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Unsymmetrical Nonlinear Vibration Systems under Random Loading*

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In this paper the author analyzes the effects of the unsymmetrical nonlinearity of restoring force and the mean square value of a random loading on the mean square value and spectrum of displacement in a vibration system which has a single degree of freedom.

In this analysis an iteration method is applied to solve the equation of motion, assuming that the nonlinearity and loading have small values, and the approximate mean square value and spectrum of displacement are obtained from an equivalent loading spectrum.

The random loading is expressed by Fourier series, and in the case of white spectrum an experiment is made using an analog computer.

1. Introduction

In this paper, the response problem of a nonlinear system which is irregularly excited by a random loading is treated. It is assumed that the system has single degree of freedom, and that the characteristics of restoring force exhibits unsymmetrical nonlinearity.

Though the system has originally symmetrical nonlinearity yet, it exhibits an unsymmetrical nonlinearity with respect to an equilibrium point, when a static load is superposed on the fluctuating load having zero mean. In this analysis, the unsymmetrical nonlinearity of restoring force is given as a quadratic expression of displacement in the system.

The main purpose is to obtain an approximate solution of displacement and its spectrum, and to investigate the influences of nonlinearity and of the level of loading upon the displacement. To solve the equation of motion, an iteration method is applied, assuming that the nonlinearity and loading have small values. This method has been applied to analyze the symmetrical nonlinear systems.

2. Representation of loading

If the random loading is assumed to be finite and has a discrete spectrum, it can be expressed in a form of Fourier series, as mentioned in the

previous paper treating a symmetrical system, i. e.

$$L(t) = \sum_{n=1}^{\infty} F(p_n) \cos p_n t \Delta p \dots\dots\dots (1)$$

$$F(p_n) \Delta p = \frac{2}{T} \int_0^T L(t) \cos p_n t dt \quad (n \geq 1) \dots\dots (2)$$

where $p_n = n\pi/T$, $\Delta p = p_n - p_{n-1} = \pi/T$. (A time interval T is assumed to be sufficiently large). $|F(p_n) \Delta p|$ is the amplitude of each harmonics.

The maximum value of $L(t)$ is given by the norm $\|L\|$, i. e.

$$|L(t)| \leq \|L\| = \sum_{n=1}^{\infty} |F(p_n)| \Delta p \equiv \eta < \infty \dots\dots\dots (3)$$

The mean square value of $L(t)$ is

$$\begin{aligned} \bar{L}^2 &= \frac{1}{T} \int_0^T L^2(t) dt = \frac{1}{2} \sum_{n=1}^{\infty} F^2(p_n) (\Delta p)^2 \\ &= \sum_{n=1}^{\infty} f(p_n) \Delta p \dots\dots\dots (4) \end{aligned}$$

where $f(p_n)$ is the spectrum of $L(t)$ and defined as follows:

$$f(p_n) = (1/2) F^2(p_n) \Delta p \dots\dots\dots (5)$$

3. Equation of motion

The unsymmetrical nonlinearity of restoring force $K(Z)$ is given as a quadratic expression of displacement Z in the system, i. e.

$$K(Z) = E_0 Z + E Z^2 \quad (E_0 > 0) \dots\dots\dots (6)$$

where E gives the nonlinearity of a spring. If $E > 0$, the spring is hard in the positive direction of the load or displacement, and soft in the negative direction; while if $E < 0$, it is soft in the positive direction, and hard in the negative direction. In the case of a single degree of freedom, the equation of motion of the system is given as follows:

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$$\gamma(d^2Z/dt^2) + \xi(dZ/dt) + E_0Z + EZ^2 = L(t) \quad (\xi > 0) \quad (7)$$

where γ is an inertia mass of the system, and ξ is a viscous damping coefficient.

Here the following dimensionless terms are introduced:

$$\left. \begin{aligned} L' &= L/L_D, \quad F' = (F/L_D)\sqrt{E_0/\gamma}, \\ f' &= (f/L_D^2)\sqrt{E_0/\gamma}, \quad E' = EL_D/E_0^2, \\ \xi' &= \xi/\sqrt{E_0\gamma}, \quad Z' = ZE_0/L_D, \quad t' = t\sqrt{E_0/\gamma} \\ p_n' &= p_n\sqrt{\gamma/E_0} \end{aligned} \right\} \quad (8)$$

where L_D is given as a standard load. If Eq.(7) is expressed by the dimensionless terms and dashes are omitted,

$$(d^2Z/dt^2) + \xi(dZ/dt) + Z + EZ^2 = L(t) \quad (9)$$

In the following, all terms are assumed to be dimensionless, and the dashes are omitted for convenience sake.

Putting $E=0$ in Eq.(9), a linear equation is obtained. A particular solution of the linear equation is

$$Z_1(t) = \sum_{n=1}^{\infty} G(p_n) \cos \{p_n t + \beta(p_n)\} \Delta p \quad (10)$$

where

$$G(p_n) = F(p_n)\Phi(p_n) \quad (11)$$

$$\Phi(p_n) = 1/\sqrt{(1-p_n^2)^2 + \xi^2 p_n^2} \leq \tau \quad (12)$$

$$\tau = 1/\xi\sqrt{1-(\xi/2)^2} \quad (\xi \leq \sqrt{2}) \quad (13)$$

$$\tan \beta(p_n) = -\xi p_n/(1-p_n^2) \quad (14)$$

This solution is given in the form

$$Z_1(t) = RL(t) \quad (15)$$

where R is the operation defined as follows: the amplitude of each frequency component of the function $L(t) = \sum_{n=1}^{\infty} F(p_n) \cos p_n t \Delta p$ is multiplied by $\Phi(p_n)$ and the phase of each frequency component shifted by $\beta(p_n)$. This operation is defined for any function having finite norms.

In case $E \neq 0$, $Z(t)$ is also given in the form

$$Z(t) = R\{L(t) - EZ^2(t)\} \quad (16)$$

with the operation R , and it is equivalent to considering the integral equation instead of Eq.(9).

4. Iteration method

Our purpose is now to perform on Eq.(16) the iterations

$$Z_{n+1} = R(L - EZ_n^2) \quad (n=0, 1, 2, \dots) \quad (17)$$

starting with $Z_0=0$. $Z(t)$ is a limit function, when $n \rightarrow \infty$, and the convergence of the sequence Z_n is established, if the following inequality is satisfied:⁽¹⁾

$$|E| < 1/4\tau^2\gamma = \xi^2 \{1 - (\xi/2)^2\} / 4\|L\| \quad (\xi \leq \sqrt{2}) \quad (18)$$

Putting $Z_0=0$ in Eq.(17), it is reduced that $Z_1=RZ$, and it is given by Eq.(15).

The iteration given by Eq.(17) can be rewritten in the following expressions:

$$Z_n = RL_{n-1}, \quad Z_{n+1} = RL_n \quad (19)$$

$$L_n = L - E(RL_{n-1})^2 \quad (20)$$

where L_n is an equivalent load of the n th order, and Z_{n+1} is the n th approximate solution. The concept of the equivalent load is the same as that of a symmetrical nonlinear system, which was illustrated in the previous paper. The equivalent load of zero order means the actual load L , and the approximate solution of zero order means the displacement of the linear system Z_1 .

Now considering the 1st approximate solution $Z_2(t)$ from Eq.(19),

$$Z_2(t) = RL_1(t) \quad (21)$$

The 1st equivalent load $L_1(t)$ is

$$L_1(t) = L(t) - E\{RL(t)\}^2 \quad (22)$$

from Eq.(20). $L_1(t)$ can be expressed as a series

$$L_1(t) = \sum_{n=0}^{\infty} F_1(p_n) \cos \{p_n t + \gamma(p_n)\} \Delta p \quad (23)$$

Then the spectrum of $L_1(t)$ is defined as follows:

$$f_1(p_n) = (1/2)F_1^2(p_n)\Delta p \quad (24)$$

So the mean square value of $L_1(t)$ is

$$\bar{L}_1^2 = \sum_{n=0}^{\infty} f_1(p_n) \Delta p \quad (25)$$

From the substitution of Eq.(23) into Eq.(21), it follows that

$$Z_2(t) = \sum_{n=0}^{\infty} G_1(p_n) \cos \{p_n t + \gamma(p_n) + \beta(p_n)\} \Delta p \quad (26)$$

where

$$G_1(p_n) = F_1(p_n)\Phi(p_n) \quad (27)$$

Hence, the spectrum of $Z_2(t)$ is

$$g_2(p_n) = (1/2)G_1^2(p_n)\Delta p = f_1(p_n)\Phi^2(p_n) \quad (28)$$

from Eq.(24), and then the mean square value of $Z_2(t)$ is

$$\bar{Z}_2^2 = \sum_{n=0}^{\infty} g_2(p_n) \Delta p = \sum_{n=0}^{\infty} f_1(p_n)\Phi^2(p_n) \Delta p \quad (29)$$

5. Estimation of $Z_1^2(t)$

In view of Eq.(22), the square of the displacement in the linear system $Z_1=RL$ must be estimated, in order to obtain the 1st equivalent load $L_1(t)$.

From Eq.(10),

$$\begin{aligned} Z_1^2(t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} G(p_n)G(p_m) \cos \{p_n t + \beta(p_n)\} \\ &\quad \times \cos \{p_m t + \beta(p_m)\} (\Delta p)^2 \\ &= (1/2) \{y_1(t) + y_2(t)\} \end{aligned} \quad (30)$$

where

$$y_1(t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} G_n G_m \cos (p_n + m t + \beta_n + \beta_m) (\Delta p)^2 \quad (31)$$

$$y_2(t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} G_n G_m \cos (p_n - m t + \beta_n - \beta_m) (\Delta p)^2 \quad (32)$$

using abbreviations

$$G_n \equiv G(p_n), \quad \beta_n \equiv \beta(p_n), \quad p_{n \pm m} \equiv p_n \pm p_m$$

At first, putting $n+m=i$, i. e. $m=i-n>0$ in

Eq.(31),

$$\begin{aligned} y_1(t) &= \sum_n \sum_i G_n G_{i-n} \cos(p_i t + \beta_n + \beta_{i-n}) (\Delta p)^2 \\ &= \sum_{i=1}^{\infty} A_i^{(1)} \cos p_i t \Delta p - \sum_{i=1}^{\infty} B_i^{(1)} \sin p_i t \Delta p \dots (33) \end{aligned}$$

where

$$\left. \begin{aligned} A_i^{(1)} &= \sum_{n=0}^i G_n G_{i-n} \cos(\beta_n + \beta_{i-n}) \Delta p \\ B_i^{(1)} &= \sum_{n=0}^i G_n G_{i-n} \sin(\beta_n + \beta_{i-n}) \Delta p \end{aligned} \right\} (i>0) \dots (34)$$

$$(G_0 \equiv 0).$$

Secondly, putting $n-m=i$, i. e. $m=n-i>0$ in

Eq.(32),

$$\begin{aligned} y_2(t) &= \sum_n \sum_i G_n G_{n-i} \cos(p_i t + \beta_n - \beta_{n-i}) (\Delta p)^2 \\ &= \sum_{i=-\infty}^{\infty} A_i^{(2)} \cos p_i t \Delta p - \sum_{i=-\infty}^{\infty} B_i^{(2)} \sin p_i t \Delta p \dots (35) \end{aligned}$$

where

$$\left. \begin{aligned} A_i^{(2)} &= \sum_{n=i}^{\infty} G_n G_{n-i} \cos(\beta_n - \beta_{n-i}) \Delta p \\ B_i^{(2)} &= \sum_{n=i}^{\infty} G_n G_{n-i} \sin(\beta_n - \beta_{n-i}) \Delta p \end{aligned} \right\} (i>0) \dots (36)$$

$$\left. \begin{aligned} A_i^{(2)} &= \sum_{n=0}^{\infty} G_n G_{n-i} \cos(\beta_n - \beta_{n-i}) \Delta p \\ B_i^{(2)} &= \sum_{n=0}^{\infty} G_n G_{n-i} \sin(\beta_n - \beta_{n-i}) \Delta p \end{aligned} \right\} (i \leq 0) \dots (37)$$

From the fact that G is an even function and β is

$$Z_1^2(t) = \frac{1}{2} \sum_{i=-\infty}^{\infty} (A_i \cos p_i t - B_i \sin p_i t) \Delta p = \frac{1}{2} \left\{ A_0 + \sum_{i=1}^{\infty} (u_i \cos p_i t - v_i \sin p_i t) \right\} \Delta p \dots (43)$$

where

$$A_0 = \sum_{n=1}^{\infty} G_n^2 \Delta p \dots (44)$$

$$\begin{aligned} u_i &= A_i + A_{-i} = \sum_{n=1}^{\infty} G_n \{ G_{i-n} \cos(\beta_n + \beta_{i-n}) + G_{i+n} \cos(\beta_n - \beta_{i+n}) \} \Delta p \\ &= \sum_{n=-\infty}^{\infty} G_n G_{i-n} \cos(\beta_n + \beta_{i-n}) \Delta p \dots (45) \end{aligned}$$

$$\begin{aligned} v_i &= B_i - B_{-i} = \sum_{n=1}^{\infty} G_n \{ G_{i-n} \sin(\beta_n + \beta_{i-n}) - G_{i+n} \sin(\beta_n - \beta_{i+n}) \} \Delta p \\ &= \sum_{n=-\infty}^{\infty} G_n G_{i-n} \sin(\beta_n + \beta_{i-n}) \Delta p \dots (46) \end{aligned}$$

While, from Eqs.(12) and (14),

$$\left. \begin{aligned} \cos(\beta_n + \beta_m) &= \Phi_n \Phi_m \{ (1-p_n^2)(1-p_m^2) - \xi^2 p_n p_m \} \\ \sin(\beta_n + \beta_m) &= -\xi \Phi_n \Phi_m \{ p_n(1-p_m^2) + (1-p_n^2)p_m \} \end{aligned} \right\} \dots (47)$$

Hence, Eqs.(45) and (46) are reduced to

$$u_i = \sum_{n=-\infty}^{\infty} F_n \Phi_n^2 F_{i-n} \Phi_{i-n}^2 \{ (1-p_n^2)(1-p_{i-n}^2) - \xi^2 p_n p_{i-n} \} \Delta p \dots (45)'$$

$$v_i = -\xi \sum_{n=-\infty}^{\infty} F_n \Phi_n^2 F_{i-n} \Phi_{i-n}^2 \{ p_n(1-p_{i-n}^2) + (1-p_n^2)p_{i-n} \} \Delta p \dots (46)'$$

an odd function,

$$G_{-i} = G_{i-n}, \quad \beta_{-i} = -\beta_{i-n}$$

hence Eqs.(36) and (37) are

$$\left. \begin{aligned} A_i^{(2)} &= \sum_{n=-i}^{\infty} G_n G_{i-n} \cos(\beta_n + \beta_{i-n}) \Delta p \\ B_i^{(2)} &= \sum_{n=-i}^{\infty} G_n G_{i-n} \sin(\beta_n + \beta_{i-n}) \Delta p \end{aligned} \right\} (i>0) \dots (36)'$$

$$\left. \begin{aligned} A_i^{(2)} &= \sum_{n=0}^{\infty} G_n G_{i-n} \cos(\beta_n + \beta_{i-n}) \Delta p \\ B_i^{(2)} &= \sum_{n=0}^{\infty} G_n G_{i-n} \sin(\beta_n + \beta_{i-n}) \Delta p \end{aligned} \right\} (i \leq 0) \dots (37)'$$

Substituting Eqs.(33) and (35) in Eq.(30),

$$\begin{aligned} Z_1^2(t) &= \frac{1}{2} \left\{ \sum_{i=1}^{\infty} (A_i^{(1)} + A_i^{(2)}) \cos p_i t \Delta p \right. \\ &\quad \left. - \sum_{i=1}^{\infty} (B_i^{(1)} + B_i^{(2)}) \sin p_i t \Delta p \right. \\ &\quad \left. + \sum_{i=-\infty}^0 A_i^{(2)} \cos p_i t \Delta p - \sum_{i=-\infty}^0 B_i^{(2)} \sin p_i t \Delta p \right\} \dots (38) \end{aligned}$$

From Eqs.(34) and (36)',

$$A_i^{(1)} + A_i^{(2)} = \sum_{n=0}^{\infty} G_n G_{i-n} \cos(\beta_n + \beta_{i-n}) \Delta p \dots (39)$$

$$B_i^{(1)} + B_i^{(2)} = \sum_{n=0}^{\infty} G_n G_{i-n} \sin(\beta_n + \beta_{i-n}) \Delta p \dots (40)$$

Eqs.(39) and (40) are identical with $A_i^{(2)}$ and $B_i^{(2)}$ of Eq.(37)', respectively. Hence, if we put

$$A_i = \sum_{n=0}^{\infty} G_n G_{i-n} \cos(\beta_n + \beta_{i-n}) \Delta p \dots (41)$$

$$B_i = \sum_{n=0}^{\infty} G_n G_{i-n} \sin(\beta_n + \beta_{i-n}) \Delta p \dots (42)$$

then we obtain, from Eq.(38),

where

$$\Phi_n \equiv \Phi(p_n), \quad F_n \equiv F(p_n)$$

6. Spectrum of the 1st order equivalent load

In view of Eqs.(28) and (29), the spectrum $f_1(p_n)$ of the 1st equivalent load $L_1(t)$ must be obtained, in order to evaluate the spectrum $g_2(p_n)$ and the mean square value of the 1st approximate solution $Z_2(t)$. If we calculate the mean square value of Eq.(22) and substitute the spectrum $f(p_n)$ of the actual load $L(t)$ in it, $f_1(p_n)$ is expressed as a function of $f(p_n)$.

From Eq.(22), a relation between the mean square values of L_1 , L and Z_1^2 are given by

$$\bar{L}_1^2 = \bar{L}^2 - 2E\bar{L}\bar{Z}_1^2 + E^2\bar{Z}_1^4 \dots \dots \dots (48)$$

The 3rd term of the right hand side of Eq.(48) has a sufficiently small value compared with the other terms, as long as the condition of convergence given by Eq.(18) is satisfied. If the 3rd term is neglected, the mean square value of the 1st equivalent load is approximated by

$$\bar{L}_1^2 = \bar{L}^2 - 2E\bar{L}\bar{Z}_1^2 \dots \dots \dots (49)$$

in which $\bar{L}\bar{Z}_1^2$ is from Eqs.(1) and (43)

$$\begin{aligned} \bar{L}\bar{Z}_1^2 = & \frac{1}{T} \int_0^T L(t) Z_1^2(t) dt = \frac{1}{2} \sum_{n=1}^{\infty} F_n \Delta p \left[A_0 \frac{1}{T} \int_0^T \cos p_n t dt + \sum_{i=1}^{\infty} \left\{ u_i \frac{1}{T} \int_0^T \cos p_n t \cos p_i t dt \right. \right. \\ & \left. \left. - v_i \frac{1}{T} \int_0^T \cos p_n t \sin p_i t dt \right\} \right] \Delta p = \frac{1}{4} \sum_{i=1}^{\infty} (F_i \Delta p) (u_i \Delta p) \dots \dots \dots (50) \end{aligned}$$

while

$$F_i \Delta p = \pm \sqrt{2f_i \Delta p} \quad (f_0 \equiv 0, f_{-i} \equiv f_i) \dots \dots \dots (51)$$

from Eq.(5). Substituting Eq.(51) in Eq.(45)', u_i can be expressed by f_i . Hence Eq.(50) is reduced to

$$\bar{L}\bar{Z}_1^2 = \pm \frac{1}{\sqrt{2}} \sum_{i=1}^{\infty} \sqrt{f_i \Delta p} \sum_{n=-\infty}^{\infty} \sqrt{f_n \Delta p f_{i-n} \Delta p} \Phi_n^2 \Phi_{i-n}^2 \{ (1-p_n^2)(1-p_{i-n}^2) - \xi^2 p_n p_{i-n} \} \dots \dots \dots (52)$$

where the positive sign is taken if $F_i > 0$, and the negative sign if $F_i < 0$.

Substituting $\bar{L}_1^2$ of Eq.(25), $\bar{L}^2$ of Eq.(4) and $\bar{L}\bar{Z}_1^2$ of Eq. (52) into Eq. (49), the relationship of the spectra is obtained as follows:

$$f_1(p_i) = f_i \mp \sqrt{2} \frac{E}{\Delta p} \sqrt{f_i \Delta p} \sum_{n=-\infty}^{\infty} \sqrt{f_n \Delta p f_{i-n} \Delta p} \Phi_n^2 \Phi_{i-n}^2 \{ (1-p_n^2)(1-p_{i-n}^2) - \xi^2 p_n p_{i-n} \} \dots \dots \dots (53)$$

The above expression gives the spectrum of the 1st equivalent load $f_1(p_i)$.

In case the spectrum of external load f_i is uniform over a frequency band under consideration,

$$f_i \equiv f_c \dots \dots \dots (54)$$

Hence, from Eq.(53),

$$\begin{aligned} \frac{f_1(p_i)}{f_c} = & 1 \mp E \sqrt{2f_c \Delta p} \sum_{n=-\infty}^{\infty} \Phi_n^2 \Phi_{i-n}^2 \{ (1-p_n^2)(1-p_{i-n}^2) - \xi^2 p_n p_{i-n} \} \dots \dots \dots (55) \\ & (\Phi_0 \equiv 0) \end{aligned}$$

where the positive sign is taken if $F < 0$, and the negative sign if $F > 0$.

7. Spectrum and mean square value of displacement

The spectrum $g_2(p_i)$ of the 1st approximate solution of displacement $Z_2(t)$ is obtained by the substitution of Eq.(53) in Eq.(28). In the case of the uniform spectrum of load, Eq.(55) is used instead of Eq.(53). The spectrum of displacement in the linear case is given as follows:

$$g_1(p_i) = f_i \Phi_i^2 \dots \dots \dots (56)$$

The mean square value of the 1st approximate solution of displacement $\bar{Z}_2^2$ is obtained by Eq.(29). In the case of the uniform spectrum of load, it is

$$\bar{Z}_2^2 = f_c \Delta p \sum_{i=1}^{\infty} \Phi_i^2 \mp E f_c \Delta p \sqrt{2f_c \Delta p} \sum_{i=1}^{\infty} \Phi_i^2 \sum_{n=-\infty}^{\infty} \Phi_n^2 \Phi_{i-n}^2 \{ (1-p_n^2)(1-p_{i-n}^2) - \xi^2 p_n p_{i-n} \} \dots \dots \dots (57)$$

The mean square value of displacement in the linear case is

$$\bar{Z}_1^2 = f_c \Delta p \sum_{i=1}^{\infty} \Phi_i^2 \dots \dots \dots (58)$$

Hence, from Eq.(57),

$$\bar{Z}_2^2 - \bar{Z}_1^2 = \mp \sqrt{2} E (f_c \Delta p)^{3/2} \sum_{i=1}^{\infty} \Phi_i^2 \sum_{n=-\infty}^{\infty} \Phi_n^2 \Phi_{i-n}^2 \{ (1-p_n^2)(1-p_{i-n}^2) - \xi^2 p_n p_{i-n} \} \dots \dots \dots (59)$$

$$(\Phi_0 \equiv 0)$$

where the positive sign is taken if $F_i < 0$, and the negative sign if $F_i > 0$.

8. Numerical example

A numerical example of the spectrum of the 1st equivalent load Eq.(55) is shown in Fig.1. In this example, the damping coefficient $\xi = 0.5$, and the load spectrum is uniform ($\Delta p = 0.1$).

From this figure, we can see the following facts: In the case of $E > 0$ and $F_i > 0$ or of $E < 0$ and $F_i < 0$, the low frequency components decrease and the components in the vicinity of $p_i = 2$ increase. On the other hand, in the case of $E < 0$ and $F_i > 0$ or of $E > 0$ and $F_i < 0$, the low frequency components increase and the components in the vicinity of $p_i = 2$ decrease. These facts imply that if the loading has a maximum peak value in the direction of the hard spring, the low frequency components decrease; while if the loading has a maximum peak value in the direction of the soft spring, the low frequency components increase. As a matter of course, the larger the load spectrum f_c is, the larger the

Fig. 1 Difference between 1st equivalent load spectrum $f_1(p_i)$ and load spectrum f_c

Fig. 2 Difference between 1st equivalent load spectrum $f_1(p_i)$ and load spectrum f_c

difference $f_1(p_i) - f_c$ is.

In Fig.2 $f_1(p_i) - f_c$ is shown, where the damping coefficient $\xi = 0.5$, the nonlinearity $E = 0.3$, the load spectrum $f_c = 1 \times 10^{-4}$ ($\Delta p = 0.1$) and $F_i < 0$. In this example, the amplitude of each frequency component of loading $F_i \Delta p = -\sqrt{2 f_c \Delta p} = -0.0045$, and if the frequency range of loading is $0.1 \leq p_i \leq 4.0$, the maximum value of loading $\|L\| = \sum_{i=1}^{40} |F_i| \Delta p = 0.18$ and $\bar{L}^2 = \sum_{i=1}^{40} f_c \Delta p = 0.0004$, $\sqrt{\bar{L}^2} = 0.02$.

The difference between the spectrum $g_2(p_i)$ of the 1st approximate solution of displacement and the spectrum $g_1(p_i)$ of displacement in the linear case is shown in Fig.3, from which we can see the following facts: In the case of $E F_i > 0$, the frequency components of displacement decrease for the frequencies below the resonance and slightly increase for the higher frequencies. In the case of $E F_i < 0$, these facts are reversed.

In Fig.4 $g_2(p_i) - g_1(p_i)$ is shown, where $\xi = 0.5$, $E = 0.3$, $f_c = 1 \times 10^{-4}$ ($\Delta p = 0.1$) and $F_i < 0$. If $0.1 \leq p_i \leq 4.0$ in this example, the mean square value of displacement in the linear system $\bar{Z}_1^2 = f_c \sum_{i=1}^{40} \Phi_i^2 \Delta p = \sum_{i=1}^{40} g_1(p_i) \Delta p = 3.086 \times 10^{-4}$, $\sqrt{\bar{Z}_1^2} = 0.0176$.

The difference of the mean square value of

Fig. 3 Difference between 1st approximate displacement spectrum $g_2(p_i)$ and displacement spectrum in linear system $g_1(p_i)$

Fig. 4 Difference between 1st approximate displacement spectrum $g_2(p_i)$ and displacement spectrum in linear system $g_1(p_i)$

displacement is evaluated from Eq.(59) and shown in Figs.5 and 6, in which the values of $(\bar{Z}_2^2 - \bar{Z}_1^2)$ are plotted against the nonlinearity E for $F_i < 0$ and $F_i > 0$, respectively. From these figures, it is shown that the mean square value of displacement is smaller in the case of $EF_i > 0$ and larger in the case of $EF_i < 0$ than that of a linear system. These results are more remarkable in the system under heavy loading.

9. Experiment by analog computer

The mean square values of displacement obtained as a result of the 1st order approximation are now compared with experimental results obtained through an analog computer of low speed type.

Fig. 5 Difference of mean square values of displacement between 1st approximation $\bar{Z}_2^2$ and linear case $\bar{Z}_1^2 (F_i < 0)$

Fig. 6 Difference of mean square values of displacement between 1st approximation $\bar{Z}_2^2$ and linear case $\bar{Z}_1^2 (F_i > 0)$

In this experiment, the number of frequency components of load spectrum to be generated is restricted within 6, and then the load function is

$$L(t) = \sum_{i=1}^6 A_i(t), \quad A_i(t) = V \cos p_i t \quad \dots\dots\dots (60)$$

where $A_i(t)$ is generated by solving the equation: $\ddot{A}_i + p_i^2 A_i = 0$ ($A_i = V$, $\dot{A}_i = 0$ at $t = 0$). If the voltage of V is 15 volts and 100 volts is adopted as a reference voltage, the value of amplitude of each component V is uniformly equal to 0.15. The frequencies of each component are $p_1 = 0.25$, $p_2 = 0.50$, $p_3 = 0.75$, $p_4 = 1.00$, $p_5 = 1.25$, $p_6 = 1.50$ rad/sec ($\Delta p = 0.25$ rad/sec). From these values, $f_c = V^2/2\Delta p = 0.0450$, $\|L\| = 6V = 0.90 (< 1.00)$, $\bar{L}^2 = 6(V^2/2) = 0.0675$ and $\sqrt{\bar{L}^2} = 0.26$.

Fig. 7 Block diagram

Fig. 8 Example of records

Fig. 9 Relation between displacement and nonlinearity

Putting the damping coefficient $\xi=1$ and the nonlinearity $E=0, 0.05, 0.10, 0.20, 0.40$, the influence of nonlinearity upon the displacement is computed. The equation to be analyzed is

$$\ddot{Z} + \xi \dot{Z} + Z + EZ^2 = L(t) \quad (61)$$

from Eq.(9); the block diagram is shown in Fig. 7.

Z^2, Z, L^2, L are simultaneously recorded, and an example of recording is shown in Fig. 8. The discrepancies of the mean square values of displacement with the linear case are plotted in Fig. 9, in which the theoretical value of the 1st order approximation is expressed by a curve.

The experimental values obtained through the analog computer do not perfectly agree with the theoretical values, on account of the errors in estimation of mean values from the records and the limited accuracy of computer units. However, the main feature of results shows a good agreement. For large values of nonlinearity, the 1st approximate value is, of course, unavailable.

Reference

- (1) J. J. Stoker: *Nonlinear Vibrations in Mechanical and Electrical Systems*, Appendix II (1950), Interscience Publishers.

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Heat Conduction in a Semi-Infinite Solid Heated by a Moving Heat Source along the Boundary*

By Takashi NAKADA** and Seiya HASHIMOTO***

1. When a heat source travels along the surface of a semi-infinite solid with a constant velocity v , the temperature θ in the semi-infinite solid is given by

$$\frac{\partial^2 \theta}{\partial w^2} + \beta \frac{\partial \theta}{\partial w} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

where $w = x - vt$, $\beta = v/a$, and the temperature distribution of the heat source can be expressed as $\theta(w)$.

2. Though the eigen-values of this partial differential equation are complex numbers, the solution can be obtained in the form of a real Fourier's integral. Thus the temperature distribution in a semi-infinite solid and heat transmission through the surface are calculated.
3. This Fourier's integral was numerically calculable by a digital computer.
4. It was much easier to solve the above partial differential equation by the accelerated Liebmann's method using a digital computer than to solve it by the Fourier's integral method, and the result obtained by one method coincided with the result obtained by the other.

1. Introduction

The problems of heat conduction in an infinite

solid or a plate which is heated by a moving heat source with a constant velocity have been treated by H.S. Carslaw and J.C. Jaeger⁽¹⁾, D. Rosenthal⁽²⁾ and P.S. Schneider⁽³⁾. Rosenthal assumed that a line source of heat is on the z -axis of a rectangular coordinate system (x, y, z) , and the line heat source moves with a constant velocity v along the x -axis. He used the moving coordinate system,

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