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Steady, Forced Vibration of Unsymmetrical

Piecewise-linear System*

(2nd Report, Comparison with Analog-
Computer Experiments)

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Results of analysis, by means of the Fourier series method introduced in the first report, of steady, forced vibrations in an unsymmetrical piecewise-linear system, whose characteristics are composed of two distinct linear regions in relation not only to the restoring force but also to the damping force, are compared with those of the analog-computer experiments performed for the above mentioned system with the help of the auxiliary apparatus composed of electronic limiters, a voltage comparator etc.

1. Introduction

In the 1st report, we have given an analysis of steady, forced vibrations in an unsymmetrical piecewise-linear system with one degree of freedom by means of a new method utilizing a perfect Fourier series⁽¹⁾.

In this report, we compare the results of the analytical calculation with those of an experiment conducted for forced vibrations of the above system by means of an analog computer with the auxiliary apparatus.

2. Experimental apparatus and method of experiment

We employed for the experimental apparatus a high speed type analog computer (Philbrick Co., 1953) at the Electrotechnical Laboratory of the Ministry of International Trade and Industry, which is supplemented by some auxiliary apparatus composed of electronic limiters, a voltage comparator etc.

As mentioned in the first report, the characteristics of our system are composed of two distinct regions separated from each other by the specified value e_0 of the displacement z , so that the values of spring constant and the coefficients of viscous damping in two regions are different from each other as shown in Fig. 1(a) and Fig. 1(b).

The equations of motion for two regions are

$$\begin{aligned} m\ddot{z} + c\dot{z} + kz &= f \cos \omega t, & z \leq e_0 \\ m\ddot{z} + (c+C)\dot{z} + kz + K(z-e_0) &= f \cos \omega t, & z > e_0 \end{aligned} \quad \dots\dots\dots (1)$$

Our experiments were performed for the case

$$K/k=1.0, \quad c=0, \quad C/\sqrt{mk} \doteq 0.5 \quad \dots\dots\dots (2)$$

and the block diagram for them is shown in Fig. 2(a) and Fig. 2(b). In Fig. 2(a), C_1 and C_2 are the constant multipliers, J_1 and J_2 the integraters, A_1 , A_2 and A_3 the adders, and the inverse triangles with 1 in their centres are the sign changers. And G_1 , G_2 and G_3 are the electronic limiters specially constructed for our purpose, which enable us to switch over from one region of the system characteristics to another. Fig. 2(b) shows the voltage comparator⁽²⁾, the electronic limiters⁽³⁾ etc. If the displacement z becomes greater than e_0 , it acts through the phase shifter P. S. upon the comparator, which sends the positive signals to the electronic limiters to set them in "on" condition. Inversely, if the displacement z becomes less than e_0 , the negative signals from the comparator set the limiters in "off" condition. The voltages of

Fig. 1

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negative and positive signals are

$$E_{c1} = \frac{r_1}{r_1 + r_1' + r_2 + r_2'} V_0, \quad E_{c2} = \frac{r_2}{r_1 + r_1' + r_2 + r_2'} V_0 \quad \dots\dots\dots (3)$$

where r_1, r_1', r_2 and r_2' are resistances far greater than R , and A is the D.C. amplifier in Fig. 2(b). Being A_1, A_2 and A_3 the sign changers, according as the signals transmitted from the comparator are positive or negative, the limiters G_1, G_2 and G_3 are in the state of "on" and "off". If G_1 is closed, the original restoring force kz is augmented by Kz ($=kz$), and if G_2 is closed, Ke_0 ($=ke_0$) is subtracted from the above sum. Thus by means of closing or opening of G_1 and G_2 , the switching over action of the restoring force characteristics is realized, so that,

$$kz \rightleftharpoons kz + K(z - e_0) \quad \dots\dots\dots (4)$$

Similarly, by closing or opening of G_3 , the switching over of the damping force characteristics can be obtained as,

$$0 \rightleftharpoons C\dot{z} \quad \dots\dots\dots (5)$$

The phase shifter P. S. is introduced for minute adjustment and compensation of the inevitable errors, when the closing or opening of the limiters is not exactly performed at the specified value e_0 .

We are compelled to maintain the exciting frequency ω at the constant value 50 c/sec. So, to change the frequency ratio Ω ,

$$\Omega = \omega/\omega_0 \quad \dots\dots\dots (6)$$

where

$$\omega_0 = \sqrt{k/m} \quad \dots\dots\dots (7)$$

we must adjust the value of C_2 and change the

natural frequency ω_0 . At the same time, we should adjust the value of C_1 correctly to keep the value of the twice damping ratio $C/\sqrt{mk}$ constant.

The differential equations written for the machine variable Z are

$$\left. \begin{aligned} p^2 Z + \frac{C_2}{T^2} Z &= F \cos \omega t, & \text{Region I} \\ p^2 Z + \frac{C_1}{T} pZ + \frac{C_2}{T^2} (2Z - e_0) &= F \cos \omega t, & \text{Region II} \end{aligned} \right\} \quad \dots\dots (8)$$

where T ($=4.02 \times 10^{-3}$ sec) is the time constant of the two integrators J_1 and J_2 . The natural frequency ω_0 and the twice damping ratio $C/\sqrt{mk}$ are given by

$$\omega_0 = \sqrt{C_2}/T \quad \dots\dots\dots (9)$$

$$C/\sqrt{mk} = C_1/\sqrt{C_2} \quad \dots\dots\dots (10)$$

We have performed six series of experiments for six values of the frequency ratio Ω . Table 1 give the frequency ratios and the twice damping ratios for these six series of experiments. In each series, we change the voltage representing the gap e_0 , to make the system start from the state of no contact (in region I only), then pass through the state of partial contact (partially in region I and partially

Table 1

No. of exp.	C_2	Ω	C_1	$C/\sqrt{mk}$
①	1.947	0.907	0.619	0.444
②	1.710	0.967	0.580	0.443
③	1.600	1.000	0.556	0.440
④	1.455	1.049	0.529	0.439
⑤	1.340	1.093	0.505	0.436
⑥	1.222	1.144	0.484	0.438

(a) Block diagram

(b) Comparator, electronic limiters etc.

Fig. 2

in region II), and lastly come in the state of full contact (in region II only). At the same time, the wave forms at the points ①, ②, ⑤, ⑥ and ⑦ in Fig. 2(a) are observed and photographed on the cathode-ray oscilloscope. An example of these photographic records is also shown in Fig. 2(a).

3. Relation between gap and amplitude under constant excitation

From the formula (21) of 1st report, we have for the displacement z ,

$$z = \Gamma \left[\cos \theta - x_0 - \sum_{n=2}^{\infty} \frac{x_n}{1-n^2\Omega^2} \cos n\theta - \sum_{n=2}^{\infty} \frac{y_n}{1-n^2\Omega^2} \left(\sin n\theta - \frac{\sin \frac{n\theta_0}{2}}{\sin \frac{\theta_0}{2}} \sin \theta \right) \right] \dots\dots\dots (11)$$

where θ is the phase angle measured from the middle point of the contact interval, θ_0 the breadth of the contact interval, x_0 , x_n and y_n ($n=1, 2, 3, \dots$) the non-dimensional Fourier coefficients for the nonlinear part of the restoring and damping force and Γ is the amplitude of the cosine part of the fundamental harmonic.

For the degree of nonlinearity as of our cases ($K/k=1.0$ and $C/\sqrt{mk}=0.5$), the deviation of the waveform from the exact sine wave is so small that we can safely regard Γ as the total amplitude (the distance between the positive and negative extreme value of the displacement)⁽⁴⁾. Similarly, by neglecting higher harmonics thoroughly we may put

$$e_0/\Gamma = \cos \frac{\theta_0}{2} - x_0 \dots\dots\dots (12)$$

Now, from Eqs. (3), (18) and (19) of the 1st report, we have

$$x_0 = a_0/k\Gamma = \frac{1}{2\pi} \int_{-\theta_0/2}^{\theta_0/2} g(\theta) d\theta = \frac{1}{2\pi k\Gamma} \int_{-\theta_0/2}^{\theta_0/2} \{K(z-e_0) + C\dot{z}\} d\theta \dots\dots\dots (13)$$

then x_0 can be obtained graphically on the photographic record at the point ⑥ as shown in Fig. 3, that is,

$$x_0 = \frac{(\text{Area } A_1) - (\text{Area } A_2)}{H \times L} \dots\dots\dots (14)$$

And the contact angle θ_0 can be measured also there.

The statical deflection δ_{st} is represented by the machine variable as

$$\delta_{st} = F/C_2 \dots\dots\dots (15)$$

and we have for the specific amplitude Γ/δ_{st} ,

$$\Gamma/\delta_{st} = C_2(D/F) \dots\dots\dots (16)$$

where D and F are the amplitudes of the waveforms on the records at the points ① and ⑤. Lastly, the specific gap ratio e_0/δ_{st} can be calculated from Eqs. (12) and (16) as

Fig. 3 Obtaining x_0 from waveform at point ⑥

Fig. 4 Specific amplitude Γ/δ_{st} versus specific gap magnitude e_0/δ_{st}

$$e_0/\delta_{st} = (e_0/\Gamma) \times (\Gamma/\delta_{st}) \quad \dots\dots\dots (17)$$

Thus we can obtain the relation between the gap magnitude e_0 and the amplitude of the resulting vibration Γ under constant excitation for each of the series of experiments. Fig. 4 gives these experimental values compared with the theoretical values calculated by means of our method advocated in the 1st report.

The calculation is performed according to the 1st approximation of the direct method.

4. Waveforms for displacement and velocity

The theoretical formulas for displacement and velocity are given by Eq. (11) and by its time derivative as

$$z/\Gamma = \cos \theta - x_0 - \sum_{n=2}^{\infty} \frac{x_n}{1-n^2\Omega^2} \cos n\theta - \sum_{n=2}^{\infty} \frac{y_n}{1-n^2\Omega^2} \left(\sin n\theta - \frac{\sin \frac{n\theta_0}{2}}{\sin \frac{\theta_0}{2}} \sin \theta \right) \quad \dots\dots\dots (18)$$

$$\dot{z}/\Gamma\omega = -\sin \theta + \sum_{n=2}^{\infty} \frac{nx_n}{1-n^2\Omega^2} \sin n\theta - \sum_{n=2}^{\infty} \frac{y_n}{1-n^2\Omega^2} \left(n \cos n\theta - \frac{\sin \frac{n\theta_0}{2}}{\sin \frac{\theta_0}{2}} \cos \theta \right) \quad \dots\dots\dots (19)$$

And, for the improved 1st approximation, which discards the higher coefficients above ξ_0 , ξ_1 and η_1 thoroughly, we have from Eq. (36) of the 1st report

$$\left. \begin{aligned} x_0 &= \frac{\theta_0}{2\pi} \left(\xi_0 + \frac{2}{\pi} \xi_1 \right) \\ x_n &= \frac{\xi_0}{n\pi} \sin \frac{n\theta_0}{2} + \frac{2}{\pi} \frac{\theta_0}{\frac{\pi^2}{\theta_0^2} - n^2} \xi_1 \cos \frac{n\theta_0}{2} \\ y_n &= \frac{2}{\pi} \frac{n}{\frac{\pi^2}{\theta_0^2} - n^2} \eta_1 \cos \frac{n\theta_0}{2} \end{aligned} \right\} \quad \dots\dots\dots (20)$$

To compare the waveforms on the photographic records and those calculated from the theory, we measure the contact (nonlinear) ratio $\theta_0/2\pi$ on the record. With this and the frequency ratio Ω , we can solve the simultaneous equations (47) of the 1st report to give ξ_0 , ξ_1 and η_1 . By calculating x_0 , x_n , and y_n in Eqs. (20) using the values of ξ_0 , ξ_1 and η_1 thus obtained and by introducing the results into the expressions (18) and (19), we obtain the theoretical wave forms corresponding to those on the photographic records.

Alternatively, we transform the infinite expressions (18) and (19) into finite terms by substituting the expressions (20) into them and performing the summation of the resulting infinite series. The summation is made possible by the application of the Eqs. (43) of the first report and similar ones for the range $(-2\pi < \theta_0 < 0)$.

The results are as follows,

$$\begin{aligned} z/\Gamma &= \cos \theta + \xi_0 \left[\frac{2}{\pi} \frac{\sin \frac{\theta_0}{2}}{1-\Omega^2} \cos \theta + \frac{\sin \frac{1}{\Omega} \left(\pi - \frac{\theta_0}{2} \right)}{\sin \pi/\Omega} \cos (\theta/\Omega) - 1 \right] \\ &+ \frac{\xi_1}{\theta_0} \left[\frac{2 \cos \frac{\theta_0}{2}}{(1-\Omega^2) \left\{ \left(\frac{\pi}{\theta_0} \right)^2 - 1 \right\}} \cos \theta + \frac{\pi}{\Omega^2 \frac{\pi^2}{\theta_0^2} - 1} \left\{ \frac{\theta_0}{\pi} \cos \frac{\pi}{\theta_0} \theta - \Omega \frac{\cos \frac{1}{\Omega} \left(\pi - \frac{\theta_0}{2} \right)}{\sin (\pi/\Omega)} \cos (\theta/\Omega) \right\} \right] \\ &+ \frac{\eta_1}{\Omega^2 \frac{\pi^2}{\theta_0^2} - 1} \left[\sin \frac{\pi}{\theta_0} \theta - \frac{\cos \frac{1}{\Omega} \left(\pi - \frac{\theta_0}{2} \right)}{\sin (\pi/\Omega)} \sin (\theta/\Omega) - \frac{\sin \frac{1}{\Omega} \left(\pi - \frac{\theta_0}{2} \right) \cos \frac{\theta_0}{2\Omega}}{\sin (\pi/\Omega) \sin \frac{\theta_0}{2}} \sin \theta \right] \\ &\Omega \neq 1.0, \quad -\frac{\theta_0}{2} \leq \theta \leq \frac{\theta_0}{2} \quad \dots\dots\dots (21) \end{aligned}$$

$$\begin{aligned}
 z/\Gamma = & \cos \theta + \xi_0 \left[\frac{2}{\pi} \frac{\sin \frac{\theta_0}{2}}{1 - \Omega^2} \cos \theta - \frac{\sin(\theta_0/2\Omega)}{\sin(\pi/\Omega)} \cos\left(\frac{\pi - \theta}{\Omega}\right) \right] \\
 & + \frac{\xi_1}{\theta_0} \left[\frac{2 \cos \theta_0/2}{(1 - \Omega^2) \left(\frac{\pi^2}{\theta_0^2} - 1 \right)} \cos \theta - \frac{\pi \Omega}{\Omega^2 \frac{\pi^2}{\theta_0^2} - 1} \frac{\cos(\theta_0/2\Omega)}{\sin(\pi/\Omega)} \cos\left(\frac{\pi - \theta}{\Omega}\right) \right] \\
 & + \frac{\eta_1}{\Omega^2 \frac{\pi^2}{\theta_0^2} - 1} \left[\frac{\cos(\theta_0/2\Omega)}{\sin(\pi/\Omega)} \sin\left(\frac{\pi - \theta}{\Omega}\right) - \frac{\cos(\theta_0/2\Omega)}{\sin(\pi/\Omega)} \frac{\sin \frac{1}{\Omega} \left(\pi - \frac{\theta_0}{2} \right)}{\sin \theta_0/2} \sin \theta \right] \\
 & \Omega \neq 1.0, \quad \frac{\theta_0}{2} < \theta < 2\pi - \frac{\theta_0}{2} \dots \dots \dots (22)
 \end{aligned}$$

$$\begin{aligned}
 e_0/\Gamma = & \cos \frac{\theta_0}{2} + \xi_0 \left[\frac{1}{\pi} \frac{\sin \theta_0}{1 - \Omega^2} - \frac{\sin(\theta_0/2\Omega) \cos \frac{1}{\Omega} \left(\pi - \frac{\theta_0}{2} \right)}{\sin(\pi/\Omega)} \right] \\
 & + \frac{\xi_1}{\theta_0} \left[\frac{1 + \cos \theta_0}{(1 - \Omega^2) \left(\frac{\pi^2}{\theta_0^2} - 1 \right)} - \frac{\pi \Omega}{\Omega^2 \frac{\pi^2}{\theta_0^2} - 1} \frac{\cos(\theta_0/2\Omega) \cos \frac{1}{\Omega} \left(\pi - \frac{\theta_0}{2} \right)}{\sin(\pi/\Omega)} \right] \dots \dots \dots (23)
 \end{aligned}$$

Because these expressions become indeterminate when $\Omega=1.0$, we take the limiting values for them as Ω tends to 1.0.

The expressions for the velocity can be obtained by differentiating the corresponding expressions for displacement with respect to time⁽⁵⁾, so we refrain from writing them down here.

Two examples of comparison are given, one of which is for

$$\left. \begin{aligned} K/k=1.0, \quad c=0, \quad C/\sqrt{mk}=0.436 \\ \Omega=1.093, \quad \theta_0/2\pi=0.30 \end{aligned} \right\} \dots \dots (24)$$

and the other is for

$$\left. \begin{aligned} K/k=1.0, \quad c=0, \quad C/\sqrt{mk}=0.440 \\ \Omega=1.00, \quad \theta_0/2\pi=0.39 \end{aligned} \right\} \dots \dots (25)$$

(a) Waveform of displacement for $K/k=1.0, \quad c/\sqrt{mk}=0.0, \quad C/\sqrt{mk}=0.436$
 $\Omega=1.093, \quad \theta_0/2\pi=0.3$ (theoretical)

(b) The same as (a) (experimental)
 Fig. 5

Fig. 5(a) and Fig. 5(b) are the wave forms for displacement, while Fig. 6(a) and Fig. 6(b) are those for velocity of the first example. And Fig. 7(a) and Fig. 7(b) are the waveforms for displacement, while Fig. 8(a) and Fig. 8(b) are those for velocity of the second example.

5. Conclusion

We have given the comparison of the results of analog computer experiments performed of steady, forced vibrations in an unsymmetrical piecewise-linear system with those of the theoretical analysis by means of a new perfect Fourier series method

(a) Waveform of velocity for $K/k=1.0, \quad c/\sqrt{mk}=0.0, \quad C/\sqrt{mk}=0.436$
 $\Omega=1.093, \quad \theta_0/2\pi=0.3$ (theoretical)

(b) The same as (a) (experimental)
 Fig. 6

(a) Waveform of displacement for
 $K/k=1.0$, $c/\sqrt{mk}=0.0$, $C/\sqrt{mk}=0.440$
 $\Omega=1.00$, $\theta_0/2\pi=0.390$ (theoretical)

(b) The same as (a) (experimental)
 Fig. 7

advocated in the 1st report. Main conclusions are as follows.

(1) The agreement of the theory with experiment is satisfactory for the relationship between the gap magnitude and vibration amplitude under constant excitation, and also for the wave forms of displacement and velocity.

(2) The first approximation after convergence improvement has a sufficient accuracy for practical use.

(3) The expressions of an infinite number of terms for displacement and velocity were transformed into those of a finite number of them including only ξ_0 , ξ_1 and η_1 of the improved 1st approximation.

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(a) Waveform of velocity for
 $K/k=1.0$, $c/\sqrt{mk}=0.0$, $C/\sqrt{mk}=0.440$
 $\Omega=1.00$, $\theta_0/2\pi=0.390$ (theoretical)

(b) The same as (a) (experimental)
 Fig. 8

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