}^2} \frac{B_0}{C_0}} \dots \quad (2.4)$$

The last formula coincides with that of the ordinary theory of coiled springs under buckling load⁽¹⁰⁾. We must notice that l_{cr} , B_{cr} , C_{cr} are the total length, flexural and shearing rigidities respectively under a load just equal to R_{cr} and depending on R_{cr} . So R_{cr} cannot be determined directly by Eq. (2.4) and its calculation requires the solution of an equation of third order given below Eq. (2.f).

Again transforming the variable in Eq. (1.D) by Eq. (2.a) we get, just as before,

$$\frac{l_0}{2} = \frac{l}{\sqrt{\mu}} \int_0^{\pi/2} \frac{d\varphi}{1 - \lambda + 2\lambda \sin^2 \varphi \sin^2 \frac{\alpha}{2}} \cdot \frac{1 - \tau + 2\tau \sin^2 \varphi \sin^2 \frac{\alpha}{2}}{\sqrt{1 - \sin^2 \varphi \sin^2 \frac{\alpha}{2}} \sqrt{1 - \tau + \tau(1 + \sin^2 \varphi) \sin^2 \frac{\alpha}{2}}}$$

or, expanding and taking note of the calculation in Eq. (2.2), we have

$$\begin{aligned}\frac{l_0}{l} &= \frac{2}{\sqrt{\mu}} (1-\lambda)^{-1} (1-\tau)^{-1/2} \int_0^{\pi/2} \left\{ 1 - \frac{2\lambda}{1-\lambda} \sin^2 \varphi \sin^2 \frac{\alpha}{2} + \frac{4\lambda^2}{(1-\lambda)^2} \sin^4 \varphi \sin^4 \frac{\alpha}{2} - \dots \right\} \\ &\quad \times \left[(1-\tau) + \frac{1}{2} \{-\tau + (1+2\tau) \sin^2 \varphi\} \sin^2 \frac{\alpha}{2} + \dots \right] d\varphi\end{aligned}$$

or

$$\frac{l_0}{l} = (1-\lambda)^{-1} \left[1 + \frac{1}{S_1} \left\{ -\frac{1-\tau}{1-\lambda} \lambda \sin^2 \frac{\alpha}{2} - \frac{\lambda}{8} \frac{(3-15\lambda+2\tau+10\lambda\tau)}{(1-\lambda)^2} \sin^4 \frac{\alpha}{2} - \dots \right\} \right]$$

or, finally,

$$\frac{l_0}{l} = (1-\lambda)^{-1} \left[1 - \frac{S_2}{S_1} \right],$$

where

$$S_2 = \lambda \frac{1-\tau}{1-\lambda} \sin^2 \frac{\alpha}{2} + \frac{\lambda}{8} \frac{(3-15\lambda+2\tau+10\lambda\tau)}{(1-\lambda)^2} \sin^4 \frac{\alpha}{2} + \dots$$

When $\alpha=0$, i.e. before buckling, we have $S_2=0$, therefore

$$\frac{l_0}{l} = (1-\lambda)^{-1} \quad (2.6)$$

This formula contains Eq. (2.C) in it. Denote δ_y the deflection of the spring before buckling, by $n=l_0/h_0$ the number of turns and d the diameter of wire, then

$$\delta_y = l_0 - l = \lambda l_0 = \frac{R}{A_0} l_0 = \frac{\pi r^3 l_0}{G I h_0} R$$

$$\text{or} \quad \delta_y = \frac{64 n r^3}{d^4 G} R \quad (2.7)$$

which coincides with the ordinary formula of coiled springs.

Now, in order to compute the load and deformation after buckling we must, at first, determine the relation between τ and $\sin \alpha/2$.

Noting Eq. (1.g), we get

$$\frac{\sqrt{\mu}}{l} = \frac{1}{r} \frac{\sqrt{\tau}}{\sqrt{f}}$$

where

$$f = \frac{3m+2}{2m+1}$$

m being Poisson's number.

In the following we take $m=10/3$, then

$$\sqrt{f} = 1.25 \quad (2.d)$$

From Eqs. (2.2) and (2.5), we get

$$\sqrt{\mu} \frac{l_0}{l} = \pi (1-\lambda)^{-1} (1-\tau)^{-1/2} [S_1 - S_2]$$

or

$$\frac{1}{\pi \sqrt{f}} \frac{l_0}{r} \sqrt{\tau(1-\tau)} (1-\lambda) = S_1 - S_2$$

Fig. 2

When $m=10/3$, from Eq. (1.e) we get

$$\frac{\lambda}{\tau} = \frac{\lambda}{\lambda + \nu} = \frac{1}{1 + \frac{G}{E}} = \frac{3m+2}{2m+2} = 0.722 \quad (2.e)$$

So the above formula may be expressed as follows:

$$a_0 = a_1 \sin^2 \frac{\alpha}{2} + a_2 \sin^4 \frac{\alpha}{2} + \dots$$

If we can neglect the terms of $\sin^6 \alpha/2$ and higher powers, we have

$$\sin^2 \frac{\alpha}{2} = \frac{-a_1 + \sqrt{a_1^2 + 4a_0a_2}}{2a_2},$$

where

$$\left. \begin{aligned} a_0 &= \frac{1}{\pi} \frac{1}{1.25} \frac{l_0}{r} \sqrt{\tau(1-\tau)} (1-0.722\tau) - (1-\tau) \\ a_1 &= \frac{1}{4} - 0.722\tau \frac{1-\tau}{1-0.722\tau} \\ a_2 &= \frac{1}{64} \frac{9-8\tau}{1-\tau} - \frac{0.722}{8} \tau \frac{3-8.33\tau+7.22\tau^2}{(1-0.722\tau)^2} \end{aligned} \right\} \quad (2.8)$$

This formula expresses the dependence of $\sin \alpha/2$ on τ . When $a_0=0$, $\sin \alpha/2=0$, and this corresponds to buckling load, we have

$$\frac{\pi^2 r^2}{l_0^2} f = \frac{\tau_{cr}}{1-\tau_{cr}} (1-\lambda_{cr})^2$$

If we put

$$M = \frac{\pi^2 r^2}{l_0^2} \frac{1}{1 + \frac{2}{m}}, \quad (1-\lambda_{cr}) = w$$

and note Eqs. (2.d) and (2.e), we get

$$\frac{\pi^2 r^2}{l_0^2} f = M \frac{3m+2}{m}$$

and

$$\begin{aligned} \frac{\tau_{cr}}{1-\tau_{cr}} (1-\lambda_{cr})^2 &= \frac{(3m+2)\lambda_{cr}}{2m+2 - (3m+2)\lambda_{cr}} (1-\lambda_{cr})^2 \\ &= \frac{(1-w)w^2}{-1 + (3+2/m)w} \frac{3m+2}{m} \end{aligned}$$

then

$$M = \frac{(1-w)w^2}{-1 + (3+2/m)w}$$

or

$$w^3 - w^2 + (3+2/m)Mw - M = 0 \quad (2.f)$$

which coincides with the current formula for the coiled spring under buckling load⁽¹⁰⁾.

By Eq. (2.f), we can determine the value τ_{cr} for each value of l_0/r .

On the other hand, if we put

$$K = \frac{R}{R_{cr}} = \frac{\tau}{\tau_{cr}} \quad (2.g)$$

and regard K as a parameter, we obtain Fig. 2 which shows the dependence of τ on l_0/r .

Noting this dependence, the value of $\sin \alpha/2$ for each l_0/r and K is determined from the Eq. (2.8). And we obtain Fig. 3.

The case in which $K=1$ (Fig. 2) corresponds exactly to the Euler's curve of columns. Ordinarily, λ is taken as ordinate⁽¹²⁾. In the limit $l_0/r \rightarrow 0$, $\tau=1$ independently of m , but λ depends on m . The reason why the abscissa is limited in the Figure of this paper is that in the only case of practical importance as for buckling there holds $l_0/r \geq 8$ and when $l_0/r > 15$ the curve coincides nearly with the horizontal asymptote.

Here we cite Eq. (1.B):

$$x_l = \sqrt{f} r \left[\frac{1}{\tau} \sqrt{(1-\tau \cos \alpha)^2 - (1-\tau)^2} \right]$$

or

$$\frac{x_l}{l_0} = 2\sqrt{f} \frac{r}{l_0} \frac{\sin \alpha/2}{\sqrt{\tau}} \sqrt{1-\tau+\tau \sin^2 \frac{\alpha}{2}} \dots (2.9)$$

If we replace the variable in Eq. (1.C) by φ following Eq. (2.a), we get

$$y_l = \frac{1}{\sqrt{\mu}} \int_0^{\pi/2} \left(1 - 2 \sin^2 \varphi \sin^2 \frac{\alpha}{2} \right) \left[(1-\tau) + \frac{1}{2} \{ -\tau + (1+2\tau) \sin^2 \varphi \} \sin^2 \frac{\alpha}{2} + \dots \right] d\varphi$$

or

$$2y_l = l \left[1 - \frac{1}{S_1} \left\{ (1-\tau) \sin^2 \frac{\alpha}{2} + \frac{1}{8} (3+2\tau) \sin^4 \frac{\alpha}{2} + \dots \right\} \right]$$

finally

$$\frac{2y_l}{l_0} = (1-\lambda) \frac{S_1 - S_3}{S_1 - S_2},$$

where

$$S_3 = (1-\tau) \sin^2 \frac{\alpha}{2} + \frac{1}{8} (3+2\tau) \sin^4 \frac{\alpha}{2} + \dots \quad (2.10)$$

Fig. 4 shows the relations expressed by the Eq. (2.9)

Fig. 3

and Eq. (2.10), the results of Figs. 2 and 3 being taken into consideration. In it, x_l/l_0 is represented by the family of curves which intersect the left ordinate at $K=1$. And $2y_l/l_0$ is represented by the family of curves which intersect the right ordinate at $K=0$. As there holds $2y_l + \delta_y = l_0$ we can measure the abscissa of this family of curves by δ_y/l_0 , then its origin will be at the right corner as is shown in the bracket () under the abscissa. For comparison, a curve corresponding to the elastica is shown in chain line and one by the theory of stability in broken line.

3. Comparison of the calculated results of deflection after buckling with the experiments

The author has performed experiments on the deflection of coiled spring after buckling. The measured results will be compared with the above calculations.

[a] Method of experiments

(1) Test machine used

The author used a Olsen type universal testing machine whose loading system is of screw type, not of oil pressure type. So it allows convenient measurements of deformation of springs which are set between two tables movable in parallel and may be fixed at any desired pressed position.

(2) Dimensions of tested springs

Fig. 4

The author experimented on the 3 kinds of cylindrical coiled springs, as shown in Table 1.

These experiments require only the constancy of the pitch of the coils under no load and are independent of the magnitude. The length of the springs was adapted to the experimental method described in (3) Method of measurement. As the properties of materials other than Poisson's ratio are known not to affect the measured results of this experiment, these were not investigated in detail.

(3) Method of measurement

Fig. 5 shows the deformations of buckled coiled springs. In this Figure the dimensions which are almost invariable and coincide practically with those under no loads shall be denoted by suffices 0. That is, $D_0 = D$, $d_0 = d$. Again the denotations in section 2. are maintained: $l' = 2y_l$, $l'_0 = l_0$, $l_0 - l' = \delta_y$ and we have $x_{l,0} = 0$. But the errors in $x_{l,0}$ originating in the manufacture of the springs are of the order of 1/10 mm and cause discrepancies between theory and experiment near the buckling point, as described below.

C in Fig. 5 is a metallic part fixing the two ends of the spring in exact direction of its central line, and on its cylindrical surface is cut a spiral groove into which the spring may be just screwed. P is a plate to protect to the displacement of C

Table 1

	D mm	d mm
Spring 1	25.9	2.1
Spring 2	24.65	4.0
Spring 3	20.4	5.0

where D: diameter of coil in mm, d: diameter of wire in mm

Fig. 5

when springs are buckled and fixes C on it.

The free length $2l_0$ of springs under no load is fixed at $2l_0/D = l_0/r = 14, 10, 8$ respectively, these values being used for calculation in section 2, and cut springs with margins to enable their two ends to be screwed in and to be fixed by the metal part C.

(4) Kinds of experiment

The author has performed various experiments as shown in Table 2.

[b] Results of experiment

(1) Method of treatment of results

Fig. 6 shows curves expressing the relations between x_l/l_0 and δ_y/l_0 which are obtained from Fig. 4 calculated in section 2. In the Fig. 6 we plot the measured values $2\delta_y$ and $2x_l$ divided by $2l_0$, $2l'$ are the measured L minus the height of C and thickness of P.

(2) Discussion of the results

The causes of the discrepancy between the experimental and the theoretical curve near buckling may be on one side the errors in dimensions of springs, such as $x_{l,0} \neq 0$ described above and others, and, on the other side, the incompletely fixed conditions of support which consists particularly in the asymmetric arrangement of springs relative to the central measuring point of springs caused by the accurate adjustment of free length.

Table 2

	Spring 1	Spring 2	Spring 3
$2l_0/D = l_0/r$	10, 8	10, 8	14, 10

Fig. 6

Except for this non-coincidence, the experimental points nicely fit the theoretical curves.

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Appendix

Put $\cos\theta=z$ into the integrals on the right of Eq. (1 · A, B, C, D), noting $d\theta=dz/(-\sqrt{1-z^2})$ we have

$$\begin{aligned} I_0 &= \int_0^\theta d\theta/\sqrt{F(\theta)} = - \int_1^z dz/\tau\sqrt{G(z)} \\ I_1 &= \int_0^\theta \cos\theta d\theta/\sqrt{F(\theta)} = - \int_1^z z dz/\tau\sqrt{G(z)} \\ I_2 &= \int_0^\theta \cos^2\theta d\theta/\sqrt{F(\theta)} = - \int_1^z z^2 dz/\tau\sqrt{G(z)} \\ J &= \int_0^\theta d\theta/(1-\lambda\cos\theta)\sqrt{F(\theta)} = \int_1^z dz/\lambda\tau(z-1/\lambda)\sqrt{G(z)} \end{aligned}$$

where

$$\begin{aligned} F(\theta) &= (1-\tau\cos\alpha)^2 - (1-\tau\cos\theta)^2 \\ G(z) &= (z-1)(z+1)(z-\cos\alpha)(z+\cos\alpha-2/\tau) \end{aligned}$$

Then, we get after W. Gröbner und N. Hofreiter: *Integraltafel 2 teil*, (1958) S. 47, noting $1 > \cos\alpha > -1$, $\infty > 2/\tau - \cos\alpha > 1$; $0 < \tau \leq 1$, for example

$$\begin{aligned} I_0 &= \int_{\theta=\alpha}^1 \frac{dz}{\tau\sqrt{G(z)}} = \frac{1}{\tau\sqrt{1/\tau - \cos\alpha}} K(k) \\ I_1 &= \int_{\theta=\alpha}^1 \frac{dz}{\tau\sqrt{G(z)}} = \frac{1}{\tau\sqrt{1/\tau - \cos\alpha}} \left[(1+\cos\alpha) \Pi\left(\frac{\cos\alpha-1}{z}, k\right) - K(k) \right] \end{aligned}$$

where

$$k^2 = \frac{(2/\tau - \cos\alpha + 1)(1 - \cos\alpha)}{4(1/\tau - \cos\alpha)}$$

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