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General Considerations on the Compression of a Wedge by a Rigid Flat Die*

(2nd Report, Solutions in Unsymmetrical Condition
and with Smooth Die)

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The load on and deformation of a wedge compressed by a smooth rigid flat die were investigated. Seven types of slip line field have been predicted for the wedge which has any inclination to the die face. These fields are classified into three modes of deformation where

- (a) the material flows separately on the contact face,
- (b) dead metal zone is in contact with the die and
- (c) the material flows only on one side of the wedge.

Four of the seven fields are complete solutions. The width of the contact face, load or its upper and lower limits and the mean strain of the wedge were compared with these modes of deformation, and it has been revealed that the mode (a) exists in a very narrow range of the inclination and thickness of the wedge and is hardly realized and that the load of the mode (b) is larger than that of mode (c) but as the mean strain is small, mode (b) will be realized rather than mode (c) when the inclination of wedge is small and the thickness of wedges moderately large and when the material is in the state of strain hardening.

1. Introduction

In this report, the compression of a wedge by a smooth die in unsymmetrical condition which is represented by three parameters θ_1 , θ_2 and α shown in Fig. 1 is investigated. Since no frictional force is assumed, the tangential component of the velocity of the flat die effects no change in the geometry of the slip line field and the generality holds even if α is assumed to be zero. Predicted slip line fields are shown in Fig. 2 where the foci are shown, too, regarding these as the unit diagram. These fields shall be represented by the same notation as that in the former papers⁽¹⁾⁽²⁾. Detailed investigation on these fields is as follows.

2. Slip line fields and calculation of load

2.1 Field a (Fig. 2 a)

Both angles ψ of the centered fans $A_1^1 D_1^1 E_1^1$ (double suffixes represent both of $A_1 D_1 E_1$ and $A_2 D_2 E_2$). The same rule applies correspondingly to the following parts) must be equal in order that the

yield criterion may not be violated in the rigid region $D_1 B D_2$. In order that the points C_1^1 may lie on the initial surface of the wedge, the relation

$$\tan \theta_1^1 = \frac{(h_1^1/c)(1 + \sin \psi) \mp \tan \delta}{1 + (h_1^1/c) \cos \psi} \quad \dots\dots\dots (1 \cdot a)$$

must be satisfied. (Double suffixes and signs should be taken in the same order.) As the foci H_1^1 should lie on $A_1^1 C_1^1$, we get

$$h_1^1/c = \sec \psi + \tan \psi \pm \tan \delta \quad \dots\dots\dots (2 \cdot a)$$

Therefore, Eq. (1·a) becomes

$$\tan \theta_1^1 = \frac{(\sec \psi + \tan \psi)^2 \pm \tan \psi \tan \delta}{2 \sec \psi + \tan \psi \pm \tan \delta} \quad \dots\dots\dots (3 \cdot a)$$

The field a or c shown in Fig. 1 of the former paper⁽¹⁾ is the special case $\theta_1 = \theta_2 = \theta$, and $\delta = 0$ or θ of this field. However, if $\theta_1 = \theta_2 = \theta$, δ is necessarily equal to zero from Eq.

(3·a) ($\delta = \theta$ is a special case where one of Eq. (3·a) vanishes). Therefore, an intermediate field between a and c cannot be a complete solution.

This field is a complete solution for $|\delta| \leq \pi/4$ and statically admissible for

Fig. 1 Unsymmetrical condition

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$-\theta_1 \leq \delta < \pi/4$ or $\pi/4 < \delta < \theta_2$. The load is given by

$$P/2k = 2(1+\psi)(\sec\psi + \tan\psi) \dots (4 \cdot a)$$
 for $c=1$, where $c=OB_1$ is the normal component of the displacement of die OG. When $-\theta_1 \leq \delta < \pi/4$ or

$\pi/4 < \delta \leq \theta_2$, Eq. (4·a) may be a lower bound by Drucker's extremum principle⁽³⁾, though this principle cannot be applied, strictly speaking.

Fig. 2 Slip line fields in unsymmetrical condition and no friction

2.2 Field b (Fig. 2 b')

Since no state of equilibrium exists in the region A_1DA_2 and the yield criterion is violated in the region E_1DE_2 , this field is kinematically admissible and it may be considered that an upper bound is given by this. Similarly to the above,

$$\tan \theta_2 = \frac{(h/c)(1 \mp \tan \delta + 2 \sin \psi_2) \mp \tan \delta}{1 + 2(h/c) \cos \psi_2} \dots (1.b)$$

and

$$\frac{h}{c} = \frac{1}{2 \cos \psi_2} + \frac{\tan \psi_2 \pm \tan \delta}{1 \mp \tan \delta} \dots (2.b)$$

are obtained. Therefore, δ is given by

$$\left. \begin{aligned} \tan \delta &= \{b - \sqrt{b^2 + 4a(a+c)}\} / (2a) \\ a &= (\sec \psi_1 - \sec \psi_2) / 2, \\ b &= 2 + \tan \psi_1 + \tan \psi_2, \\ c &= \tan \psi_1 - \tan \psi_2. \end{aligned} \right\} \dots (3.b)$$

The upper bound of the load for $c=1$

$$P/2k \leq (h/c) \{ (1 + \psi_1)(1 - \tan \delta) + (1 + \psi_2)(1 + \tan \delta) \} \dots (4.b)$$

is obtained from the theorem (2) in the former paper⁽¹⁾.

2.3 Field c (Fig. 2 c)

Similarly

$$\tan \theta_1 = \frac{(2h/c)(1 + \sin \psi) - \tan \theta_2}{1 + (2h/c) \cos \psi} \dots (1.c)$$

$$2h/c = \sec \psi + \tan \psi + \tan \theta_2 \dots (2.c)$$

and therefore,

$$\tan \theta_1 = \frac{(\sec \psi + \tan \psi)^2 + \tan \psi \tan \theta_2}{2 \sec \psi + \tan \psi + \tan \theta_2} \dots (3.c)$$

are obtained. The load is

$$P/2k = (1 + \psi)(\sec \psi + \tan \psi + \tan \theta_2) \dots (4.c)$$

for $c=1$.

2.4 Field d (Fig. 2 d)

From the condition on the points C_2^1 ,

$$h_1/c = (\tan \theta_2 \pm \tan \delta) / (1 - \tan \theta_2) \dots (2.d)$$

It is clear from this equation that θ_2 should be smaller than $\pi/4$. The load is

$$P/2k = h_1/c + h_2/c \dots (4.d)$$

for $c=1$. Although δ is arbitrary within the range $-\theta_1 \leq \delta \leq \theta_2$ (or its great portion), the field f which minimizes the load P may be realized, practically. When the field f cannot hold, δ is limited to the smaller value than θ_2 by the condition $F_1B \geq A_1B$, or $1 + \tan \delta \geq h_1/c$, that is,

$$\tan \delta \leq \cot \theta_1 - 2 \dots (5.d)$$

by Eq. (2.d). (It is considered above that δ nears θ_2 . If δ nears $-\theta_1$, θ_1 must be changed for θ_2 and the sign of δ must be inverted. But this should be considered only when $\theta_2 < \theta_1$.)

In symmetrical condition, all of the intermediate fields between the fields d and f are complete solutions and give an equal load $P/2k$, while these are complete solutions too, but their loads are different in unsymmetrical condition. Since the limit of the

field a is $\psi=0$ and then $P/2k=2$ from Eq. (4.a), the field d may be considered in the range $P/2k \leq 2$. Therefore, the boundary between a and d is given by

$$\cot \theta_1 = \cot \theta_2 + 4 \dots (6.d)$$

putting $P/2k=2$ in Eq. (4.d) and assuming the equality holds in (5.d). Field d may be realized within a narrow range between Eqs. (6.d) and (5.f) which gives the range of the field f.

2.5 Field e (Fig. 2 e* and e')

Fig. 2 e* shows a statically admissible field. Similarly to analysis in the field d,

$$\tan \theta_2 = \{(h/c)(1 \mp \tan \delta) \mp \tan \delta\} / \{1 + 2(h/c)\} \dots (1.e^*)$$

Solving this,

$$h/c = (\tan \theta_1 + \tan \theta_2) / \{2(1 - \tan \theta_1 - \tan \theta_2)\} \dots (2.e^*)$$

and

$$\tan \delta = (\tan \theta_2 - \tan \theta_1) / (2 - \tan \theta_1 - \tan \theta_2) \dots (3.e^*)$$

An upper bound may be given by

$$P/2k \geq (\tan \theta_1 + \tan \theta_2) / (1 - \tan \theta_1 - \tan \theta_2) \dots (4.e^*)$$

for $c=1$. F_2^1 must lie on the extension of $\overline{A_1A_2}$, not within $\overline{A_1A_2}$. Therefore, the condition of F_2 is $\angle OBJ_2 \geq \angle A_2C_2J_2$ or $c(c+2h)(\tan \theta_2 - \tan \delta) \geq 2h^2(1 + \tan \delta)$. This is reduced into $2h/c \leq (1 - \tan \delta) / (1 + \tan \delta)$ by Eq. (1.e*), or using Eqs. (2.e*) and (3.e*),

$$\left. \begin{aligned} \tan \theta_1 &\leq 1 - \tan \theta_2 - \sqrt{\tan \theta_2} \\ \tan \theta_2 &\leq 1 - \tan \theta_1 - \sqrt{\tan \theta_1} \end{aligned} \right\} \dots (5.e^*)$$

is obtained.

Among the eight parameters h , δ , ϕ_1^1 , φ_2^1 , ψ_1^1 contained in the kinematically admissible field shown in Fig. 2 e', four are decided by the four conditions on the points C_2^1 and H_2^1 . Here, the remaining four are assumed to be $\psi_2^1 = \pi/2$ and $\varphi_2^1 = \pi/4 \mp \delta$ for simplicity of calculation. Then,

$$\tan \theta_2 = \frac{(h/c)(\tan \phi_2^1 \mp \tan \delta) \mp \tan \delta}{(h/c)(1 + \tan \phi_2^1) + 1} \dots (1.e)$$

$$h/c = (1 \pm \tan \delta) \tan \phi_2^1 / \{1 \mp \tan \delta\} (1 + \tan \phi_2^1) \dots (2.e)$$

from which

$$\left. \begin{aligned} \tan \delta &= (\alpha_2 \beta_1 - \alpha_1 \beta_2)^2 / (\alpha_2^2 - \alpha_1^2) \\ \alpha_2^1 &= \sqrt{\tan \phi_2^1}, \beta_2^1 = \sqrt{1 + \tan \phi_2^1} \end{aligned} \right\} \dots (3.e)$$

are obtained. Eq. (2.e) reduces to

$$h/c = \alpha_1 \alpha_2 / (\beta_1 \beta_2) \dots (5.e)$$

An upper bound may be given by

$$P/2k \leq \alpha_1 \alpha_2 (\beta_1 \beta_2 - \alpha_1 \alpha_2) \dots (4.e)$$

for $c=1$.

2.6 Field be (Fig. 2 be')

In the above mentioned field e', ϕ_2 must be less than $\pi/2$. For the larger value of θ_2^1 , the field b gives a smaller upper bound, but both of ψ_1 and ψ_2 cannot be equal to zero at the same time for decreasing θ_2^1 in unsymmetrical condition. Therefore,

it is naturally considered that an intermediate kinematically admissible field be which is shown in Fig. 2 be' lies between the fields b and e, and that this field is more simple and gives a smaller upper bound of the load rather than the field e' where $\psi_2 = \pi/2$ or $\varphi_2 = \pi/4 \mp \delta$ are not assumed. Since the left and right sides of this field are similar to the fields e and b, respectively,

$$\left. \begin{aligned} \tan \theta_1 &= \{(h/c)(\tan \phi_1 - \tan \delta) - \tan \delta\} / \{(h/c)(1 + \tan \phi_1) + 1\} \\ \tan \theta_2 &= \{(h/c)(1 + \tan \delta + 2 \sin \psi_2) + \tan \delta\} / \{1 + 2(h/c) \cos \psi_2\} \end{aligned} \right\} \dots\dots (1 \cdot be)$$

and

$$\begin{aligned} h/c &= \{(1 + \tan \delta) \tan \phi_1\} / \{(1 - \tan \delta)(1 + \tan \phi_1)\} \\ &= \frac{1}{2} \sec \psi_2 + (\tan \psi_2 - \tan \delta) / (1 + \tan \delta) \end{aligned} \dots\dots\dots (2 \cdot be)$$

are readily obtained. An upper bound of the load may be given by

$$\begin{aligned} P/2k &\leq (h/c) \left\{ \frac{1}{2} (1 + \tan \phi_1)(1 - \tan \delta) \right. \\ &\quad \left. + (1 + \psi_2)(1 + \tan \delta) \right\} \dots\dots\dots (4 \cdot be) \end{aligned}$$

for $c=1$. If $\psi_1=0$ in field b, this agrees with field be', while if $\psi_2=0$ in field be', this agrees with field

e'. Therefore, $\phi_2 \leq \pi/4$ always.

2.7 Field f (Fig. 2 f)

This field is a special case, that is, the limiting state $\delta \rightarrow \theta_2$, $h_1 \rightarrow 2h$, $h_2 \rightarrow 0$, of the field d as already shown, therefore, we get from Eqs. (2·d) and (4·d)

$$2h/c = P/2k = (\tan \theta_1 + \tan \theta_2) / (1 - \tan \theta_1) \dots\dots\dots (2 \cdot f), (4 \cdot f)$$

If $\psi=0$, Eq. (4·c) becomes $P/2k = 1 + \tan \theta_2$. Then the field f should be considered to hold in the range $\cot \theta_1 \geq 2 + \tan \theta_2$ (5·f) obtained from Eq. (4·f); θ_1 should be less than $\pi/4$ from Eq. (2·f).

3. The mean strain

The work hardening may be estimated by the mean strain calculated from the work of deformation per unit deformed volume as shown in the former report⁽²⁾. As the normal component of the die velocity is assumed to be $c=1$ in this report,

$$\epsilon = (P/2k) / (2A)$$

should be calculated instead of Eq. (6) of the former report⁽²⁾. For the fields b, be and e, A is calculated from the kinematically admissible fields and the upper bounds are used as $P/2k$. The results calculated for each field are as follows.

Fig. 3 Stress fields in the rigid region

Field a:

$$\epsilon = \frac{(1+\psi)(1+\sin\psi)}{\left(1+\frac{1}{2}\psi\right)\cos\psi\{(\sec\psi+\tan\psi)^2+\tan^2\delta\}} \quad \dots\dots\dots(7\cdot a)$$

Field b:

$$\epsilon = \frac{(1+\phi_1)(1-\tan\delta)+(1+\phi_2)(1+\tan\delta)}{(2h/c)(3+\phi_1+\phi_2)} \quad \dots\dots\dots(7\cdot b)$$

Field c:

$$\epsilon = (1+\psi) / \left\{ \left(1+\frac{1}{2}\psi\right)(\sec\psi+\tan\psi+\tan\theta_2) \right\} \quad \dots\dots\dots(7\cdot c)$$

Field d: The stress free region should be included in A.

$$\epsilon = 1/[1+\tan\delta\{(h_1-h_2)/(h_1+h_2)\}] \quad \dots\dots\dots(7\cdot d)$$

If the equality of (5·d) holds,

$$\epsilon = 1 / \left[1 + \frac{\cot\theta_2(\cot\theta_1-2)^2}{\cot\theta_1-2+\cot\theta_2} \right] \quad \dots\dots\dots(7\cdot d)'$$

Field e:

$$\epsilon = \frac{1}{2} \beta_1^2 \beta_2^2 \frac{(\beta_1 \beta_2)/(\alpha_1 \alpha_2) - 1}{1 + \alpha_1^2 + \alpha_2^2} \quad \dots\dots\dots(7\cdot e)$$

Field be:

$$\epsilon = \frac{\frac{1}{2}(1+\tan\phi_1)(1-\tan\delta)+(1+\phi_2)(1+\tan\delta)}{(2h/c)(2+\tan\phi_1+\phi_2)} \quad \dots\dots\dots(7\cdot be)$$

Field f: $\epsilon = 1/(1+\tan\theta_2) \dots\dots\dots(7\cdot f)$

4. Stress fields in the rigid region

Constructed stress fields are shown in Fig. 3. Field a is evidently a complete solution, while, the field c is a complete solution when $\theta_2 \geq \psi$. This condition is always satisfied. All of the fields a corresponding to a series of the combination of θ_1 and θ_2 given by the variation of the ratio h_1/h_2 within the range $|\delta| \leq \pi/4$ and all of the fields c with θ_1 and θ_2 given by the movement of the apex O on the broken line OA have the same stress fields. Of course, the symmetrical condition is included. The columns ECDG (d), ECCE (e^*) and ECDE (f) shown in Fig. 3d, e^* and f are compressed by $2k$. Therefore, if $\theta_2 \geq 0$, the fields d and f are complete solutions and field e^* is statically admissible.

Above conclusions are also drawn by Hill's theorem⁽⁴⁾. It is easily verified by the method of Bishop et. al⁽⁵⁾ that the velocity in the rigid region is zero. But the bending or buckling may occur in a thin wedge.

5. Numerical results and discussions

5.1 Deformation modes

Contours of $(h_1+h_2)/c$ or $2h/c$ and δ on (θ_1, θ_2)

Fig. 4 Width of the contact surface

plane are shown in Fig. 4. All of the results are shown only for $\theta_1 \leq \theta_2$ but the mirror image with respect to the line $\theta_1 = \theta_2$ may be considered. Field a is a complete solution in the range ACDFA of Fig. 4(a) and statically admissible in the range between AFD ($\delta = 45^\circ$) and AGD ($\delta = \theta_2$). Statically admissible field a agrees with the complete solution c along the curve AGD. The field d may occur only in a portion BCD given by Eqs. (6·d) and (5·f) of the whole range OBCDEO. The deformation mode a that represents the flow of the material separately on the contact face and consists of the fields a and d may be realized only in the range ABDGA. The more the wedge inclined, the more $(h_1 + h_2)/c$ decreases. Therefore, Eq. (4·a) may be certainly a lower bound in the range AFDGA. It is noted that δ is not zero on BC where $\theta_1 = \theta_2$. That is, the field d will be unsymmetrical even in the symmetrical compression. Fig. 4(b) shows the results for the mode b that represents a dead material zone in contact with the die and consists of the fields b, be and e. These are kinematically admissible, $(h_1 + h_2)/c$ in Fig. 4(a) equal to $2h/c$ in Fig. 4(b) along the curve AKL and the latter is smaller in the region AKL. The broken lines in the range OBC* indicate $2h/c$ of the field e*. Fig. 4(c) shows the results for the mode c that represents the flow of material only on one side of the wedge and these results are exact for all combinations of θ_1 and θ_2 . The curve BDC is the boundary between the fields c and f.

5.2 Comparison of the load

Similar contours of $P/2k$ are shown in Fig. 5(a), (b) and (c) for the deformation modes a, b and c, respectively. A lower bound of $P/2k$ is given by the broken lines in Fig. 5(a) and it coincides with $P/2k$ of Fig. 5(c) along AGD. On the other hand, $P/2k$ of the field a coincides with the upper bound given by the field b along the curve AH. Consequently, the possible range ABDGA of the mode a may be reduced by the range below the curve AH.

In Fig. 5(b), the upper bound of $P/2k$ of the mode b which is exact only on the line AB is shown. The broken lines in the range OBC* give a lower bound of $P/2k$. In this range, the difference of the upper and lower bounds is smaller than that of $2h/c$ given by the fields e' and e*. Therefore, an exact value may be predicted between both bounds. Since $2h/c$ and the upper bound of $P/2k$ of the mode b for the combination of θ_1 and θ_2 far from $\theta_1 = \theta_2$ are much larger than those of the mode c shown in Fig. 5(c), this upper bound may be considerably larger than the exact solution. $P/2k$ of the mode c shown in Fig. 5(c) is an exact solution and this is least for all of θ_1 and θ_2 among three modes. An inclination of the wedge which minimizes the $P/2k$ of the mode c

Fig. 5 Load $P/2k$

Fig. 6 Mean strain ϵ

exists.

5.3 The mean strain ϵ

Similar contours of ϵ are shown in Fig. 6(a), (b) and (c); ϵ has peaks at the points O and C in Fig. 6(a). A little inclination of the wedge decreases ϵ markedly and increases the possibility of the mode a, probably. In the former report⁽²⁾, ϵ of the field d was equal to unity, because of the assumption $\delta=0$. ϵ for $\delta \neq 0$ is apparently smaller along BC, being equal to $1/\sqrt{2}$ at B in Fig. 6(a). ϵ in Fig. 6(b) is calculated from the kinematically admissible fields but this is smaller than those of the modes a and c, excepting the range OIJ. Therefore, when the inclination of the wedge is small and the wedge is not quite thin, the mode b may be realized in the work hardening material, though $P/2k$ of the mode c is least. ϵ of the mode c is shown in Fig. 6(c). Since both $P/2k$ and ϵ of the mode a are larger than those of the mode c, the mode a may be hardly realized in the work hardening material.

6. Conclusions

In this report, the compression of a wedge by a rigid flat die in unsymmetrical condition was investigated. Seven types of slip line field were classified into three modes of deformation, that is,

(a) the material flows separately on the contact face (fields a and d),

(b) the dead material zone is in contact with the die (fields b, be and e) and

(c) the material flows only on one side of the wedge (fields c and f).

The complete solution of the mode b was never found and accordingly it is not clear whether the limit of this mode exists or not. Of the three modes, $P/2k$ of mode c is least but ϵ of b is least for most of the combinations of θ_1 and θ_2 . Generally, fields b and f may be most probable in the work hardening material.

The case where either θ_1 or θ_2 is negative is a different problem.

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