the frequency were recognized, though their quantitative comparison was difficult on account of the existence of the Coulomb friction.

(5) A distortion of the acceleration figure of the main mass was clearly seen on the oscillogram taken by means of an unbonded type (accelerometer) of strain gauge and a certain kind of transistor D.C. amplifier, of which the shape is regarded to have a tendency affirming the validity of the theory.

Acknowledgement

The authors wish to thank Mr. Syōsaku Kojima and Mr. Takashi Kaneko for helping in the calculations, experiments and other works. Thanks are also due to Mr. Koko Ichimura and Mr. Shunei Kubo, of Tokiko & Co. Ltd. for furnishing several helpful literatures and data.

References

(2) H.J. Hoffmann: ATZ, Jg. 60, Hr. 10 (1958), S. 289.
(3) K. Lampe: VDI-Z, Bd. 95, Nr. 34 (1953), S. 1151.

621–52:534.014.5

A Proposal of a New Method of Phase Analysis of
On-off Control Systems
with Relation to Sinusoidal Input

By Sitiro MINAGAWA

A phase plane method of analysis of the on-off control system with relation to sinusoidal input is proposed. The behaviour of a representative point in the phase plane is broken down into two separate movements, one of which is its rotation around the circumference of the ellipse, and the other is the travelling of the ellipse along a curve. This method is very useful especially in the analysis of on-off control system by reason of the fact that such an ellipse can be treated similarly to the representative point in the phase analysis of the system with relation to the step input. Several numerical examples are given to make this clear.

Introduction

A great deal of work has been done on the analysis and synthesis of on-off control systems with relation to the step input by means of the method of phase plane. However, for the complexities in the process of estimation, the usefulness of this method being one in the analysis of the system with relation to the sinusoidal input, the method of approximate frequency transfer function is often used. On the other hand, in such case as the estimation of the subharmonic oscillation produced in the output of an optimum on-off control system by a sinusoidal input, the approximate method may miss the sight of the essential part of the problem, because the switching line is given as a curve in the phase plane.

The aim of this paper is to propose a new method of systematic approach to the performance of the second order on-off control system with relation to the given sinusoidal input by means of the phase plane. One of such proposals has been offered by R. N. Buland; he named his method the “delta method”.

In our method the movement of the representative point is divided into two parts, one of which is its rotation around the circumference of an ellipse and the other the movement of the ellipse. The ellipse may be said to be the replacement of the representative point in the ordinary phase analysis of the system with relation to the step input. Since, in the analysis of on-off control systems, the movement of the ellipse is independent of the rotation of the
representative point around its circumference excepting the instant when the switching is performed, such a replacement will become extremely meaningful.

In this method we shall use the virtual switching line which has been introduced by the author(4) in the estimation of the cycling performance of the second order system, and it will also be defined in §4 of this paper.

The ellipse being given by the input sinusoidal signal and the position of its center at the instant when the switching is performed being given on the virtual switching line, the trajectories of the steady state oscillation produced in the output of the system are given, as will be stated in the last section.

1. Basic system

The systems to be treated in this paper are such as shown in Fig.1, where the controlled element is of second order and the output of the on-off controller is taken to be +1 or -1 according to the sign of the function \( f(\varepsilon, \dot{\varepsilon}) \), which is calculated by the computing device. We may also consider the case where the dead-zone, backlash or dead-time is accompanied by the operation of the on-off controller.

One example of such systems is a remote position control system by means of a direct current motor with a tachometer feedback circuit. The transfer function between the motor shaft and the applied voltage is\(^{(5)}\)

\[
\frac{1}{Ks(Ts+1)}, \tag{1}
\]

where

\[
T = \frac{JR}{K_t K},
\]

\( K_t \): torque constant of the motor

\( K \): back emf coefficient of the motor

\( J \): inertia of the motor shaft connecting the load

\( R \): resistance of the electric circuit

Using the absolute value of the applied voltage to the motor as the unit of the manipulated variable, and \( T \) and \( T/K \) as the units of the time and output, Eq. (1) is replaced by

\[
\frac{1}{s(s+1)}, \tag{2}
\]

The control function \( f(\varepsilon, \dot{\varepsilon}) \) is given by the design principle of the tachometer feedback circuit. For example, in the so-called optimum on-off control of the controlled element indicated by the transfer function (2) it takes the form

\[
f(\varepsilon, \dot{\varepsilon}) = \log (1+|\dot{\varepsilon}|) - \text{sign } \dot{\varepsilon}(\varepsilon + \dot{\varepsilon}).
\]

2. Method of phase-plane

The control signal is given by

\[
\varepsilon(t) = x_1(t) - x_0(t), \tag{3}
\]

where \( x_1(t) \) and \( x_0(t) \) are the input and the output respectively. So that, the behaviour of the representative point in the phase-plane, in which \( \varepsilon \) and \( \dot{\varepsilon} \) (= \( d\varepsilon/dt \)) are used as coordinates, is broken down into two movements, one of which is given by \( x_1(t) \) and the other by \( x_0(t) \). The input being given by

\[
x_1(t) = a \sin(\omega t + \varphi), \tag{4}
\]

the former is a rotation around the circumference of an ellipse whose radii are \( a \) and \( a\omega \) along \( \varepsilon \)-axis and \( \dot{\varepsilon} \)-axis respectively. On the other hand, the line described by \( -x_0(t) \) is equal to the trajectory of this system with relation to the step input. Thus, the movement of the representative point on the phase-plane is divided into two parts, one of which is a rotation around the circumference of an ellipse and the other is the travelling of its center along a trajectory of this system with relation to the step input.

Such an ellipse is extremely useful in the analysis of the on-off control system to the sinusoidal input, as well as the representative point with relation to the step input, because the movement of the ellipse is independent of the rotation of the representative point around the circumference of the ellipse excepting the instance when the switching is performed. In Fig. 2, these two movements are indicated by the arrows. For brevity, the trajectory of the center of the ellipse is called the "\( \varepsilon \)-trajectory".

3. Steady oscillation in on-off control system

We shall discuss the harmonic or subharmonic

![Fig. 1 On-off control system](image)

![Fig. 2 Trajectories and \( \varepsilon \)-trajectories](image)
oscillations which are generated in the output by
the input sinusoidal signal and the generating
conditions thereof by means of the method above
stated.

Let us beforehand note that the $\varepsilon$-trajectory of the
steady oscillation is given by a closed curve, and let
us restrict ourselves to such cases as when this
closed curve is further single. If the center of the
ellipse returns to the initial position following this
single closed curve while a representative point
takes $n$ turns around the circumference of this
ellipse, the fundamental frequency of the output
is $1/n$ of that of the input. In the case where $n \geq 2$,
a subharmonic oscillation of order $n$ is produced, on
the other hand, in the case where $n = 1$, a harmonic
oscillation.

Furthermore, the time for taking the representa-
tive point one turn around the circumference of the
ellipse equals to the period of the input, thus we
obtain the following:

1) The $\varepsilon$-trajectory is a closed curve.

2) The time for taking the center of the
ellipse one turn around the closed curve is an
integral multiple of the period of the input.

Since the switching is performed only at the
instant when the representative point is on the
switching line, we further get:

3) A representative point is on one side of
the switching line between any switching instant and
the succeeding.

We adopt a stable oscillation among those
satisfying the above three conditions.

4. Virtual switching line

The first condition may be replaced by the
concept of the virtual switching line which we have
introduced for the practical purpose to calculate the
cycling performance of the on-off control system to
the step input.10

This was defined there as follows;

If this line is set in the system as its switching
line, the system becomes conservative one, i.e. the
phase-plane trajectory of this system to any step
input is a single closed curve.

It should be noted that the virtual switching
line is decided only by the controlled element in
such cases as stated in Fig 1. For example, let the
transfer function of the controlled element be $1/s^2$, then the $\varepsilon$-trajectories are the parabolas which are
symmetric with respect to $\varepsilon$-axis to themselves in
each side of the switching line, and further these
two groups of parabolas are symmetric to each
other with respect to $\varepsilon$-axis. Therefore, if we adopt
$\varepsilon$-axis as the switching line, the trajectory of this
system with relation to any step input becomes a
single closed curve, such as shown in Fig. 3 (a).
Thus, $\varepsilon$-axis is the virtual switching line for such
a system.

If the transfer function of the controlled element
is $1/(s+1)$, then the trajectories of the center of
ellipse are given by

\[
\log \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right) + \varepsilon = \text{const.}
\]

or

\[
\log \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right) + \varepsilon = \text{const.}
\]

in each side of the switching line. Therefore, if we
adopt a line

\[
\log \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right) = -2(\varepsilon + \varepsilon)
\]

as the switching line, then the trajectories of this
system become single closed curves. That is, Eq.
(6) is of the virtual switching line for such a
system (see, Fig. 3 (b)).

The first condition stated in the preceding section
is replaced by that the switching is performed at
the instant when the center of the ellipse is on the
virtual switching line.

Furthermore, the time for taking the center of
the ellipse one turn along a closed trajectory is
given by $\int \frac{d\varepsilon}{\varepsilon}$, where the integration is taken
along this closed trajectory. Meanwhile, this time
is equal to an integral multiple of the period of the
input by virtue of the condition (2). Therefore,
the position of the center of the ellipse at the instant when the switching is performed, or a closed curve along which the center of the ellipse travels, is determined for any order of oscillation.

The ellipse is decided by the input, and the position of the center of the ellipse at the instant when the switching is performed is given by the input period and the order of the output oscillation. Therefore, the intersections of the ellipse with the given switching line at the instant switching is performed and they are the positions of the representative point at that instant (see Fig. 4).

When the ellipse does not intersect with the given switching line, there does not exist any steady state oscillation of that order with relation to the given sinusoidal input.

In the next section, we shall state some numerical examples of the steady oscillation produced in the system such as in Fig. 1

\[ G(\omega) = \frac{1}{s^2} \quad \text{or} \quad G(\omega) = \frac{1}{s(s+1)}. \]

5. Numerical examples

5.1 When \( G(\omega) = 1/s^2 \)

The virtual switching line for such cases is \( \dot{e} \)-axis as has been stated in the preceding section. Let the center of the ellipse be at \((0, E)\) at a switching instant, then the time required for it to travel from \((0, E)\) to \((0, -E)\) along one of the \( e \)-trajectories is given by

\[ \int_{E}^{E'} \frac{dE}{\dot{E}} = - \int_{-E}^{E} d\dot{e} = 2E. \]

Therefore, by virtue of the condition \((2)\), in order to have a subharmonic oscillation of \((2n+1)\)-th order in the output, we obtain

\[ E = \frac{2n+1}{2} \cdot \pi \cdot \frac{\omega}{\omega_0}. \]

Thus, the position of the center of the ellipse at the instant when the switching is performed is

\[ \left(0, -\frac{2n+1}{2} \cdot \pi \cdot \frac{\omega}{\omega_0} \right) \quad \text{or} \quad \left(0, -\frac{2n+1}{2} \cdot \pi \cdot \frac{\omega}{\omega_0} \right) \]

in order to have the subharmonic oscillation of \((2n+1)\)-th order in the output.

We shall contend now that one of the two intersections of the ellipse with the given switching line at the instant when the switching is performed is impossible to achieve by the instability by means

\* We should note here that, if the switching line is symmetric to itself with respect to the origin of phase plane, the subharmonic oscillations of the even order cannot be achieved. Because the representative point takes integral turns around the circumference of the ellipse in the even order subharmonic oscillation until the center of the ellipse arrives at the virtual switching line, and therefore, it is not in a symmetric position of that at the preceding instant switching was performed with respect to the origin of phase plane.

of Fig. 5. Let the center of the ellipse turn around a closed curve \( O_1Q_0Q_1 \) in a steady state, in which the switching is performed at the instant when the center of the ellipse is in \( O_1 \) or \( O_2 \) and the representative point in \( A_1 \) (or \( B_1 \)) or \( A_2 \) (or \( B_2 \)). Provided that the center of the ellipse starts from the position \( O_1' \), and the representative point from \( A_1' \) or \( B_1' \) then a greater length of time is required until the center of the ellipse arrives at \( \dot{e} \)-axis in \( O_2' \) than that required for it to travel from \( O_1 \) to \( O_2 \). Therefore, when a representative point starting from \( A_1' \) arrives at the switching line in \( A_4' \), the center of the ellipse is in the position \( O_2'' \) on the right side of \( \dot{e} \)-axis. On the other hand, when the representative point starting from \( B_1' \) arrives at the switching line in \( B_4' \), the center of the ellipse is in the position \( O_4''' \) on the left of \( \dot{e} \)-axis. The center of the ellipse approaches nearer to the closed curve \( O_1Q_0Q_1 \) if it starts from \( O_2''' \), and withdraws from that if it starts from \( O_4''' \), as is shown in Fig. 5. This indicates that the intersection \( B_1 \) is not impossible to achieve by the instability.
5-1.1 When the switching line is a straight line.

Let the switching line be \( \phi + k \dot{\phi} = 0 \), \( (\dot{\varepsilon}, \ddot{\varepsilon}) \) the position of the representative point, and \( \varphi \) the phase lag between the input and the output at the instant when the switching is performed, then it follows that

\[
\begin{align*}
\varepsilon_t &= a \sin \varphi \\
\dot{\varepsilon}_t &= \frac{2n+1}{2} \pi + a \omega \cos \varphi \\
\varepsilon_t + k \dot{\varepsilon}_t &= 0,
\end{align*}
\]

(7)

considering the relation between the ellipse and the switching line.

As a consequence of Eq. (7), we obtain

\[
a \omega^2 \sin (\varphi + \tan^{-1}(k \omega)) = -\frac{2n+1}{2} \pi \sqrt{1+(k \omega)^2}.
\]

(8)

The relations between \( a \omega^2 \) and \( \varphi \) for several values of \( k \omega \) are indicated in Fig. 6, in which the values on the dotted lines cannot be achieved because of instability.

This figure indicates that there exists a minimum value in the amplitude of the input in order to have steady oscillation for the given values of \( k, \omega \) and \( n \).

If the amplitude of the input is smaller than this minimum value, a steady oscillation does not occur for such values of \( k, \omega \) and \( n \). Especially, when \( n = 0 \), any steady oscillation does not occur in the output. In these cases, there exists an instant such that the representative point goes back to the same side of the switching line after the switching is performed. This is contradictory to the switching condition, and therefore, the representative point must be stopped there. This point is called the "end point" (9).

In the practical case, however, the switching is performed at an instant a little later than the representative point across the switching line under the influence of the dead time, neutral zone or backlash, i.e., in the other side of the switching line, and then the representative point can return to the preceding side of the switching line and further go on. Thus, we obtain the output such as shown in Fig. 6. Such phenomena have been treated by Ya. B. Dorgorenko.

5-1.2 When the switching line is the optimum one of the system In this case, the switching line is as follows (see, for example, (2)):

\[
\varepsilon + \frac{1}{2} |\dot{\varepsilon}| \dot{\varepsilon} = 0,
\]

thus, we obtain

\[
a \omega^2 \cos \varphi = -\frac{2n+1}{2} \pi \cos \varphi
\]

\[
\pm \sqrt{\sin^2 \varphi + (2n+1) \pi \sin \varphi \cos \varphi}
\]

(9)

through a similar line of discussion as above.

The relation between \( a \omega^2 \) and \( \varphi \) is indicated in Fig. 8 for \( n = 0 \). When the amplitude and the angular frequency of the input are such that \( a \omega^2 \) is smaller than the minimum value of this figure, it can be followed by the system such as shown in Fig. 7.

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**Fig. 6** Relations between \( a \omega^2 \) and \( \varphi \) of 5-1.1

**Fig. 7** Output following the input

**Fig. 8** Relations between \( a \omega^2 \) and \( \varphi \) of 5-1.2
One example of the trajectories in the steady state are shown in Fig. 10, in which the thick full line indicates a realizable steady oscillation.

**5-2 When \( G_r = 1/s(s+1) \) and the switching line is the straight line**

In respect to the stability and that the oscillation of the even order does not occur, the above discussion should be referred.

Let the switching be performed on the straight line

\[ \varepsilon + k \dot{\varepsilon} = 0, \]

then the following relation is obtained in a similar manner as above.

\[ a \sqrt{1 + (k \omega)^2} \left( \sin \varphi + \tan^{-1}(k \omega) \right) \]
\[ = \frac{2n+1}{\omega} \left( 1 - k \right) \tanh \left( \frac{2n+1}{2 \omega} \right). \]

In Figs. 9 (a) \(-\) (d), the relations between \( \varphi \) and \( \omega \) are stated for several values of \( a \) and \( k \). In the lower regions of the thick broken lines of these figures, the phenomena stated in Fig. 7 are obtained.

**5-3 Transient response**

Let us state here some examples of the transient performance of the system such as treated in 5-1.2.

From Eq. (7), we obtain

\[ \left( \begin{array}{c} \frac{d \varepsilon}{d \varphi} \\ \frac{d \omega}{d \varphi} \end{array} \right)_{r, \omega} = \frac{1 + a \omega^2 \sin \varphi}{\dot{\varepsilon}}. \]

![Figure 9](image-url) Relations between \( \omega \) and \( \varphi \) of 5-2.
On the other hand, the gradient of the switching line is given by $-\frac{1}{\xi}$.

Accordingly, in order that the representative point continues its travelling after the switching is performed, the following condition has to be fulfilled between $\phi$ and $\xi$ at the instant when the representative point arrives at the switching line, (the condition that this point does not become the end point) i.e.,

0 < $\phi$ < 180 when $\xi > 0$

or

180 < $\phi$ < 360 when $\xi < 0$

Three examples such as stated in Fig. 11 (a), (b) and (c) may be obtained. In these figures, the thick lines indicate the trajectories of the representative point, while the thin lines indicate those of the center of the ellipse. In Fig. 11 (b) or (c), the starting point becomes the end point unlike in Fig. 11 (a). However, by the influence of the time lag, backlash etc. in the switching device, the representative point being in A, if it is disturbed, goes to B, and then to C. It travels from C in a similar manner as in Fig. 11 (a).

**Concluding remarks**

A new method of approach to the response of the on-off control system with relation to the sinusoidal input has been proposed. The ellipse representing the input sinusoidal signal plays an important part of the method, this is very similar to the part of the representative point in the ordinary phase analysis with relation to the step input. The other important concept is the virtual switching line, which has been introduced fictitiously for the sake of convenience in the estimation of the response of the second order on-off control system.

The conditions of the steady oscillation of the

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**Fig. 10** Trajectories of steady oscillation

**Fig. 11** Trajectories of transient responses
second order system are as follows;

1) The center of the ellipse travels along a closed curve.

2) While the representative point takes integral turns around the circumference of the ellipse, the center of the ellipse takes one turn along the closed curve above stated.

3) The center of the ellipse started from a neighborhood of the closed curve converges into it, that is, the cycling performance of the center of the ellipse is stable.

4) The representative point is in one side of the switching line between an instant when the switching is performed and the next.

The conditions 1), 2), and 3) have been estimated in Figs. 5, 8 and 9. Although the last condition has not been referred, since the condition that the system follows the input such as stated in Fig. 7 is obtained, our discussion is meaningful in the practical design of the on-off control system.

We have restricted ourselves to the analysis of the second order system in §5, however this method is also applicable to the first order system. In such cases the virtual switching line is needless to use in the analysis. Examples will be stated in the forthcoming paper.

References


Discussion

H. Akashi: It is interesting to find that some of the conditions for the occurrence of subharmonic oscillations are obtained by the author’s method of approach. Eq. (9) gives the lower limit of input amplitude, which is common to the harmonic and subharmonic response of the system. According to my results(a), there is an upper limit of input amplitude for each order subharmonic response. Has the author found similar results by his method?

K. Iizawa: This interesting method would also possibly be applicable to the on-off control systems with sampling action. It is interesting to note that it is unnecessary or of less value to trace the system response at every instant for a particular input, and it would be more desirable and sufficient to determine several key characteristics such as maximum and minimum values, period of oscillation, maximum rate of change in the system variables for a set of expectable input signals. The discusser feels that the author’s method would fit with this approach. The author is requested to clarify whether the conditions given in §5 are necessary and sufficient for a periodic motion.

Author’s closure

The author wishes his thanks to Dr. Akashi and to Dr. Iizawa for their very helpful suggestions and criticisms.

As Prof. Akashi points out, there exists an upper limit to the amplitude of an input sinusoidal

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Append-Fig. 1 Region of $a_0$ for the third order subharmonic oscillation in the system of 5:1:1

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Radial $\varphi$

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Upper limit

Existence region

Prohibited region
not of the author’s ellipse, must be calculated.

The author has obtained such upper limit by numerical calculations. One example is stated in append-Fig. 1, where the broken line indicates the upper limit to the value of $a_0^2$ for the occurrence of the third order subharmonic oscillation in the output of the system of 5.1.1.

In this respect, the author’s brief discussion in “Preprint for the 9-th General Meeting of the Tokai Branch of the Japan Soc. Mech. Engr., March, 1960” should be referred.

Dr. Izawa’s suggestion in regard to the application of the author’s method to the analysis and synthesis of the sampled data control system with relation to sinusoidal input and an idea of adjusting the system for a set of expectable inputs are very meaningful and instructive, and the author is now willing to seize an opportunity to study them.

The conditions treated in this paper are necessary for the periodic motion. In order to afford the sufficient condition, the movement of the representative point on the phase plane must be given in detail. This can not be attained by the author’s method. However, the sufficient condition is not always needed, for example, in designing an on-off control system so as to follow the given sinusoidal input. The author’s closure to Prof. Akashi’s discussion should also be referred.

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“Summed and Differential Harmonic” Oscillation in Non-Linear Vibratory Systems

(Systems with Unsymmetrical Nonlinearity)

By Toshio YAMAMOTO**

The author has studied “summed and differential harmonic” oscillations occurring in rotating shaft systems, i.e., vibratory systems with gyroscopic terms. In the present paper, the author treats “summed and differential harmonic” oscillations appearing in rectilinear vibratory systems, the nature of which is quite different from that of systems with gyroscopic terms. By analytical discussion, it is concluded that summed types only can take place in rectilinear vibratory systems while both summed and differential type can occur in rotating shaft systems.

Further, experimental results for two experimental apparatus having different construction from each other verify that “summed and differential harmonic” oscillations actually appear in rectilinear vibratory systems, and summed type only can take place.

1. Introduction

The motions appearing in the systems having gyroscopic terms are different from those of the other ordinary vibratory systems without gyroscopic terms. For instance, the motions of the former systems are not rectilinear vibrations, but are always whirling motions. The author has studied experimentally and theoretically “summed and differential harmonic” oscillations occurring in rotating shaft systems which are systems with gyroscopic terms.(1)–(5).

In the present paper, the author treats “summed and differential harmonic” oscillations of rectilinear vibratory systems. When multiple-degree-of-freedom systems have non-linear spring characteristics, this nonlinearity results in occurrence of sub-harmonic and “summed and differential harmonic” oscillations, even if the systems are rectilinear vibratory systems. The author will discuss analytically the condition of the possibility of occurrence of “summed and differential harmonic” oscillation, and will determine the modes of vibrations which can take place, and equations of response curves. Further the author

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