A Study of Boundary Layers in the Downstreams of a Cylinder Row in Motion*
(2nd Report, Theoretical Investigation on the Boundary Layers of a Flat Plate)

By Naomichi Hirayama**

A unique turbulent boundary layer—forced turbulent boundary layer—which is found on a flat plate, located at zero incidence angle in the downstream of a cylinder row in motion, as dealt with in the 1st Report, was investigated theoretically in this report. As the result:

1) An unknown function in the experimental formula of velocity distribution in the boundary layer under transition to forced turbulent flow was calculated by means of the equation of mean motion.
2) The mean friction coefficient in the transition region was calculated by means of the equation of momentum.
3) The position where the transition completes was estimated theoretically.
4) In the region where the forced turbulent boundary layer is fully developed, the mean friction coefficient was estimated theoretically.

1. Introduction

Experimental results of a boundary layer on a flat plate located in the downstream of a cylinder row in motion (the angle of incidence is zero) are analyzed theoretically in this report. As this boundary layer (including the forced boundary layer and the transition region to the former) is unique, the conventional method of analysis, which is useful for solving an ordinary turbulent boundary layer, cannot be applied. In this report, the empirical formula of velocity distribution in the boundary layer (as given in 1st Report) is used, and as a means of obtaining the unknown function \( c \) in the formula a simplified equation of motion is used. In addition, the momentum equation of boundary layer is used to calculate the mean friction drag coefficient. Exact analysis is impossible, because this theory is based on the assumption of the empirical velocity distributions.

Eventually, this report is aimed at giving a theoretical foundation to various trends in the phenomenon which were recorded in the preceding experiment. Such analysis would be valuable for solving theoretically unsettled problems.

2. Transition to forced turbulent flow

Emmons\(^{(1)}\) and Schubauer & Klebanoff\(^{(2)}\) have confirmed that when the turbulence level in the main flow \( \sqrt{\frac{V}{2}}U_1 \) is as low as 0.003, the transition from laminar to turbulent begins in the boundary layer on a flat plate at Reynolds number \( Re = 2.6 \times 10^6 \), where the symbol \( x \) denotes the distance from the leading edge. In the initial period of the transition, turbulence takes place in relatively narrow regions—turbulent spots—which flow downwards with an increasing size. Accordingly, turbulent and laminar flows are alternately recorded by a hot-wire located in such boundary layers. The ratio of turbulent period to the entire time—the intermittency factor—serves as an explanation for the mean velocity distribution in the transition region\(^{(3)}\).

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** Assistant Professor, Faculty of Engineering, Tokyo Metropolitan University, Setagaya-ku, Tokyo.
Compared with such ordinary transition, the transition to the forced turbulent flow occurs in such a basically different way stated as follows.

1) No distinct intermittency is found in the boundary layer under transition, as may be seen in an example of hot-wire records, in Fig. 2.

2) Fig. 3 shows examples of velocity distributions of an ordinary boundary layer under transition measured by Schubauer & Klebanoff (full lines) and those plotted based on this experiment, Fig. 8 in the 1st Report (dotted lines). The nomenclature used is illustrated diagrammatically in Fig. 1. In the case of a main stream with low turbulence, the velocity distribution approaches the line A (the velocity distribution of a fully developed turbulent boundary layer) as transition proceeds, from the inner region of the boundary layer, leaving an outer layer, which does not coincide with the line A even after the boundary layer has become fully turbulent. But, in the case of a boundary layer with fluctuating main flow, the velocity distribution becomes parallel to the line A from rather outer part of the layer and approaches the line A in a state parallel to it. No distinct outer layer is found in this case.

From these facts, the transition to forced turbulent flow may be a substantially different phenomenon and accordingly, the transition theory of intermittency cannot be applied.

3. The equation of motion of the mean flow

The equation of motion of mean flow with fluctuation velocity is shown below on the basis of a two-dimensional incompressible fluid.

\[ U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + \frac{1}{\rho} \frac{\partial (U^2 - \bar{v}^2)}{\partial x} + \frac{1}{\rho} \frac{\partial vU}{\partial y} = \frac{\partial^2 U}{\partial y^2} \]  \hspace{1cm} (1)

where \( x \) and \( y \) are the coordinates as illustrated in Fig. 1; \( U \) and \( V \) are the mean velocities, \( \bar{u} \) and \( \nu \) the fluctuation velocities parallel and perpendicular to the surface of the flat plate, and \( \nu \) the kinematic viscosity. Each term of Eq. (1) is calculated to compare the magnitudes with each other. It is convenient to use the experimental velocity distribution in the boundary layer under transition as obtained in the 1st Report, which is;

\[ \frac{U}{U_r} = - \frac{1}{K} \left( \ln \frac{y_1}{y_r} + c \right) \]  \hspace{1cm} (2)

where \( K \) is a constant \((=0.42)\), and \( c \) a function of only \( z \) and \( U_r \) the friction velocity. If Eq. (2) is exact up to the outer boundary of the layer, discon-
continuity should exist at the outer edge of the boundary layer as shown with a dotted line in Fig. 4. This is not reasonable physically. In fact, the velocity distribution curve in the boundary layer joins smoothly with the main flow curve. It is slightly off the dotted line at the edge of the layer. Eq. (2) is not exact also in the laminar sub-layer. For these reasons, the magnitudes of the terms are calculated in the region confined by two lines A and B in Fig. 4. Each term is calculated by the following equations.

\[
U \frac{\partial U}{\partial x} = U \left( \frac{1}{K} \frac{dU_r}{dx} \ln \frac{U_r}{\nu} + \frac{1}{K} \frac{dU_r}{dx} \right) + U \left( \frac{1}{U_r} \frac{dU_r}{dx} \right)
\]

\[
= U \left( \frac{dU_r}{dx} \left( \frac{U}{U_r} + \frac{1}{K} \right) + \frac{U_r}{K} \frac{dU_r}{dx} \right)
\]

Eq. (3)

\[
V \frac{\partial U}{\partial y} : \text{From the equation of continuity} \quad \frac{\partial V}{\partial y} = -\frac{\partial U}{\partial x},
\]

\[
V = -\int_{y}^{y} \left( \frac{\partial U}{\partial x} \right) dy
\]

For an approximate estimation, \( U/U_r = U_r y/\nu \) is used in the laminar sub-layer \( 0 < y < \delta_i \) (\( \delta_i \) : the thickness of the laminar sub-layer). Then,

\[
V = -2 \int_{0}^{y} \left( \frac{U_r y}{\nu} \frac{dU_r}{dx} - dU_r \right) dy - \int_{y}^{\infty} \left( \frac{U_r y}{\nu} \frac{dU_r}{dx} + \frac{U_r}{K} \frac{dU_r}{dx} \right) dy
\]

\[
= -\left( \frac{U_r}{U_r} + \frac{U_r}{K} \frac{dU_r}{dx} \right) (y - \delta_i) + \frac{U_r}{\nu} \frac{dU_r}{dx} \delta_i^2
\]

\[
\frac{\partial U}{\partial y} = \frac{U_r y}{K} \quad : \quad V \frac{\partial U}{\partial y} = -\frac{U_r}{K} \left( \frac{U_r}{U_r} + \frac{U_r}{K} \frac{dU_r}{dx} \right) \left( 1 - \frac{\delta_i}{y} \right) - \frac{U_r^2}{K \nu} \frac{dU_r}{dx} \delta_i^2
\]

Eq. (4)

Though Eq. (4) is an approximate estimation, as the magnitude of this term is relatively small (see Fig. 5), it does not affect the general conclusion.

iii) \( \frac{\partial (\bar{U}^2 - \bar{V}^2)}{\partial x} \) is calculated from the measured values.

iv) \( \nu \frac{\partial U}{\partial y} = -\nu \frac{U_r}{K} \frac{1}{y^2} \)

Eq. (5)

v) \( \frac{\partial w}{\partial y} \): This term cannot be estimated directly as long as \( w \) is not measured, but it can be estimated from other terms in Eq. (1).

Though it is difficult to obtain correct values of \( dU_r/\delta z \) in Eqs. (3) and (4), errors are permissible for the purpose of comparison along the y axis. An example of calculated values of each term of Eq. (1) is given in Fig. 5. The dotted lines are the parts of curves outside the range confined between the lines A and B, and they are not correct. Fig. 5 indicates that the third term of the left side and the term of the right side of Eq. (1) is negligibly small. Among other terms, the 1st and 4th term are predominant. This tendency remains unchanged under other conditions.

4. Analysis of transition by means of simplified equation of motion

The most simplified equation of motion may be given as

\[
U \frac{\partial U}{\partial x} = \frac{\partial w}{\partial y}
\]

Eq. (6)

This equation is also valid in two-dimensional flows where turbulent viscosity surpasses the fluid viscosity, such as in two-dimensional wakes. But the nature of the right side of the equation is not clarified. It is thus impossible to obtain the velocity distribution by solving this equation. But as the form of the velocity distribution is known by Eq. (2), Eq. (6) will be used as a means to solve the unknown function \( c(x) \).

The left side of Eq. (6) is given by Eq. (3). In the right side, \(-\bar{w}\) is expressed as follows by introducing the mixing length \( l \):

\[
-\bar{w} = l \sqrt{\bar{v}^2} \frac{\partial U}{\partial y}
\]

Eq. (7)

where \( l \) and \( \sqrt{\bar{v}^2} \) are functions of \( x \) and \( y \) in general. Accordingly,
\[-\frac{\partial \bar{w}}{\partial y} = \frac{\partial}{\partial y} (\sqrt{\bar{v}^2}) - \frac{1}{\sqrt{\bar{v}^2}} \frac{\partial}{\partial y} \frac{\partial \bar{U}}{\partial y} + \frac{1}{\sqrt{\bar{v}^2}} \frac{\partial \bar{U}}{\partial y} \] ......(8)

\[\partial U/\partial y, \text{ and } \partial^2 U/\partial y^2 \text{ calculated from Eq. (2) are substituted into Eq. (8), to form ;} \]

\[-\frac{\partial \bar{w}}{\partial y} = \frac{U_r}{K} \left\{ \frac{1}{y} \frac{\partial}{\partial y} (\sqrt{\bar{v}^2}) - \frac{1}{\sqrt{\bar{v}^2}} \frac{\partial \bar{U}}{\partial y} \right\} \] ......(9)

Eqs. (3) and (9) are substituted into Eq. (6),

\[\frac{KU_r}{U_r} \left( \frac{dU_r}{ds} \left( \frac{U_r}{K} + \frac{U_r}{U_r} \frac{dc}{ds} \right) \right) \]

\[= \frac{1}{y} \frac{\partial}{\partial y} (\sqrt{\bar{v}^2}) - \frac{1}{\sqrt{\bar{v}^2}} \frac{\partial \bar{U}}{\partial y} \] ......(10)

As Eq. (10) cannot be solved exactly, attempts were made to determine the relations among factors which have predominant effects on the terms of Eq. (10). The both sides of Eq. (10) are functions of \(x\) and \(y\) in general, but circumstance is somewhat different at a relatively outer part of the layer, for instance, in the vicinity of the line B. At this place, the variation of \(U/U_r\) is slight and may be considered approximately equal to \(U_r/U_r\). Though \(U_r/U_r\) changes to a considerable extent along \(y\) axis (see Fig. 8 in the 1st Report), it is caused by variation of \(U_r\) rather than \(U\). Here, \(U_r/U_r = U_r/U_r\) is assumed. The value \(1/K\) is neglected compared with \(U_r/U_r\) (This neglect will be discussed later). This neglect minimizes a possible error due to \(U_r/U_r = U_r/U_r\). In this region, \(\sqrt{\bar{v}^2}\) is equal to \(\sqrt{\bar{v}^2}\) (Fig. 10, 1st Report).

So, Eq. (10) becomes as follows.

\[\frac{KU_r}{U_r} \left( \frac{U_r}{K} \frac{dU_r}{ds} + \frac{U_r}{U_r} \frac{dc}{ds} \right) \]

\[= \sqrt{\bar{v}^2} \left( \frac{1}{y} \frac{dl}{dy} - \frac{1}{y^2} \frac{dc}{ds} \right) \] ......(11)

Though \(\sqrt{\bar{v}^2}\) will vary slightly with \(y\) except when \(\beta\) (Fig. 6, 1st Report) is equal to 90°, it is experimentally confirmed that the value is approximately constant in the narrow (towards \(y\)-direction) band region where Eq. (11) covers. Therefore the value \(\sqrt{\bar{v}^2}\) may be a function of only \(x\). The left side of Eq. (11) is also a function of \(\beta\). If \(l\) is the function of \(y\), only, Eq. (11) will be represented by;

\[\frac{KU_r}{U_r} \left( \frac{U_r}{K} \frac{dU_r}{ds} + \frac{U_r}{U_r} \frac{dc}{ds} \right) = -A \sqrt{\bar{v}^2} \] ......(12)

where \(-A\) is a constant. As explained above, Eq. (12) is induced from the two assumptions; i) the left side of Eq. (10), that is, \(KU_r/\partial U_r/\partial x\) is nearly constant along \(y\) axis and ii) the mixing length \(l\) is the function of \(y\) only. As in Fig. 5, \(U_r/\partial U_r/\partial x\) is nearly constant. The 1st assumption will be approximately satisfied. As \(U_r/\partial U_r/\partial y\) is negative, the sign minus is added to the constant \(A\). The second assumption is explained as follows. According to the experiment by Nikuradse(3), the mixing length \(l\) in a fully turbulent pipe flow is expressed by \(l/R\)

\[= f(y/R) \] (where \(R\) is the inner radius) irrespective of Reynolds number and the roughness of the pipe. As \(R\) indicates the maximum possible size of turbulence, \(R\) will be suitable for a reference scale of turbulence. In the case of the inner layer of an ordinary turbulent boundary layer, as the velocity deficiency law is similar to a pipe flow, \(l\) will be expressed by the same expression if a proper dimension of the boundary layer itself is substituted for \(R\). Though similar reasoning will be possible for a forced turbulent boundary layer, the case is somewhat different in the vicinity of the line B, because both intensity and scale of turbulence depend on those of the main flow at this station, and are independent of the characteristics of the boundary layer itself (accordingly, Kármán’s analogy concerning the mixing length is not effective here). Even if the following expression,

\[l/\rho = f(y/\rho)\]

is possible as in the pipe flow, the scale of mixing length \(l_0\) must be independent of this boundary layer and must be a scale which has something to do with the main flow turbulence (for instance, the spacing of the cylinder row in motion). Because of these reasons, \(l\) is assumed to be a function of \(y\) only. Furthermore, from Eq. (11), \(l = \alpha y - A y^2\) \((\alpha : \text{an integral constant})\) can be obtained. The value \(dU/\partial y\) must decrease as the value \(y\) increases just as the case of a circular pipe flow. Though the measurement of \(w\) in Eq. (7) will be useful to confirm whether \(l\) is the function of \(y\) only, the measurement was not possible due to the limited capacity of the experimental equipments used. Another method to confirm the nature of \(A\) may be considered when each term of Eq. (11) other than \(\partial (\bar{w})/\partial y\) is known, but it was found ineffective because of the scattering of the value \(dU_r/\partial x\) as stated before. Thus, the preceding analysis is to be further developed here, and its results will be discussed later. From Eq. (12),

\[\frac{dc}{dx} = -A \sqrt{\bar{v}^2} \frac{KU_r}{U_r} \frac{dU_r}{dx} \] ......(13)

The solution is

\[c = KU_r \frac{U_r}{U_r} \frac{dU_r}{dx} \] ......(14)

where \(\beta\) is an integration constant (and does not represent an angle). On the other hand, when the condition \(U_r = U_r\) at \(y = \beta\) is inserted into Eq. (2),

\[U_r = \frac{U_r}{U_r} \left( \frac{KU_r}{U_r} + \frac{c}{r^2} \right) \] ......(15)

\[\therefore \ln \frac{dU_r}{\nu} = \frac{KU_r}{U_r} \frac{dU_r}{dx} \] ......(15)

From Eqs. (14) and (15),

\[\ln \frac{dU_r}{\nu} = \frac{KU_r}{U_r} \frac{dU_r}{dx} \] ......(16)
The values of \( \ln(\partial U_1/\nu) \) and \( (KU_1U_2) - c \) are plotted against \( x \) in Fig. 6. The right side of the dotted lines in the figure are the fully developed forced turbulent region, where Eq. (16) cannot be applied. Though it is dangerous to draw general conclusions from few experimental data, at least the following may be established: Eq. (16) is approximately satisfied, and each term in Eq. (16) varies only slightly within the transition region. Therefore, it is possible to assume that the right side of Eq. (16) is a linear function of \( x \). This is equivalent to assuming \( \sqrt{\nu_r^2}/U_1 \) as a constant, it will be permissible when \( \sqrt{\nu_r^2} \) does not attenuate rapidly in the range in question or when the transition region is not so wide. Putting the mean value of \( \sqrt{\nu_r^2}/U_1 \) in the transition region into \( |\sqrt{\nu_r^2}/U_1|_{m} \), Eq. (16) will be expressed by

\[
\ln(\partial U_1/\nu) = \frac{KU_1}{U_1} - c = A \sqrt{\nu_r^2} \ln \left( \frac{\nu_r^2}{U_1} \right) + \beta \quad \cdots (17)
\]

From Eq. (17),

\[
\beta = \frac{\nu}{U_1} e^{A\sqrt{\nu_r^2}/U_1|m^2} \quad \cdots (18)
\]

The analysis so far made is based on the assumption of the velocity distribution as expressed by Eq. (2). And the value of \( \beta \) does not represent exactly the whole thickness of the boundary layer, but expresses the station where \( U \) attains \( U_1 \) in Eq. (2). Accordingly, the laminar sub-layer and the outer layer where the velocity distributions are not expressed by Eq. (2) are neglected.

Through such analysis, the exact value of, for instance, the thickness of the boundary layer cannot be obtained. But it is possible to discuss general tendency of momentum losses or of friction drag coefficients, because in the sub-layer and the outer layer, either velocity deficit or thickness is very small. In fact, the momentum thickness calculated by Eq. (2) is about 90% in magnitude of the exact value.

5. Mean friction drag coefficient

in the boundary layer

The momentum thickness of a boundary layer \( \delta \) is expressed as follows.

\[
\delta = \int_{U_2}^{U_1} \frac{dU}{U_1} \quad \cdots (19)
\]

The velocity defect law is led from Eq. (2) and (15).

\[
1 - \frac{U}{U_1} = \frac{1}{K} \ln \left( \frac{U}{U_1} \right) \quad \cdots (20)
\]

Accordingly,

\[
1 - \frac{U}{U_1} = \frac{U}{KU_1} \ln \left( \frac{U}{U_1} \right) \quad \cdots (21)
\]

Substituting Eq. (21) into Eq. (19),

\[
\delta = \frac{U_2}{KU_1} \int_{U_2}^{U_1} \ln \left( \frac{U}{U_1} \right) dU \quad \cdots (22)
\]

Performing integration,

\[
\int_{U_2}^{U_1} \ln \left( \frac{U}{U_1} \right) dU = - \delta \quad \cdots (23)
\]

Substituting Eq. (23) into Eq. (22),

\[
\delta = \left( \frac{U_2}{KU_1} \right)^2 - 2 \left( \frac{U_2}{KU_1} \right) \delta \quad \cdots (24)
\]

Putting \( KU_1/U_2 = z \), and from Eq. (18),

\[
\delta = \frac{\nu}{U_1} e^{A\sqrt{\nu_r^2}/U_1|m^2} \left( 1 - \frac{2}{z} \right) e^{A\sqrt{\nu_r^2}/U_1|m^2} \quad \cdots (25)
\]

Substituting Eq. (25) into Eq. (24),

\[
\delta = \left( 1 - \frac{2}{z} \right) \frac{\nu}{U_1} e^{A\sqrt{\nu_r^2}/U_1|m^2} \quad \cdots (26)
\]

The equation of momentum in the boundary layer of a flat plate is

\[
\frac{KU_1}{\nu} = \frac{d^2 \delta}{dx^2} \quad \cdots (27)
\]

Substituting Eq. (26) into Eq. (27),

\[
\frac{KU_1}{\nu} = \frac{d\left( 1 - \frac{2}{z} \right) e^{A\sqrt{\nu_r^2}/U_1|m^2} \delta}{dx^2} \quad \cdots (28)
\]

Arranging it,

\[
\frac{KU_1}{\nu} = \frac{d^2 \delta}{dx^2} \left( 1 - \frac{2}{z} \right) e^{A\sqrt{\nu_r^2}/U_1|m^2} \quad \cdots (29)
\]

Rearranging Eq. (29),

\[
\frac{d^2 \delta}{dx^2} = \frac{\nu}{KU_1} \left( 1 - \frac{2}{z} \right) e^{A\sqrt{\nu_r^2}/U_1|m^2} \quad \cdots (30)
\]

This non-linear equation of 1st order is called "Riccati's equation", and cannot be solved with any elemental function.\(^{[6]}\)
Empirically, the value $z$ varies around $7\sim 13$ in the transition region, so that $(1-2/z)$ varies within the range $0.85\sim 0.72$. As this variation is minor compared with that of other terms in Eq. (30), it is assumed approximately constant. The first term is also far smaller than the second term. The following assumptions are made.

$$\left(1-\frac{2}{z}\right) = B\left(n=0.8, \quad 2\frac{dz}{ds} + \frac{\sqrt{\frac{U_1}{U_1}}}{U_1} \right) = B$$

These assumptions will be discussed later.

From Eqs. (30) and (31),

$$z = \left(\frac{R^k}{e^{\theta}} \cdot \frac{A}{A-B \cdot \sqrt{\frac{\theta}{U_1}}/U_1} \right)^{1/2}
\times e^{-z/2} \cdot \frac{A}{A-B \cdot \sqrt{\frac{\theta}{U_1}}/U_1} \times \nu$$

Substituting Eq. (32) into Eq. (26),

$$\frac{B}{(1-2/z)} = \frac{K}{e^{\theta}} \cdot \frac{A}{A-B \cdot \sqrt{\frac{\theta}{U_1}}/U_1} \times \frac{e^{\theta}}{KU_1}$$

$$\times e^{\theta/2} \cdot \frac{A}{A-B \cdot \sqrt{\frac{\theta}{U_1}}/U_1} \times \nu$$

while the equations from Eq. (27) to (33) are necessary in order to calculate $z$, the value of $\theta$ will be obtained directly from Eq. (26) on the assumption $(1-2/z) = B$,

$$\frac{B}{(1-2/z)} = \frac{K}{e^{\theta}} \cdot \frac{A}{A-B \cdot \sqrt{\frac{\theta}{U_1}}/U_1} \times \nu$$

Between Eq. (33) and Eq. (34), there is the same difference as that between $(1-2/z)$ and $B$, which will be discussed later. The change of $\theta$ in Eq. (33) due to $z$ depends, as in the case of Eq. (34), mainly on the last term $A^{1/2} \cdot \sqrt{\frac{\theta}{U_1}}/U_1 \times \nu$. This is because of $e^{\theta/2} \cdot \frac{A}{A-B \cdot \sqrt{\frac{\theta}{U_1}}/U_1} \times \nu$ where $\theta$ is of order 10 as indicated in Eq. (32). For simplicity, Eq. (34) is used in the following analysis.

The mean friction coefficient (not local friction coefficient) is expressed as follows.

$$C_D = 2B \frac{\theta}{R_s}$$

Substituting Eq. (34) into Eq. (35),

$$C_D = 2B \frac{\theta}{K} \cdot \frac{A}{A-B \cdot \sqrt{\frac{\theta}{U_1}}/U_1} \cdot \nu$$

where,

$$R_s = U_1 \cdot \frac{\theta}{\nu}$$

As the constant $A$ has the dimension of the reciprocal of length, as can be seen from Eq. (17), $A \nu/U_1$ is a non-dimensional parameter. Putting the Reynolds number $R_e$ where $C_D$ is minimum $R_{e_{mn}}$, it is calculated on the condition of $dC_D/dR_s = 0$. That is

$$R_{e_{mn}} = \frac{U_1}{A \nu} \cdot \frac{\theta}{\sqrt{\frac{\theta}{U_1}}/U_1}$$

In Fig. 7, the result of experiment is compared with Eq. (37). The experimental values are estimated from $C_D-R_s$ curves (Data on the condition $\theta = 45^\circ$ are plotted from Fig. 11 of the 1st Report, and those at $\theta = 60^\circ$ are omitted). As stated in the previous report, when the main stream fluctuation is intense, $C_D-R_s$ curves have no minimum point, and the experimental values decrease in number. In Fig. 7, the experimental values agree well with the theoretical values, when the value of $A$ is suitably selected. This agreement will be one of the proofs that the assumptions hitherto made are reasonable to a considerable extent. But the actual range where Eq. (37) can be applied is yet to be determined. This limit of application will exist, because, Eq. (37) is based on assumptions that $\sqrt{\theta/\nu} = \sqrt{\theta/\nu}$ in the outer part of the boundary layer, and that the velocity distribution is expressed by Eq. (2). Furthermore, the numerical value $A$ depends on the property of mixing length $l$ as may be understood from Eqs. (11) and (12). In this report, $l$ is assumed as a function of $y$, but, other influences, for instance, of the scale of turbulence, may occur under other conditions. In addition, the range of the intensity of turbulence in the measurement covers the entire region in practice, the mean velocity $U_1$ varies within an only relatively low velocity range and only one kind of cylinder row was adopted.

Now, irrespective of whether there are minimums in $C_D-R_s$ curves, Eq. (37) may be considered as a parameter, and can be substituted into Eq. (36).

$$C_D = 2B \frac{\theta}{K} \cdot \frac{A}{A-B \cdot \sqrt{\frac{\theta}{U_1}}/U_1}$$

$$R_s = U_1 \cdot \frac{\theta}{\nu}$$

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Now, irrespective of whether there are minimums in $C_D-R_s$ curves, Eq. (37) may be considered as a parameter, and can be substituted into Eq. (36).

$$C_D = 2B \frac{\theta}{K}$$

$R_s = R_{e_{mn}}$ is substituted as a boundary condition in Eq. (38),

$$\theta = K \cdot C_D e^{-p/R_{e_{mn}}}$$

From Eqs. (38) and (39),
\[ C_D = C_{D_0} \frac{R_{E_{0x}} (R_e - R_{E_{0x}})}{R_e} \]  

Eq. (40) can be compared with the experiment for suitable values of \( R_{E_{0x}}, C_{D_0} \) and \( R_{E_{0x}} \) in Fig. 8. The agreement is fair.

In order to find the points where \( c \) attains the final value, 2.4. \( z = K U_l / U_r \) is introduced into Eq. (17).

\[ c = z - A \sqrt{\frac{\sqrt{v}}{U_1}} \cdot \frac{e^8}{B} \cdot z - \beta \]  

From Eq. (37) and Eq. (32),

\[ z = \left( \frac{K^2 \cdot R_{E_{0x}} \cdot R_{E_{0m}}}{e^8 \cdot B} \right)^{1/2} e^{-1/2 \cdot R_e / R_{E_{0m}}} \]  

According to Eq. (39),

\[ z = \left( \frac{2K^2 \cdot R_{E_{0m}}}{C_{D_0} \cdot R_{E_{0x}}} \right)^{1/2} e^{-1/2 \cdot R_e / R_{E_{0m}}} \]  

Eqs. (39), (37), and (42) are introduced into Eq. (41),

\[ c = \left( \frac{2K^2 \cdot R_{E_{0m}}}{C_{D_0} \cdot R_{E_{0x}}} \right) e^{-(R_e - R_{E_{0x}}) / R_{E_{0m}}} \]  

Eq. (43) is calculated on the same conditions as those in Fig. 8, and is compared with the experimental values in Fig. 9. The theory gives slightly small values than the experiment. The points where \( c \) attains 2.4 (the dotted line in Fig. 9) are illustrated with arrows in Fig. 8. It is theoretically explained that the bigger the value of \( C_{D_0} \) \( c \) attains the final value at smaller values of \( R_e \). On the right side of the arrow marks, Eq. (40) is not applicable, therefore, the curves are illustrated with dotted lines.

Here, the assumptions employed are discussed.

Eq. (42) is calculated on the conditions as used in Fig. 8, and the value of \( z \) is found to vary from 8 to 14 in the transition region. Though \( 1/K \) was neglected against \( U_1 / U_r \) it is not unreasonable because of \( (1/K) \cdot (U_1 / U_r) = 1/8 \sim 1/14 \). It is important to note that this neglect will compensate the error due to the assumption \( U_l / U_r = c_0 \) in Eqs. (10) and (11). Next, the value of \( (1 - 2/\alpha) \) changes within the range 0.75 to 0.88, so that the assumption \( (1 - 2/\alpha) = B = 0.8 \) in Eq. (31) will be permissible. The value \( 2(2K^2 \cdot R_{E_{0m}} / C_{D_0} \cdot R_{E_{0x}}) \) in the second assumption of Eq. (31) is calculated by Eq. (42),

\[ \frac{dz}{dz} = \left( \frac{2K^2 \cdot R_{E_{0m}}}{C_{D_0} \cdot R_{E_{0x}}} \right)^{1/2} e^{-1/2 \cdot R_e / R_{E_{0m}}} \frac{(1/2)}{R_{E_{0m}}} \]  

From Eq. (37),

\[ \frac{dz}{dz} = -z \cdot A \sqrt{\frac{v}{U_1}} \]  

So that,

\[ \frac{dz}{dz} = -z \cdot B = \frac{1}{z \cdot B} = \frac{1}{6.4} = \frac{1}{11.2} \]  

Therefore, the value \( 2dz/dz \) is very small as anticipated from Eq. (31).

6. Fully developed forced turbulent boundary layer

The velocity distribution in a fully developed forced turbulent boundary layer is given by the following experimental formula in the previous report.

\[ \frac{U}{U_r} = \frac{1}{K} \left( \ln \frac{U_r \cdot y}{c_0} + \delta \right), \quad c_0 = 2.4 \]  

The momentum thickness \( \delta \) is expressed just as in the preceding section.

![Fig. 8](image_url)  
Fig. 8 Mean friction drag coefficients due to theory and experiment

![Fig. 9](image_url)  
Fig. 9 Calculated and measured values of \( c \)
Fig. 10  $C_D$–$R_e$ curves of fully developed forced turbulent boundary layer

\[ \delta = \frac{U_c}{k} \left[ 2 \left( \frac{U_c}{k} \right)^2 \right] \delta \]

When $z = kU_1 U_c$ is introduced,

\[ \delta = \frac{1}{z^2} \left( \frac{1}{z} \right) \delta \] \hspace{1cm} (45)

After deciding $\delta$ by substituting the boundary condition $y = \delta$, $U = U_1$ into Eq. (44),

\[ z = \ln \frac{\partial U_c}{\nu} + c_0 \] \hspace{1cm} (46)

Putting $c_0 = \ln D$

\[ z = \ln \frac{\partial U_c D}{\nu} \], therefore, \[ \frac{\nu Z}{kU_1} \delta \] \hspace{1cm} (47)

Eq. (47) is substituted into the momentum equation of a flat plate, Eq. (27), and is arranged as follows.

\[ DK \frac{U_1}{\nu} dz = \frac{1}{2} \left( 1 - \frac{2}{z} \right) \epsilon \] \hspace{1cm} (48)

\[ DK \frac{U_2}{\nu} dz = \frac{1}{2} \left( 1 - \frac{2}{z} \right) \epsilon + 2 \int (z - 2) \epsilon dz - E \] \hspace{1cm} (49)

where, $E$ is an integral constant. Eq. (50) is rewritten:

\[ DK^3 R_e + E = \epsilon \left( z^2 - 4z + 6 \right) \] \hspace{1cm} (50)

Eq. (47) is introduced into the relation

\[ C_D = \frac{2}{DK} \frac{1}{R_e} \epsilon \left( 1 - \frac{2}{z} \right) \] \hspace{1cm} (52)

Elimination of $z$ from Eqs. (51) and (52) gives the relation between $R_e$ and $C_D$ with $E$ as a parameter. The value $z$ is eliminated graphically. The result is illustrated in Fig. 10, together with experimental values. Though the comparison is difficult when $u_l / \bar{U}_1$ is small because the forced turbulent boundary layer is completed at relatively large $R_e$, the data on the condition $u_l / \bar{U}_1 = 0.743$ agree well with the theoretical values. The curves in Fig. 8, which have completed transition at the points indicated by arrows, will shift to the corresponding curves in Fig. (10).

7. Conclusion

Through the theoretical discussion about the experimental results concerning the boundary layer on the flat plate located in the downstream of a cylinder row in motion in the previous report the following conclusions are obtained.

1) The unknown function of $x, c(x)$, in Eq. (2)—the velocity distribution formula in the boundary layer under transition to the forced turbulent boundary layer—is solved theoretically, and is found to depend on the non-dimensional friction velocity $U_c / U_1$ and the intensity of the mean ratio of fluctuation to mean velocity $[\sqrt{\bar{U}^2}] / U_1$.

2) The mean friction coefficient $C_D$ of the boundary layer under transition into the forced turbulent flow is expressed by Eq. (40). This is, of course, a function of $R_e$ but is dependent on $R_{em}$ as well as $R_{e_m}$ and $R_{em}$ is expressed as in Eq. (37), and depends on fluctuation velocity and mean velocity in the main flow, the physical meaning of which is a Reynolds number where the mean friction coefficient is locally minimum.

3) The mean friction coefficient of the fully developed forced turbulent boundary layer is calculated theoretically.

4) The stations where the transition is completed are calculated theoretically, and are compared with the experiment.

This analysis is based on the proposition of experimental formula concerning the velocity distribution in the boundary layer, Eq. (2), but the limit of application of the formula is not confirmed in the preceding report owing to the experimental restrictions. Furthermore, when the main stream fluctuation around the leading edge is extremely intense, the thickness of the boundary layer increases rapidly.
near the leading edge. Though this phenomenon is treated as a boundary condition in this theory, it contains unsolved problems such as unsteady separation, and they were not discussed. These are the problems yet to be solved in the future.

In this report, by starting from the relations among the factors which work dominantly on the boundary layer with fluctuating mainstream, theoretical backgrounds to the tendency of the phenomena have been given at least qualitatively.

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Research on the Flow in a Centrifugal Pump Impeller*

(1st Report, A Theory on Straight Cascades of Thin Wings)

By Susumu Murata**

In this report, a method is presented for the calculation of a potential flow through straight cascades of thin wings. The calculation of the flow has been based on the singularity method: normal velocities \( v \) are cubic functions of abscissae \( z \), and tangential velocities \( u \), corresponding to normal velocities \( v \) \((=0, 1, x, x^2, x^3)\), are determinable by a conformal mapping of the cascade into a unit circle. As the author has introduced the recurrence formulae for velocity functions, an improved accuracy has been obtained by using the velocities \( u \) of higher degree. This method will be applied to the calculation of the flow through a centrifugal pump impeller in the following reports. In this report, the author has applied it to the calculation of the flow through a straight cascade of thin wings, comparing with the results hitherto published in numerical examples.

1. Introduction

In order to analyse the performances of a centrifugal pump impeller hydrodynamically, a singularity method is expounded in this report for the calculation of the potential flow through straight cascades of thin wings. In the next reports, the flow through a centrifugal pump impeller with vanes of approximately logarithmic-spiral forms, the effects of variation of impeller breadth, and the effects of inclination or curvature of flow path, will be discussed by applying the results of this report. In the analysis of the flow through a pump impeller, it is necessary to calculate the displacement flow which has a given velocity \( v(x) \) in the \( y \) direction on the chord of wings (Fig. 1).

One of the singularity methods has been treated by Ishihara\(^2\), Scholz\(^3\), and Schlichting\(^4\), in which the normal velocity \( v(x) \) is given by the integral equation including the circulation distribution \( f(x) \) as an integrand. It is difficult to solve the integral

* Received 1st July, 1960.
** Assistant Professor, Faculty of Engineering, Yamashita University, Kofu.