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On the Plane Stress Problem in a Ring of Orthotropic Material*

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In the present paper, the two-dimensional stress distribution in an orthogonally aeolotropic plate, bounded by two concentric circular contours, is treated by the perturbation method using the orthogonal co-ordinates, similar to that of the previous papers.

Numerical examples are worked out for three kinds of material, oak, plywood and isotropic plate. The results show that considerably large stress concentration may occur in some aeolotropic plates even in the case of uniform inner pressure.

1. Introduction

As is well-known, the stress distributions in a ring or a thick walled tube of isotropic material can be analyzed easily by the use of the polar co-ordinates. In the case of the aeolotropic plate, however, the same subject will be as troublesome as other problems in the multiply connected domain, because the differential equation concerning the stress function in terms of the polar co-ordinates is of the complicated form for the directionality of its elastic properties and the general solution will not be obtained easily.

In the previous paper⁽¹⁾, the author showed that the stresses in a perforated orthotropic strip, which is a kind of the multiply connected domain, may be calculated by the use of the orthogonal co-ordinates. In the present paper, the general expression of the stress function in a ring of orthotropic material is given in the same way, and some examples are evaluated by the perturbation method.

2. Fundamental formulae

Consider an orthotropic plate bounded by two concentric circular contours with their center at the origin of the x - y plane and having a radius of unit length and $\lambda < 1$ respectively as shown in Fig. 1. Suppose the x and y axes are chosen parallel to the elastic axes of the plate in a state of the generalized plane stress, then the displacements u, v , the rotation ω , and the stress components in the plane are given by the following

equations⁽²⁾⁽³⁾.

$$\left. \begin{aligned} (p_1 + p_2) \widehat{xx} + p_1 p_2 (p_1 + p_2) \widehat{yy} + 2i E_x \omega \\ = (p_2^2 - p_1^2) \{ p_1 \varphi_1''(z_1) + p_2 \varphi_2''(z_2) \}, \\ p_1 p_2 \widehat{yy} - \widehat{xx} + i(p_1 + p_2) \widehat{xy} \\ = (p_1 + p_2) \{ p_1 \varphi_1''(z_1) + p_2 \varphi_2''(z_2) \} \\ E_x u = -R_e \{ (p_1^2 + \nu_x) \varphi_1'(z_1) + (p_2^2 + \nu_x) \varphi_2'(z_2) \} \\ E_x v = I_m \{ p_1 (p_2^2 + \nu_x) \varphi_1'(z_1) + p_2 (p_1^2 + \nu_x) \varphi_2'(z_2) \} \end{aligned} \right\} \dots\dots (1)$$

$$\dots\dots (2)$$

where

E_x, E_y : Modulus of elasticity in the x, y directions, respectively

G_{xy} : Modulus of rigidity of the plate

ν_x : Poisson's ratio in the x direction

or concerning the r - θ co-ordinates defined in Fig. 1, the stress components may be written as follows.

$$\left. \begin{aligned} 4r\widehat{rr} &= -R_e \left[\sum_j \{ (1 + p_j) e^{i\theta} + (p_j - 1) e^{-i\theta} \}^2 \varphi_j''(z_j) \right] \\ 4r\widehat{r\theta} &= I_m \left[\sum_j \{ (1 + p_j)^2 e^{2i\theta} - (p_j - 1)^2 e^{-2i\theta} \} \varphi_j''(z_j) \right] \\ 4\widehat{\theta\theta} &= R_e \left[\sum_j \{ (1 + p_j) e^{i\theta} - (p_j - 1) e^{-i\theta} \}^2 \varphi_j''(z_j) \right] \end{aligned} \right\} \dots\dots (3)$$

Fig. 1

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where $\sum_j$ means the summation for $j=1$ and 2 .

Considering the stresses, the displacements, and the rotation to be one-valued functions, we may express the stress function on the outside of the inner circle in the following form

$$\left. \begin{aligned} \varphi_j''(z_j) &= \sum_{n=0}^{\infty} (a_{n,j} + ib_{n,j}) z_j^n + \sum_{n=2}^{\infty} (a_{-n,j} + ib_{-n,j}) z_j^{-n} \\ &\quad + \frac{1}{2\pi p_j(p_j^2 - p_k^2)} \{ (p_j^2 + \nu_x) X + ip_j(p_k^2 + \nu_x) Y \} \\ &\quad j=1, 2, \quad k=2, 1, \quad j \neq k \\ M &= 2\pi(b_{-2,1} + b_{-2,2}) \end{aligned} \right\} \dots\dots\dots (4)$$

where X, Y mean the resultant forces acting at the rim of the inner hole in the x, y directions respectively, and M means the resultant moment about the origin.

For simplicity sake, consider first the stresses distributed symmetrically with respect to x and y axes, then the stress function and the boundary conditions may be written as follows.

Stress function

$$\varphi_j''(z_j) = \sum_{n=-\infty}^{\infty} a_{2n,j} z_j^{2n} \dots\dots\dots (5)$$

Boundary conditions:

$$\text{at } r=\lambda \quad \widehat{rr} = \frac{R_0}{2} + \sum_{n=1}^{\infty} R_{2n} \cos 2n\theta, \quad \widehat{\theta\theta} = \sum_{n=1}^{\infty} \Theta_{2n} \sin 2n\theta \dots\dots\dots (6)$$

$$\text{at } r=1 \quad \widehat{rr}' = \frac{R_0'}{2} + \sum_{n=1}^{\infty} R_{2n}' \cos 2n\theta, \quad \widehat{\theta\theta}' = \sum_{n=1}^{\infty} \Theta_{2n}' \sin 2n\theta \dots\dots\dots (7)$$

3. The expressions for the boundary conditions at the rim of the hole

A circle of radius λ with its center at the origin of the physical plane, as is well-known, can be transformed into a unit circle $r_j=1$, $\theta_j=\theta$ on the w_j plane by the mapping function

$$z_j = \frac{\lambda}{2} \{ (1+p_j)w_j + (1-p_j)w_j^{-1} \}, \quad w_j = r_j e^{i\theta_j} \dots\dots\dots (8)$$

and the stress function may be written in the following form

$$\varphi_j''(z_j) = f_j(w_j) = \sum_{n=-\infty}^{\infty} A_{2n,j} w_j^{2n} \dots\dots\dots (9)$$

From the boundary conditions (6), we have the following relations

$$\left. \begin{aligned} A_{-2n,j} &= \left(\frac{1-p_j}{1+p_j} \right)^n A_{0,j} + \sum_{r=1}^n \left(\frac{1-p_j}{1+p_j} \right)^{n-r} \left[\frac{p_j+p_k}{p_k-p_j} \left\{ A_{2(r-1),j} + \frac{p_j-1}{1+p_j} A_{2r,j} \right\} \right. \\ &\quad \left. + \frac{2p_k}{p_k-p_j} \left\{ \frac{1+p_k}{1+p_j} A_{2(r-1),k} + \frac{p_k-1}{1+p_j} A_{2r,k} \right\} + K_{j(r)} \right] \end{aligned} \right\} \dots\dots\dots (10)$$

where

$$K_{j(r)} = \frac{1}{(p_k-p_j)(1+p_j)} \{ (1-p_k)(\Theta_{2(r-1)} + R_{2(r-1)}) + (1+p_k)(R_{2r} - \Theta_{2r}) \}$$

Furthermore, we have the relations between the coefficients in the stress function φ_j'' and f_j , which are derived from the mapping function (8), as⁽¹⁾

$$\left. \begin{aligned} A_{2n,j} &= \left(\frac{1+p_j}{1-p_j} \right)^n \sum_{r=n}^{\infty} a_{2r,j} (1-p_j)^r \left(\frac{\lambda}{2} \right)^{2r} \binom{2r}{r-n} \quad n \geq 0 \\ a_{-2n,j} &= (1-p_j^2)^n \left(\frac{\lambda}{2} \right)^{2n} \sum_{r=1}^n \left\{ A_{-2r,j} - \left(\frac{1-p_j}{1+p_j} \right)^r \sum_{s=r}^{\infty} A_{2s,j} (1-p_j^2)^s \left(\frac{\lambda}{2} \right)^{2s} \binom{2s}{s-r} \right\} \\ &\quad \times \left(\frac{1+p_j}{1-p_j} \right)^r \frac{r}{n} \binom{2n}{n-r} \end{aligned} \right\} \dots\dots\dots (11)$$

Then, substituting Eq. (11) into Eq. (10), we have as the boundary conditions at the rim of the hole, a set of linear equations with respect to the coefficients in the stress function $\varphi_j''(z_j)$ as

$$a_{-2n,j} = \left(\frac{\lambda}{2} \right)^{2n} \left[K_{-2n,j} + \sum_{r=0}^{\infty} \{ a_{2r,j} \frac{2n}{2r} + a_{2r,k} \frac{2n}{2r} \} \left(\frac{\lambda}{2} \right)^{2r} \right] \quad n \geq 1 \dots\dots\dots (12)$$

and the symbols $K_{-2n,j}$, ${}^{2n}\gamma_{2r,j}$ and ${}^{2n}\delta_{2r,j}$ are the constants defined as below

$$\left. \begin{aligned} K_{-2n,j} &= (1-p_j^2)^n \sum_{q=1}^n \left(\frac{1+p_j}{1-p_j} \right)^q \binom{2n-1}{n-q} K_j(q) \\ {}^{2n}\gamma_{2r,j} &= (1-p_j^2)^{n+r} \sum_{q=1}^n \left\{ 1 + \frac{p_j+p_k}{p_k-p_j} \left(\frac{1+p_j}{1-p_j} \right)^{2q-1} \right\} \frac{2q-1}{2r+1} \binom{2n-1}{n-q} \binom{2r+1}{r+q} \\ {}^{2n}\delta_{2r,j} &= (1-p_j^2)^n (1-p_k^2)^r \frac{2p_k(1-p_k)}{(p_k-p_j)(1+p_j)} \sum_{q=1}^n \left\{ \frac{(1+p_j)(1+p_k)}{(1-p_j)(1-p_k)} \right\}^q \frac{2q-1}{2r+1} \binom{2n-1}{n-q} \binom{2r+1}{r+q} \end{aligned} \right\} \dots\dots\dots (13)$$

4. The expression of the conditions at the rim of the outer boundary

By the similar transformation

$$z_j = \frac{1}{2} \{ (1+p_j)w_j' + (1-p_j)w_j'^{-1} \}, \quad w_j' = r_j' e^{i\theta_j'} \dots\dots\dots (14)$$

the unit circle on the physical plane, which is the outer boundary of the domain under consideration, will be transformed into a unit circle on the w_j' plane. In this case, however, it should be noted that the mapping function (14) has a branch point on the physical plane in the inner side of the outer boundary. Therefore, we separate the stress function into the two parts, one possesses the singularity in the physical plane and the other does not, as follows

$$\varphi_j'' = \varphi_{jS}'' + \varphi_{jR}'', \quad \varphi_{jS}'' = \sum_{n=1}^{\infty} a_{-2n,j} z_j^{-2n}, \quad \varphi_{jR}'' = \sum_{n=0}^{\infty} a_{2n,j} z_j^{2n} \dots\dots\dots (15)$$

or considering the mapping function (14), they can be rewritten as

$$\left. \begin{aligned} \varphi_{jS}'' &= f_{jS}(w_j') = \sum_{r=1}^{\infty} C_{-2r,j} w_j'^{-2r} \\ \varphi_{jR}'' &= f_{jR}(w_j') = B_0 + \sum_{r=1}^{\infty} B_{2r,j} \left\{ \left(\frac{1+p_j}{1-p_j} w_j'^2 \right)^r + \left(\frac{1+p_j}{1-p_j} w_j'^2 \right)^{-r} \right\} \end{aligned} \right\} \dots\dots\dots (16)$$

where

$$C_{-2r,j} = \left(\frac{p_j-1}{1+p_j} \right)^r \sum_{n=1}^r a_{-2n,j} \left(\frac{4}{p_j^2-1} \right)^n \binom{n+r-1}{r-n} \dots\dots\dots (16 \cdot a)$$

$$B_{2r,j} = \sum_{n=r}^{\infty} a_{2n,j} \left(\frac{1-p_j^2}{4} \right)^n \binom{2n}{n+r} \dots\dots\dots (16 \cdot b)$$

The inverse transformation of Eq. (16·b), which will be used later, can be obtained in the following form.

$$a_{2n,j} = \left(\frac{4}{p_j^2-1} \right)^n \sum_{r=n}^{\infty} B_{2r,j} \binom{r+n}{r-n} \frac{2r}{r+n} (-1)^r \dots\dots\dots (17)$$

Now, the stress function expressed in terms of w_j'

$$\varphi_j'' = f_{jR} + f_{jS} = B_0 + \sum_{r=1}^{\infty} \left[B_{2r,j} \left\{ \left(\frac{1+p_j}{1-p_j} w_j'^2 \right)^r + \left(\frac{1+p_j}{1-p_j} w_j'^2 \right)^{-r} \right\} + C_{-2r,j} w_j'^{-2r} \right] \dots\dots\dots (18)$$

should satisfy the boundary conditions (7), at $w_j' = e^{i\theta}$, and the following equation is obtained

$$B_{2r,j} = \sum_{q=r}^{\infty} \left(\frac{1-p_j}{1+p_j} \right)^{q-r} \frac{K_j'(q) + C_j(q)}{(1+p_j)D_j(q)} \dots\dots\dots (19)$$

where

$$\begin{aligned} D_j(q) &= (p_k-p_j) \left\{ 1 - \frac{1-p_j}{1+p_j} \left(\frac{1-p_k}{1+p_k} \right)^{2q+1} \right\} + (p_j+p_k) \left\{ \left(\frac{1-p_j}{1+p_j} \right)^{2q+1} + \left(\frac{1-p_k}{1+p_k} \right)^{2q+1} \right\} \\ K_j'(q) &= (p_k-1) \left\{ 1 + \left(\frac{1-p_k}{1+p_k} \right)^{2q} \right\} \{ \Theta_{2(q+1)}' - R_{2(q+1)}' \} + (1+p_k) \left\{ 1 + \left(\frac{1-p_k}{1+p_k} \right)^{2(q+1)} \right\} \{ \Theta_{2q}' + R_{2q}' \} \\ C_j(q) &= \left\{ p_j+p_k + (p_j-p_k) \left(\frac{1-p_k}{1+p_k} \right)^{2q+1} \right\} [(1+p_j)C_{-2(q+1),j} + (p_j-1)C_{-2q,j} \\ &\quad + 2p_k \{ (1+p_k)C_{-2(q+1),k} + (p_k-1)C_{-2q,k} \}] \end{aligned}$$

or substituting Eqs. (16·a) and (17) into Eq. (19), we have a set of linear equations with respect to $a_{2n,j}$ for the boundary conditions at the rim on the outer boundary as

$$a_{2n,j} = \sum_{r=1}^{\infty} \{ a_{-2r,j} {}^{2n}\alpha_{2r,j} + a_{-2r,k} {}^{2n}\beta_{2r,j} \} + K_{2n,j} \quad n \geq 0 \dots\dots\dots (20)$$

and the symbols ${}^{2n}\alpha_{2r,j}$, ${}^{2n}\beta_{2r,j}$ and $K_{2n,j}$ are the constants defined as below.

$$\left. \begin{aligned} K_{2n,j} &= \left(\frac{4}{p_j^2 - 1} \right)^n \sum_{q=n}^{\infty} \left(\frac{p_j - 1}{1 + p_j} \right)^q \binom{q+n}{2n} \frac{K_j'(q)}{(1+p_j)D_j(q)} \\ {}^{2n}\alpha_{2r,j} &= \left(\frac{4}{p_j^2 - 1} \right)^{n+r} \sum_{q=n}^{\infty} \left(\frac{p_j - 1}{1 + p_j} \right)^{2q+1} \left\{ 1 + \frac{p_j - p_k}{p_j + p_k} \left(\frac{1 - p_k}{1 + p_k} \right)^{2q+1} \right\} \frac{p_j + p_k}{D_j(q)} \frac{2q+1}{2r-1} \binom{q+r-1}{2r-2} \binom{q+n}{2n} \\ {}^{2n}\beta_{2r,j} &= \left(\frac{4}{p_j^2 - 1} \right)^n \left(\frac{4}{p_k^2 - 1} \right)^r \sum_{q=n}^{\infty} \left(\frac{1 - p_j}{1 + p_j} \frac{1 - p_k}{1 + p_k} \right)^q \frac{2p_k(p_k - 1)}{(1+p_j)D_j(q)} \frac{2q+1}{2r-1} \binom{q+r-1}{2r-2} \binom{q+n}{2n} \end{aligned} \right\} \dots (21)$$

5. Stresses in a plate

In order to solve the two sets of linear Eqs. (12) and (19), it is convenient for us to use the perturbation method. If the unknown coefficients $a_{2n,j}$ and $a_{-2n,j}$ can be expanded into the power series of $\lambda/2$ as

$$a_{2n,j} = \sum_{r=0}^{\infty} {}^{2n}a_{2r,j} \left(\frac{\lambda}{2} \right)^{2r}, \quad a_{-2n,j} = \sum_{r=0}^{\infty} -{}^{2n}a_{2r,j} \left(\frac{r}{2} \right)^{2(n+r)} \dots (22)$$

they are, from Eqs. (12) and (19), as follows

$$\left. \begin{aligned} {}^{2n}a_{0,j} &= K_{2n,j} \\ {}^{2n}a_{2r,j} &= \sum_{q=1}^r \{ {}^{2n}\alpha_{2q,j} {}^{-2q}a_{2(r-q),j} + {}^{2n}\beta_{2q,j} {}^{-2q}a_{2(r-q),k} \} \quad r \geq 1 \end{aligned} \right\} \dots (23)$$

$$\left. \begin{aligned} -{}^{2n}a_{0,j} &= {}^{2n}\gamma_{0,j} K_{0,j} + {}^{2n}\delta_{0,j} K_{0,k} + K_{-2n,j} \\ -{}^{2n}a_{2r,j} &= \sum_{q=0}^r \{ {}^{2n}\gamma_{2q,j} {}^{2q}a_{2(r-q),j} + {}^{2n}\delta_{2q,j} {}^{2q}a_{2(r-q),k} \} \end{aligned} \right\} \dots (24)$$

and using the above two equations alternately, we can determine the unknown constants one by one.

Stresses in a plate are obtained from Eqs. (1), (4) or (5). In the case of the stress distributions symmetric with respect to x and y , they are, after some calculations, as follows

$$\left. \begin{aligned} \widehat{yy} &= \sum_{r=1}^{\infty} \left(\frac{\lambda}{2} \right)^{2r} \sum_j R_e \left[\sum_{s=1}^r -{}^{2s}a_{2(r-s),j} L_{2s,j} \right] + R_e \left[\sum_j \sum_{n=0}^{\infty} K_{2n,j} z_j^{2n} \right] \\ \widehat{xx} &= - \sum_{r=1}^{\infty} \left(\frac{\lambda}{2} \right)^{2r} \sum_j R_e \left[\sum_{s=1}^r -{}^{2s}a_{2(r-s),j} L_{2s,j}' \right] - R_e \left[\sum_j \sum_{n=0}^{\infty} p_j^2 K_{2n,j} z_j^{2n} \right] \\ \widehat{xy} &= \sum_{r=1}^{\infty} \left(\frac{\lambda}{2} \right)^{2r} \sum_j I_m \left[\sum_{s=1}^r -{}^{2s}a_{2(r-s),j} L_{2s,j}'' \right] + \sum_j I_m \left[\sum_{n=0}^{\infty} p_j K_{2n,j} z_j^{2n} \right] \end{aligned} \right\} \dots (25)$$

where

$$\left. \begin{aligned} L_{2s,j} &= z_j^{-2s} + \sum_{n=0}^{\infty} \{ {}^{2n}\alpha_{2s,j} z_j^{2n} + {}^{2n}\beta_{2s,k} z_k^{2n} \} \\ L_{2s,j}' &= p_j^2 z_j^{-2s} + \sum_{n=0}^{\infty} \{ {}^{2n}\alpha_{2s,j} p_j^2 z_j^{2n} + {}^{2n}\beta_{2s,k} p_k^2 z_k^{2n} \} \\ L_{2s,j}'' &= p_j z_j^{-2s} + \sum_{n=0}^{\infty} \{ {}^{2n}\alpha_{2s,j} p_j z_j^{2n} + {}^{2n}\beta_{2s,k} p_k z_k^{2n} \} \end{aligned} \right\} \dots (26)$$

The rim stress of the inner hole will be given from the stress function $f_j(w_j)$ of Eq. (9) as

$$\widehat{\theta\theta} = \widehat{xx} + \widehat{yy} - \widehat{rr} = \sum_j R_e \left[(1 - p_j^2) \sum_{n=-\infty}^{\infty} A_{2n,j} e^{2in\theta} \right] - \widehat{rr} \dots (27)$$

or after some calculations, we have

$$\begin{aligned} \widehat{\theta\theta} &= R_e \left[\sum_j \frac{(1 - p_j^2)(1 - p_k)}{(p_k - p_j) \{ 1 + p_j - (1 - p_j)e^{-2i\theta} \}} \sum_{q=1}^{\infty} \left\{ \Theta_{2(q-1)} + R_{2(q-1)} + \frac{1 + p_k}{1 - p_k} (R_{2q} - \Theta_{2q}) \right\} e^{-2iq\theta} - \widehat{rr} \right] \\ &\quad + \sum_{q=1}^{\infty} \left(\frac{\lambda}{2} \right)^{2q} \sum_j R_e \left[\sum_{s=0}^q {}^{2s}a_{2(q-s),j} M_{2s,j} \right] \dots (28) \end{aligned}$$

where

$$\begin{aligned} M_{2s,j} &= (1 - p_j^2)^{s+1} \sum_{r=0}^s \left(\frac{1 + p_j}{1 - p_j} \right)^r e^{2ir\theta} \binom{2s}{s-r} + (1 - p_j^2)^s \sum_{r=1}^{s+1} \binom{2s+1}{r+s} \frac{2r-1}{2s+1} \\ &\quad \times \left[(1 - p_j^2) \left\{ e^{-2i\theta} + \frac{p_j + p_k}{p_k - p_j} \left(\frac{1 + p_j}{1 - p_j} \right)^r e^{-2ir\theta} \right\} \left\{ 1 - \left(\frac{1 - p_j}{1 + p_j} \right) e^{-2i\theta} \right\}^{-1} \right. \\ &\quad \left. + \frac{2p_j(1 - p_j)(1 - p_k)}{p_j - p_k} \left(\frac{1 + p_j}{1 - p_j} \right)^r e^{-2ir\theta} \left\{ 1 - \left(\frac{1 - p_k}{1 + p_k} \right) e^{-2i\theta} \right\}^{-1} \right] \dots (29) \end{aligned}$$

6. Stresses in a ring under uniform inner pressure

In this case, we have

$$R_0/2 = -p_0, \quad R_0' = R_n = \Theta_n = R_n' = \Theta_n' = 0 \quad n > 1 \quad \dots\dots\dots (30)$$

and the rim stress around the inner hole is obtained from Eq. (28)

$$\frac{\widehat{\theta\theta}}{-p_0} = R_e \left[1 + \sum_j \frac{2(1-p_j^2)(1-p_k)e^{-2i\theta}}{(p_j-p_k)(1+p_j-(1-p_j)e^{-2i\theta})} \right] + \sum_{s=1}^{\infty} \left(\frac{\lambda}{2} \right)^{2s} \sum_j R_e \left[\sum_{q=0}^s {}^{2q}a_{2(s-q),j} M_{2q,j} \right] \quad \dots\dots\dots (31)$$

The stresses at an arbitrary point in a plate are

$$\frac{\widehat{yy}}{-p_0} = \sum_{s=1}^{\infty} \left(\frac{\lambda}{2} \right)^{2s} \sum_j R_e \left[\sum_{q=1}^s {}^{-2q}a_{2(s-q),j} L_{2q,j} \right] \quad \dots\dots\dots (32)$$

and so on.

Considering the limiting case for $p_j \rightarrow 1$, we have the expressions for an isotropic plate. After some calculations, the well-known formulae are obtained as

$$\frac{\widehat{\theta\theta}}{-p_0} = 1 + 2 \sum_{s=1}^{\infty} \lambda^{2s} = \frac{1+\lambda^2}{1-\lambda^2} \quad \dots\dots\dots (33)$$

$$\frac{\widehat{yy}}{-p_0} \Big|_{x=y=x} = (1+x^{-2}) \sum_{s=1}^{\infty} \lambda^{2s} = (1+x^{-2}) \frac{\lambda^2}{1-\lambda^2} \quad \dots\dots\dots (34)$$

7. Numerical examples

Figs. 2~4 show the numerical results for the plate of isotropic material, oak ($p_1=5/3$, $p_2=49/51$) and plywood ($p_1=1/p_2=5/3$, namely $E_x=E_y$). The hoop stresses around the hole or along the elastic axis of the plate are shown in Fig. 2 or 3 respectively, for $\lambda=0.2$. The maximum stress occurs at the rim of the hole and is given in terms of the power series of λ as

(a) isotropic plate

$$\frac{\widehat{\theta\theta}_{\max}}{-p_0} = (1+\lambda^2)/(1-\lambda^2) = 1 + 2\lambda^2 + 2\lambda^4 + 2\lambda^6 + 2\lambda^8 + \dots \quad \dots\dots\dots (35)$$

(b) oak plate

$$\frac{\widehat{\theta\theta}_{\max}}{-p_0} = 1.016 + 2.00\lambda^2 + 1.75\lambda^4 + 2.76\lambda^6 + 1.50\lambda^8 + \dots \quad \dots\dots\dots (36)$$

Fig. 2

Fig. 3

Fig. 4

(c) plywood

$$\frac{\widehat{\theta\theta}_{\max}}{-p_0} = 1.267 + 2.41\lambda^2 - 3.32\lambda^4 + 18.38\lambda^6 - 27.29\lambda^8 + \dots \quad \dots\dots\dots (37)$$

and the results are shown in Fig. 4. The notations a , b and c in these figures mean the stress values for isotropic material, oak and plywood, respectively.

The two facts mentioned below are remarkable in this case.

(1) The stresses in an oak plate are nearly equal to that of the isotropic one. The phenomenon which was also pointed out by Uemura and others⁽⁴⁾ in the case of an infinite plate with a circular

hole under uniform inner pressure, is generally characteristic of the orthotropic plate, one of the aeolotropic constants of which is nearly equal to 1, with axisymmetric boundary conditions.

(2) In the case of orthotropic material with aeolotropic constants different from 1, the maximum stress is much larger than that of isotropic one and, for small λ , the difference between them becomes larger with increasing λ .

8. Conclusion

The general method of analyzing the stresses in a ring of orthotropic material has been described here with the numerical results under uniform inner pressure. According to the present method, the stresses are expressed in terms of the power series of λ , and the convergence of the formulae is a question for large value of it. Therefore, the evaluation of the stresses, except for the isotropic plate, has been worked out for $\lambda < 0.5$ as shown in

Fig. 4. But, as the radius of the inner circle becomes larger and λ approaches to 1, the stress distribution should coincide with that of the thin walled cylinder which is independent of the elastic property of the material. Accordingly we may extrapolate our results in Fig. 4 by the lines asymptotic to that of the isotropic plate. The curves in Fig. 4 show the tendency as mentioned above.

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Some Plane Stress Problems of Orthotropic Material*

By Takuo HAYASHI**

As to the plane stresses of orthotropic material, various studies have been worked out by many investigators, but the stresses in a plate bounded by lines asymmetric with respect to the elastic axes of the material, have been little studied yet. In this paper, some problems of this kind are treated by the use of Beltrami-Cayley's conformal mapping method.

As examples, the stress distributions in (1) a semi-infinite plate, (2) an infinite plate with an elliptic hole, and (3) an elliptic plate, are calculated. The results obtained show significant differences from those in isotropic plate or in orthotropic plate with symmetrical boundary to the elastic axes of the plate.

1. Introduction

The plane stress problems of orthotropic material treated so far by many investigators mainly concern the stresses in the domain symmetric with respect to the elastic axes of the material, and for the other cases, only a few works have been reported up to the present by Lexnitzky⁽¹⁾, S. Moriguchi⁽²⁾, K. Washizu⁽³⁾, M. Higuchi⁽⁴⁾ and others. This is mainly because in the case of the orthotropic plate, the differential equation for the

stress function becomes rather complicated with respect to the co-ordinates other than the orthogonal ones parallel to the elastic axes of the plate and further treatment of the equation is very much difficult.

On the other hand, the stress function for the orthotropic plate, as is well known, can be given as a sum of two harmonic functions in two functional planes and when the mapping functions are found to transform the boundaries in each functional plane into simple figures, the stress problems can be treated with comparative ease. Those methods have been used so far for the treatment of the stresses in a perforated orthotropic plate⁽⁵⁾, but

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