Studies on the Characteristics of Axial-Flow Pumps

(Part 2, Analytical Method of Determining the Discharge-Theoretical Head Characteristics)

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In this paper, the flow in an axial-flow pump impeller at partial flow rates is studied by dividing it into two domains; one is near the casing wall where the radial flow is seriously affected by the wall, another is far from it where the fluid flows along impeller blades and the effect of the wall is negligibly small.

By considering about the flow in impeller blades in this way, it becomes possible to analyse an impeller action and to demonstrate the reason for the conspicuous increase of the head at partial quantities. The analytical method to find the relation between the discharge and the theoretical head is obtained. But the complexity of the flow makes it necessary to introduce some coefficients, which should be determined empirically.

1. Introduction

In the preceding report(1), as the reason for the conspicuous increase of the head of an axial flow pump toward zero capacity, it has been pointed out that the angle between the relative velocity and the rotational direction becomes larger than the discharge angle of impeller blades at the outlet from the impeller. In the present report, the flow in impeller blades is treated conveniently by separating it into two different zones; one is a domain (jet flow domain) in which the effect of the casing wall is important, while the impeller blades play a small part, another is a domain (fundamental flow domain) in which this effect is negligibly small and the fluid element flows along the blades. Only under such assumption, it becomes possible to explain the phenomenon described above, and to analyse the pump characteristics.

To determine analytically the relation between the capacity and the theoretical head, we have to study on the flow in a pump by using a one dimensional equation, because there is not a path of the fluid element on the co-axial cylindrical surface with the pump, therefore the application of the aerofoil theory seems to be impossible. In the present report, the pump performance is derived analytically by taking into account the flow in impeller blades, which makes this method differ vastly from the ones reported previously. As a result, it is recognized that the effect of the wall surface on the flow in impeller blades is very important at low flow rates. It will be shown in part 4 that the calculated results are in good agreement with those obtained from the experiment.

2. Fundamental studies on the impeller performance at partial capacities (II)

2.1 Fundamental flow domain and jet flow domain

As there is no radial flow in impeller blades designed by the free vortex theory at the design point, the effects of the casing wall on the flow are small. But at low flow rates, the radial flow in impeller blades becomes so considerable that the casing wall affects remarkably the flow in blades, and its effect extends to points fairly distant from the casing wall. In order to reveal the impeller performance at partial capacities, it seems convenient to separate the flow in blades into two parts, one corresponding to the zone in which the effect of the casing wall is small, another to the zone with large radii, in which the effect of the wall is remarkable. To make a theoretical treatment possible, it is conveniently assumed that its boundary is given by a co-axial cylindrical surface with the pump, radius $r_1$, moreover the inside part of it is

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termed "fundamental flow domain", and the outside part is termed "jet flow domain".

On the determination of radius \( r_1 \), there is no distinct basis theoretically. But, in the centrifugal domain\(^{16} \) the fluid elements, which have been turned by the casing wall in the axial direction, flow out to the suction and delivery sides of the impeller in the jet flow domain. Thereafter, a radial point where a backward flow to the suction side begins seems to give one basis of deciding \( r_1 \). We, accordingly, make it a rule that this radius gives \( r_1 \) in the centrifugal flow domain. According to the preceding report, \( r_1 \) will be determined as follows.

Since the width of annulus on the suction side of impeller blades, where the velocity distributions were measured, was larger than that of the pump annulus in part 1, the width of the former must be converted into that of the latter to determine the value of \( r_1 \). For this purpose, it is assumed that the area ratio of the fundamental flow domain to the jet flow domain keeps the same value for both the wide and the narrow annulus. The relation between the capacity and \( r_1 \) are shown in Fig. 1(a). As illustrated clearly in the figure, \( R_1 = r_1/r_0 \) (radius of the casing wall) which takes a minimum value at the shut-off point increases linearly with the discharge and coincides to one at \( \varphi_{in}=0.112 \). \( \varphi_{in}=\varphi_{in}/u_t \) (mean axial velocity on the delivery side of impeller blades, \( u_t \) : rotational velocity of the impeller tips) corresponding to the critical capacity\(^{17} \). There must, however, exist a zone in which the effect of the casing wall is not negligible in the axial flow domain\(^{16} \). Therefore, \( R_1 \) in the axial flow domain may be defined conveniently on the assumption that the linear relation (dotted line) between \( R_1 \) and \( \varphi_{in} \) should be kept at \( \varphi_{in}=(\varphi_{in})_r \), as shown in Fig. 1(a). The dotted line in Fig. 1(a) shows that it takes \( R_1 = 1 \) at \( \varphi_{in} = 0.227 \) (≈ \( \varphi_{in0} \); non-dimen-

![Fig. 1 Value of \( R_1 \)](image)

sional representation of mean axial velocity of the fluid on the delivery side of impeller blades at the design point). Accordingly, the value of \( R_1 \) in the axial flow domain may be given by a straight line connecting two points on the \( R_1-\varphi_{in} \) diagram; one is \( R_1 = 1 \) at the design capacity (which contains the leakage flow), another is \( \varphi_{in0} \) at the zero-capacity. The values of \( \varphi_{in} \) for various types of impeller blades will be decided empirically in part 4. On the effect of the hub ratio, since the axial velocity distribution in the meridional plane, on the suction side of impeller blades, is similar to the parabolic curve which takes its peak on the hub surface, \( \varphi_{in} \approx \varphi_{in0} \) can be calculated approximately by supposing that the \( \varphi_{in} \)-distribution is given by the parabolic curve. Calculated results are shown in Fig. 1(b). Experimental results shown in Fig. 1(a) corresponding to \( R_1 \) at \( R_0 = 0.491 \) (\( R_1 \) : hub ratio), its \( \varphi_{in} \approx \varphi_{in0} \) therefore, becomes nearly equal to the calculated one.

2.2 Flow in the fundamental flow domain

In this domain, since the effect of the casing wall may be neglected as above mentioned, the flow on the delivery side of the impeller can be determined without taking into account the flow in impeller blades. In order to make it possible to analyse the flow by using a one dimensional equation, we must introduce the coefficients of slip angle expressed by

\[
1 + p_1 = H_{tub}/H_{th} = c_{udm}/c_{ud}
\]

where \( H_{tub}, c_{udm}, H_{th} \) and \( c_{ud} \) represent the theoretical heads and peripheral velocity components on the delivery side of the impeller respectively with an infinite and a finite number of blades.

For centrifugal pumps, it is well known that
the values of $1+p_i$ are constant although the capacity is changed, but for axial flow pumps it is yet unknown. To prove it, values of $1+p_i$ are calculated and shown in Fig. 2 by using experimental results obtained in the preceding report, at various flows without a reverse flow on the delivery side of impeller blades. It is known from the figure, that $1+p_i$ takes almost a constant value for various capacities at respective radius, except in the case where $\varphi_4$ near the hub becomes very small. In its calculation, further, the direction of zero-lift angle is chosen as the exit angle of blades. Accordingly, the assumption that fluid elements flow along blades surface in the fundamental flow domain will be permitted, if $1+p_i$ above mentioned is used.

**2.3 Flow in the jet flow domain**

The vector diagram of flow at radius $r_1$ is shown in Fig. 3. A blade section on the co-axial cylindrical surface with the pump has a fairly large inclination to the rotational direction, in general, but the inclination of the profile on the meridian section to the radial direction is very small. Accordingly, from the condition that fluid elements flow along blade surfaces, it is easily understood that the peripheral velocity component $c_{s1}$ at radius $r_1$ increases with the reduction of the axial one $v_1$, but the effect of the radial velocity component is negligibly small. The theoretical head of an axial flow pump, moreover, is proportional to the magnitude of the peripheral velocity component of the fluid, thus, the axial velocity component affects deeply the pump head but the radial velocity one does not.

A fluid element which has the velocity components at $r_1$ as shown in Fig. 3 changes its radial velocity component $c_{r1}$ into the axial one and flows out in the jet flow domain. As the result, the axial velocity components at the outlet from impeller blades increase conspicuously. If we suppose that fluid particles flow likewise along the blade surface in the jet flow domain, the whirl component $c_{w1}$ at $r_1$ must decrease on the contrary in this domain, and there should, thereafter, exist even a case in which $c_s$ takes a negative value, as illustrated in Fig. 4. The absolute velocity $c_1$ at $r_1$, moreover, changes into $c'$ suddenly. As shown in Fig. 1, the width of jet flow domain being narrow, it seems to be an unnatural phenomenon that the flow state varies suddenly in this domain. Since, in general, axial flow pumps have fairly large pitch-chord ratios, it may be supposed that the effect of impeller blades on the flow in this domain is negligibly small and only radial velocity components are turned into axial ones by presence of the casing wall. As the result, the rotational components of fluid particles having entered into the jet flow domain are kept constant near the radius $r_1$, and only the axial components increase. The velocity triangle, therefore, becomes as shown by A in Fig. 4. According to such a consideration, fluid elements have not abrupt changes of velocity at $r_1$ and it seems that they represent actual flows. It is understood readily from the figure that the angle between the directions of the relative and rotational velocities becomes larger than the discharge angle of blades.

As an example demonstrates that the treatment above described is reasonable, vector diagrams of flows on both sides of impeller blades in the jet flow domain at low flow rates are derived from the data shown in the previous report and represented in Fig. 5. The area of pump annulus on the suction

![Fig. 3 Flow at $r_1$](image)

* It will be explained in part 4.

![Fig. 4 Variation of the flow in the vicinity of $r_1$ in the jet flow domain](image)

![Fig. 5 Flows on both sides of impeller blades in the jet flow domain](image)
side of the impeller, where the flow was measured, being larger than that on the delivery side, its difference was corrected according to the assumption that the axial velocity distribution in a wide annular section was converted into one on a narrow annular section by the potential flow theory, and whirl components by the free vortex theory, too. The flow states on both sides of the impeller were compared at the same radius as shown in the figure. In spite of a considerable difference of the exit angles of vanes to the suction and delivery sides of the impeller, it is well known that fluid elements flow out almost symmetrically to both sides of impeller blades. This phenomenon seems to illustrate that the presence of blades is not important in the jet flow domain, unless a pitch-chord ratio of the impeller blades is too small. It will, moreover, be shown empirically in part 4 that the existence of impeller blades as a boundary condition in the jet flow domain can be neglected even if the pitch-chord ratio \((l/l)\) is fairly small \((l/l=0.86\) at a mean radius). According to the thinking above described, further, the width of clearances between the propeller blade-tips and the linear ring should affect very little the flow in the jet flow domain, at low flow rates. Experimental results which will be shown in the following report illustrate that this effect is remarkable in the axial flow domain but is very little in the centrifugal flow domain.

Also, the theoretical analysis will show that the impeller blades in the jet flow domain make the whirl components and heads of fluids increase toward the radial direction.

2-4 A consideration on the critical discharge

The cause to the occurrence of a backward flow on the suction side of impeller blades can be qualitatively explained as follows. The radial velocity component at radius \(r_f\) seems to increase with the reduction of flows even in the axial flow domain. Accordingly, the effect of the casing wall on the radial flow having entered into the jet flow domain in impeller blades becomes considerable. Then the angle between the directions of the relative and peripheral velocity components of the fluid in the jet flow domain increases and the head increases, too. A fluid particle getting into impeller blades at large radius moves very little radially in the impeller, and flows out nearly along the blade surface. A fluid element getting into the impeller at small radius, consequently, has an energy larger than one getting into it at large radius. As the difference in kinematic energy of these fluid particles at the outlet from the impeller is seemingly little, a pressure difference arises and the fluid particle having entered at large radii flows out backward at last.

According to the consideration above mentioned, the reverse flow on the suction side of impeller blades is generated by an energy difference of fluids on the delivery side of the impeller; thereafter, large quantities of fluid flow out backward, suddenly, at the critical capacity.

3. Analytical method of determining the discharge-theoretical head performance

In the present chapter, the relation between the flow and the theoretical head of an axial flow pump with the impeller of free vortex type will be derived analytically by using the one dimensional theory for all flow rates from the design point to the shut-off point. Energy losses and the effect of guide vanes on the flow are neglected and calculation is performed only about the impeller to make the analysis possible.

3-1 Flow at the outlet from impeller blades in the fundamental flow domain

It is assumed that there is no whirl component \(c_{w}\) and the axial velocity component \(v_{a}\) is constant in front of the impeller. If the peripheral velocity of the impeller tips is given by \(u_{r}\), the dimensionless velocity component is shown as follows

\[ q_{a}=v_{a}/u_{r}=\text{const.} \quad \chi_{a}=c_{w}/u_{r}=0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1) \]

On the delivery side of the impeller, as the effect of the casing wall is neglected in this domain, the following fundamental equations are derived if the radial velocity component is neglected,

\[ \frac{p_{e}}{7} + \frac{v_{a}^{2}}{2g} + \frac{c_{a}^{2}}{2g} = E_{a} + H_{a} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2) \]

\[ H_{a}=\omega r\cdot c_{w}/g \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3) \]

\[ dp_{a}/dr=\rho (c_{a}^{2}/r) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4) \]

by introducing the coefficient of slip angle \(1+p_{s}\), the condition that fluid elements flow out along impeller blades is expressed by

\[ c_{w}^{2} = \frac{1}{1+p_{s}^{2}} \left\{ \frac{v_{a}}{v_{d}} \left( \frac{\omega}{\omega} \left( 1+p_{s} \right) g H_{a} \right) \right\} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (5) \]

where, \(p_{s}\) : pressure, \(v_{a}\), \(c_{w}\) : axial and peripheral velocity components, \(E_{a}\) : total head on the suction side of the impeller, \(H_{a}\) : theoretical head, \(\omega\) : angular velocity of the impeller, subscript \(d\) refers to the outlet condition, subscript \(0\) refers to the design condition.

The differential equation for \(v_{a}\) derived from Eqs. (2), (3) and (5) becomes non-linear type even provided that \(1+p_{s}=\text{const.}\). It, therefore, seems to be difficult to solve the equation analytically.
Otherwise, as we are obliged to analyse the flow in impeller blades approximately, the approximate solutions of above equations are sufficient. The numerical example carried out by others\(^5\), which corresponds to the case simplified by putting \( p_2 = 0 \), shows that in the positive flow domain \( H_{ib} \) is nearly constant radially. \( H_{ib} = H_{ib0} = \text{const.} \), thereafter, can be assumed as a first approximation, then \( p_2 \) is derived from Eqs. (3) and (4), and \( v_d \) is determined from the above equations. Non-dimensional axial velocity components represented by \( \varphi_d = v_d/n_4 \) on the exit side of the impeller is expressed by

\[
\varphi_d = \left( -B_l + \sqrt{B_l^2 - 4A_lC_l} \right) / 2A_l \tag{6}
\]

\[
A_l = \frac{1}{2} \left[ 1 + \left( \frac{R_l - \psi_{ib0}}{R_l} \right)^2 \right] \left( \frac{1 + p_1}{1 + p_1} \right)^2 \tag{7}
\]

\[
B_l = \frac{p_1}{\varphi_{ib0}} \left( \frac{R_l}{1 + p_1} \right)^2 \tag{7}
\]

Where, \( R_l = r/r_i \) (\( r_i \) : radius of impeller tips), \( \psi_{ib0} = H_{ib0}/(u_i^2/g) \), \( \varphi_{ib0} = v_{ib0}/u_i \), \( R_i = (1 + R_i)/2 \), \( R_{ib} = r/r_i \) (\( r_i \) : radius of hub), \( \psi_{ib0} = H_{ib0}/(u_i^2/g) \), \( (1 + p_1) r_i' \) : value of \( 1 + p_1 \) at radius \( R_i' \). By substituting Eqs. (6) and (7) in Eq. (5), the whirl component \( \chi_d = c_{ad}/u_i \) will be obtained, the distribution of \( \varphi_{ib} = H_{ib}/(u_i^2/g) \), thereafter, is determined by Eq. (3). In these calculations, since differentiations and integrations are not applied at all, these solutions are justifiable, although \( 1 + p_1 \) is a function of radius. By repeating such a calculation, it is possible to elevate its accuracy, but the numerical example shows that this 2nd approximate solution is accurate enough. As the radius \( r_i \) at which the backward flow occurs is given by putting \( C_i = 0 \), the dimensionless radius \( R_{ib} = (r_i/r_i) \) is given by the relation

\[
R_{ib} = \frac{1}{2 + p_2} \left[ 2\varphi_{ib0} - \left( \frac{R_l^2 - \psi_{ib0}(1 + p_1) R_i'}{R_l^2 - \psi_{ib0}(1 + p_1) R_i} \right)^2 \right]^{1/2} \tag{8}
\]

It is shown easily that we can not apply fundamental equations (2) ~ (5) to find the flow in the reverse flow domain at the outlet from impeller blades. By eliminating \( p_2 \) from Eqs. (2) ~ (4), the following relation is derived,

\[
\frac{1}{2} \frac{dv_d^2}{dr} = (\omega r - c_{ad}) \left( \frac{dc_{ad}}{dr} + c_{ad} \frac{dr}{r} \right) \tag{9}
\]

The variations of \( v_d \) and \( v_d^2 \) near radius \( r_i \) become empirically as shown in Fig. 6, therefore, \( v_d^2 \) on the left side of Eq. (9) should take a singular point at \( r_i \). But \( c_{ad} \) being a continuous function even at \( r_i \), the right side of Eq. (9) does not become discontinuous at \( r_i \). Accordingly, there is no real function which satisfies Eq. (9) at radius \( r_i \).

The existence of a reverse flow domain on the delivery side means that the fluid flowing out from the impeller blades goes into the impeller, once more. The backward flow, consequently, is affected by the conditions at the outlet from impeller blades. Flow line being given as shown in Fig. 7, the energy which has been carried by a fluid particle at \( r_i \) must be equal to that at \( r_a \), because energy losses are neglected. If a very little variation of \( H_{ib} \) in the radial direction is neglected conveniently in the reverse flow domain, the following relation can be obtained instead of the balancing equation of pressure

\[
H_{ib} = (H_{ib})_R = \text{const.} \tag{10}
\]

As energy losses are neglected, the angle between the relative velocity and the rotating direction coincides with the discharge angle of blades in the backward flow domain. If, moreover, it is assumed that \( 1 + p_1 \) takes the same value as that in the case of the positive flow, flows in the
reverse flow domain can be expressed as follows

\[ \psi_d = \psi_{d0} \frac{R_t^2 - R_{re}^2}{R_t^3 - R_{re}^3} (1 + p_{1}) \]  \tag{11} 

\[ x_t = \frac{1}{1 + (p_{1})R_{re}} \frac{R_t}{R_t^2} \]  \tag{12} 

\[ \psi_{d0} = \psi_{d0}^0 (1 + p_{1})R_{re} \]  \tag{13} 

from Eqs. (3), (5) and (10). Where, \((1 + p_{1})R_{re}\) indicates the value of \(1 + p_{1}\) at \(R_{re}\).

3.2 Radial flow at radius \(r_1\)

In order to know the flow in the jet flow domain, it is necessary to discover the radial flow at radius \(r_1\). The flow in impeller blades must be assumed to be an axial-symmetrical flow in order that it may be analysed theoretically.

On a flow line which gets into impeller blades in the fundamental flow domain and reaches the radius \(r_1\) in blades, Bernoulli’s equation is given as follows

\[ \frac{v_{1}^2}{2g} + \frac{c_{1}^2}{2g} + \frac{p_{1}}{\gamma} = E_{t} + \frac{1}{g} \frac{v_{10}}{v_{10}} \left( \frac{v_{10} - x}{v_{1} - x} (1 + p_{1})R_{1}R_{ho} \right) \]  \tag{14} 

where, subscript \(1\) indicates values at \(r_{1}\) (see Fig. 3). If it is conveniently assumed that theoretical heads increase linearly in the axial direction at the design flow rate, the boundary condition by the vane surface is expressed by

\[ c_{a1} = \frac{1}{1 + (p_{1})} \left( \frac{v_{1} - x}{v_{10} - x} \frac{g (1 + p_{1})R_{1}R_{ho}}{\gamma} \right) \]  \tag{15} 

\[ y_{R1} = c_{1}/u_{1} = \frac{\psi_{R1} - \psi_{d0}}{R_{R1}^2 - (1 + p_{1})R_{1}R_{ho}} \]  \tag{16} 

where, \(\psi_{R1} = E_{R1}/(\gamma u_{1}^2)\); \(\psi_{d0} = (\psi_{d0}^0)/(\gamma u_{1}^2)\); \(\psi_{d0}\) indicates the non-dimensional pressures at the inlet and outlet from the impeller; \(Z = x/r_{1}\); subscript \(R_{1}\) indicates a component at \(R_{1}\) (=\(r_{1}\)/\(r_{2}\)); Moreover, \(\psi_{d-R1}\) refers to the non-dimensional axial velocity on the delivery side of impeller blades at \(R_{1}\). According to Pfieiderer's, \(1 + p_{1}'\) at \(r_{1}\) is shown by

\[ 1 + p_{1} = 1 + (p_{1}/Z) \]  \tag{17} 

\(\xi\) will be determined for various types of impellers empirically in part 4, and it will be shown that \(\xi\) is nearly equal to one for the usual type of impeller in the centrifugal flow domain. But in the axial flow domain, the effect of the stagnation pressure must be superposed on the radial flow at \(r_{1}\), thereafter, \(\xi\) and \(y_{R1}\) become very small.

Accordingly, a dimensionless capacity \(Q_{R1} (= q_{1}/u_{1}r_{2}^3)\), \(r_{2}\) : radius of casing wall) entering from the fundamental flow domain to the jet flow domain is given by

\[ Q_{R1} = 2\pi R_{2}L \int_{0}^{y_{R1}} \frac{r_{2}^3}{dZ} \]  \tag{18} 

where, \(L = h_{1}/r_{2}\); \(\lambda\) : reduction ratio of the effective annular area by existence of blade sections on the cylindrical surface of radius \(r_{1}\), in the impeller. If non-dimensional capacities which have flowed into and out of the impeller in the fundamental flow domain are shown by \(Q_{s} (= q_{s}/u_{1}r_{2}^3)\) and \(Q_{d} (= q_{d}/u_{1}r_{2}^3)\), the following relation must hold as the condition of continuity.

\[ Q_{s} = Q_{d} + Q_{R1} \]  \tag{19} 

* It will be shown experimentally in part 4 that such an assumption is reasonable for the impeller designed by a usual method.

** It will be discussed in part 4.
3-3 Flow in the jet flow domain I (case of the axial flow domain)

Fluid elements, which have entered from the fundamental flow domain to the jet flow domain in the impeller, turn their direction by the casing wall and flow out. The following equation represents the condition of continuity in the jet flow domain (see Fig. 8).

\[
dq_{s}/dr = 2\pi(r(v_s - v_d)) \tag{20}
\]

and its boundary conditions are given by

\[
q_s = 0 \text{ at } r = r_2 \\
q_s = q_{r1}, \quad v_s = v_{dr1} \text{ at } r = r_1
\tag{21}
\]

If we suppose a quadratic relation of \( r \) for \( q_s \) to satisfy Eq. (21), \( v_s \) is derived as follows

\[
\varphi_s = \varphi_s - \frac{A}{\pi(r/2)} \tag{22}
\]

\[
A = \frac{2}{1 - R_1} \left\{ \frac{R_1(\varphi_s R_1 - \varphi_s)}{1 - R_1} \cdot \frac{Q_{r1}}{2\pi} \right\}
\]

\[
B = \frac{2R_2 \cdot Q_{r1}}{\pi(1 - R_1)^2} \cdot \frac{1 + R_1}{1 - R_1} \cdot 2R_1 \cdot (\varphi_s R_1 - \varphi_s) \tag{23}
\]

where,

\[
R = r/r_2, \quad R_1 = r_1/r_2
\]

Since \( \varphi_s \) should coincide with that in the fundamental flow domain, \( \varphi_s \) can be determined by the above equation.

Owing to the cause described in the preceding chapter, the existence of blades as a boundary condition may be neglected. Consequently, whirl velocity components are determined by two equations; one is Bernoulli's equation applied to a stream line which enters into the impeller in the fundamental flow domain and gets out from it in the jet flow domain, another is the balancing equation of pressure at the outlet from the impeller.

The equation to derive the \( c_s \)-distribution becomes a non-linear differential equation, which seems difficult to solve; therefore, it is desirable to analyse by approximate methods. If the following \( c_s \)-distribution is assumed as 1st approximation

\[
\chi_s = k(R_s - R_1) + \chi_s R_1 = kR_s + \alpha \tag{24}
\]

then, the following pressure distribution is derived.

\[
\psi_{s2} = \psi_{s1} R_1 + \psi^2 (R_s^2 - R_1^2)/2 + 2\alpha k(R_s - R_1)
+ \alpha^2 \log R/R_1 \tag{25}
\]

where, constants \( k \) and \( \alpha \) are given by

\[
k = -kR_1 + \chi_s R_1
\]

\[
\alpha = -kR_1 + \chi_s R_1
\]

\[
\psi^2 (R_s^2 - R_1^2)/2 + 2\alpha k(R_s - R_1)\]

\[
+ \alpha^2 \log R/R_1 \tag{26}
\]

3-4 Flow in the jet flow domain II (case of the centrifugal flow domain)

In the centrifugal flow domain, fluid particles which have got into the jet flow domain, separate into two different parts and flow out to the suction and delivery sides of blades. In this case, the condition of continuity is written as follows

\[
dq_{s}/dr = -2\pi(r(v_s + v_{ds})) \tag{28}
\]

where, \( v_{ds} \) ; axial velocity component of the backward flow on the suction side. As the boundary conditions to Eq. (28) are shown by similar relations as Eq. (21), the following relations are obtained.

\[
\varphi_s + \psi_{s2} = -(A'' + (B''/(2R_i)) \tag{29}
\]

\[
A'' = 2 \cdot \frac{R_1 \cdot Q_{r1}}{R_1(1 - R_1)^2} - \frac{1}{1 - R_1} \tag{30}
\]

\[
B'' = \frac{2R_1 \cdot Q_{r1}}{\pi(1 - R_1)^2} \cdot \frac{1 + R_1}{1 - R_1} \cdot 2R_1 \cdot \chi_s R_1 \tag{25}
\]

where, \( \psi_{s2} = v_{ds}/R_i \). In the centrifugal flow domain, it being necessary to decide \( \varphi_s \) and \( \psi_{s2} \), respectively, there must be another equation. But, as theoretical determination of it seems to be difficult, it must be determined empirically. If there were no axial velocity component of liquid at radius \( r_1 \), a fluid particle would flow out to the suction and delivery sides with nearly the same velocity, respectively, because the effect of the blade surface may be
neglected in this domain. But, in fact, there being axial velocity components, the value obtained by subtracting \( u_{z1} \) from \( u_{2} \) seems to be nearly equal to \( u_{x2} \). Then, we can put
\[
\varphi_{21}(\varphi_{x2} - \varphi_{x1R1}) = \kappa
\]
where, \( \kappa \) = 1 approximately, but it must be decided experimentally for the sake of accuracy*. From Eqs. (29)～(31), the axial velocity components are obtained by
\[
\varphi_{x2} = \frac{1}{1 + \kappa} \left\{ -\left( A'' + \frac{B''}{2R} \right) + \kappa \varphi_{x1R1} \right\} 
\]
(32)
\[
\varphi_{x1} = \frac{1}{1 + \kappa} \left\{ -\left( A'' + \frac{B''}{2R} \right) - \varphi_{x2R1} \right\}
\]
(33)

For deciding a flow at the outlet from blades, the relations Eqs. (25) (26) (27) and (3) can be utilized, as they are, if Eq. (32) is used instead of Eq. (22), in the centrifugal flow domain. Since energy losses of the backward flow on the suction side are also neglected, an energy equation can be derived from the flow line which enters into the impeller in the fundamental flow domain and gets out again to the suction side in the jet flow domain. Consequently, a similar calculation to that on the delivery side may be performed and the flow condition on the suction side is decided by Eqs. (33), (25), (27) and (3). In these equations, components on the suction side must be used instead of those on the delivery side, further \( \chi_{xR1} = 0 \).

3-5 Determination of the theoretical head

The analytical method to calculate the flow on the suction and delivery sides of impeller blades is derived as above by taking into account the flow in the impeller. Theoretical head will be determined as follows.

If \( \varphi_{x1} \) is assumed adequately, the velocity distribution on the exit side of the impeller in the fundamental flow domain can be calculated by using Eqs. (5)～(7), (8), (11) and (12), and the head distribution is also calculated by Eqs. (3) and (13). The static pressure distribution, moreover, is derived as the difference of the total head and velocity. \( R_{1} \) will be decided by the method described in 2-1, if an adequate capacity is anticipated. As the radial flow at radius \( R_{1} \) is obtained by Eq. (16), the capacity entering into the jet flow domain is obtained by a graphical integration of Eq. (18). The flow state on the delivery side in the jet flow domain can be determined by applying the methods described in 3-3 or 3-4. Since, by the means expressed above, the flows on the delivery side in the fundamental and jet flow domains are decided, the non-dimensional capacity \( Q = \frac{j_{1}u_{1}r_{1}^{2}}{g} \) and theoretical head \( \psi_{1k1} = H_{1k1}/(u_{1}^{2}/g) \) will be expressed as follows
\[
Q = \int_{J_{R1}}^{R_{1}} 2\pi R \cdot \varphi_{x} \cdot dR 
\]
(34)
\[
\psi_{1k1} = \frac{1}{Q} \int_{J_{R1}}^{R_{1}} 2\pi R \cdot \varphi_{x} \cdot \chi_{x} \cdot dR 
\]
(35)

If there is no backward flow on the delivery side, \( R_{1} \) corresponds to \( R_{s} = (r_{1}/r_{2}) \), and if there are backward flows, it will be given by
\[
\int_{J_{R1}}^{R_{1}} 2\pi R \cdot \varphi_{x} \cdot dR = 0
\]
(36)

These integrations can be performed comparatively easily by a graphical method. If the discharge calculated by Eq. (34) is far different from that assumed beforehand, the similar calculations must be repeated by using a newly decided \( Q \). But a little difference of it may be neglected, because the variation of the theoretical head by the magnitude of \( r_{1} \) is comparatively small**.

4. Conclusions

In the present paper, the following points are mainly revealed by investigating on the flow in impeller blades on the basis of the experimental results obtained in the previous report.

1. It is explained naturally that the angle between the relative velocity and the rotational direction at large radii on the delivery side of impeller blades may become larger than the exit angle of impeller blades, at low flow rates. This phenomenon seems to give the main reason for the conspicuous increase of head toward zero capacity.

2. It is shown why large quantities of water flow out backward abruptly at the critical capacity.

3. The analytical method to find the relation between the discharge and the theoretical head is obtained for all flow rates from the design point to the shut-off point.

In order to demonstrate the adequacy of assumptions which are the basis of theoretical studies and to determine some coefficients introduced, moreover, to compare the theoretical results with the experimental ones, many measurements must be performed about impellers of different types. They will be reported in the following papers.

References

(1) T. Toyokura : This Bulletin, p. 287.

** Since the effect of centrifugal force increases with \( r_{1} \), but the magnitude of \( u_{1}r_{1} \) increases simultaneously, a variation of \( c_{1}r_{1} \) is comparatively little.