Fig. 9 Relation between displacement and nonlinearity

Putting the damping coefficient $\xi=1$ and the nonlinearity $E=0, 0.05, 0.10, 0.20, 0.40$, the influence of nonlinearity upon the displacement is computed. The equation to be analyzed is

$$\ddot{Z} + \xi \dot{Z} + Z + EZ^2 = L(t) \quad (61)$$

from Eq.(9); the block diagram is shown in Fig. 7.

Z^2, Z, L^2, L are simultaneously recorded, and an example of recording is shown in Fig. 8. The discrepancies of the mean square values of displacement with the linear case are plotted in Fig. 9, in which the theoretical value of the 1st order approximation is expressed by a curve.

The experimental values obtained through the analog computer do not perfectly agree with the theoretical values, on account of the errors in estimation of mean values from the records and the limited accuracy of computer units. However, the main feature of results shows a good agreement. For large values of nonlinearity, the 1st approximate value is, of course, unavailable.

Reference

- (1) J. J. Stoker: *Nonlinear Vibrations in Mechanical and Electrical Systems*, Appendix II (1950), Interscience Publishers.

536.21

Heat Conduction in a Semi-Infinite Solid Heated by a Moving Heat Source along the Boundary*

By Takashi NAKADA** and Seiya HASHIMOTO***

1. When a heat source travels along the surface of a semi-infinite solid with a constant velocity v , the temperature θ in the semi-infinite solid is given by

$$\frac{\partial^2 \theta}{\partial w^2} + \beta \frac{\partial \theta}{\partial w} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

where $w = x - vt$, $\beta = v/a$, and the temperature distribution of the heat source can be expressed as $\theta(w)$.

2. Though the eigen-values of this partial differential equation are complex numbers, the solution can be obtained in the form of a real Fourier's integral. Thus the temperature distribution in a semi-infinite solid and heat transmission through the surface are calculated.
3. This Fourier's integral was numerically calculable by a digital computer.
4. It was much easier to solve the above partial differential equation by the accelerated Liebmann's method using a digital computer than to solve it by the Fourier's integral method, and the result obtained by one method coincided with the result obtained by the other.

1. Introduction

The problems of heat conduction in an infinite

solid or a plate which is heated by a moving heat source with a constant velocity have been treated by H.S. Carslaw and J.C. Jaeger⁽¹⁾, D. Rosenthal⁽²⁾ and P.S. Schneider⁽³⁾. Rosenthal assumed that a line source of heat is on the z -axis of a rectangular coordinate system (x, y, z) , and the line heat source moves with a constant velocity v along the x -axis. He used the moving coordinate system,

* Received 30th November, 1960.

** Professor, The Research Laboratory of Precision Machinery and Electronics, Tokyo Institute of Technology.

*** Research Fellow, Central Research Laboratory, Hitachi Ltd., Kokubunji, Tokyo.

$w = x - vt$, and the Fourier's equation of heat conduction, then got the solution $\theta = ce^{-(\beta/2)w} K_0[(\beta/2)r]$, where $r = \sqrt{y^2 + w^2}$. The isotherms of this solution are all symmetrical with respect to w -axis and intersect the w -axis at right angles.

The purpose of this paper is to investigate how the heat generated in the lubricating oil film between the mating surfaces of gears is taken away into the interior of the gear body. First, we assumed the temperature distribution in the oil film (heat source) or the distribution of the heat generation near the mating point, and assumed that the heat source travelled along the surface of the tooth profile. Further, we considered the heat transfer coefficient between the oil film and the tooth surface, and proceeded to analysis on the assumption that the gear tooth was a semi-infinite solid confined by its profile. In this case, isotherms of the interior of the semi-infinite solid generally are not perpendicular to the boundary, so it is difficult to express the temperature of the interior of the semi-infinite solid using the Rosenthal's modified Bessel function. For this reason, we assumed the boundary conditions in terms of Fourier's integral, and attempted to obtain the solution of the Fourier's integral having the eigen-value of complex variable.

On the other hand, we used a digital computer to solve the differential equation numerically and compared the solution with the result of the analysis.

Nomenclature

- θ : temperature within semi-infinite solid
 Θ : temperature of heat source
 q : heat flux across boundary to semi-infinite solid, cal/cm²sec
 k : thermal conductivity of semi-infinite solid
 α : heat-transfer coefficient of boundary
 v : velocity of moving heat source
 $a = k/(c\rho)$: thermal diffusivity
 $\beta = v/a$
 $h = \alpha/k$

Fig. 1 Dual coordinate system for moving heat source

The coordinate system oxy of Fig. 1 is fixed to the semi-infinite solid, and the differential equation for temperature inside the semi-infinite solid is expressed by,

$$\frac{\partial \theta}{\partial t} = a \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) \quad (1)$$

The temperature distribution of the heat source adjacent to the surface of the semi-infinite solid is fixed to the moving coordinate system $o' \Theta w$ and moves to the right with a constant velocity v . That is, with respect to the fixed coordinate system, the heat source is defined by $\Theta(x-vt) = \Theta(w)$. Here, the heat flux across the boundary is defined by the following equation,

$$-k \left(\frac{\partial \theta}{\partial y} \right)_{y=0} = q = \alpha \{ \Theta(x-vt) - \theta_{y=0} \} \quad (2)$$

Such a problem might arise actually in the following cases.

1. Welding and gas cutting of plates.
2. Continuous heat treatment of steel wires.
3. Local heating and refining of germanium materials.
4. Heat dissipation by lubricating oil film of gear tooth surface and temperature of lubricating oil film.
5. Fluid heat transportation.

2. Differential equation obtained by transformation of coordinate system, and its solution

If $w = x - vt$, then

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial w} \frac{\partial w}{\partial t} = -v \frac{\partial}{\partial w}, \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial w^2}$$

Substituting these relations in Eq.(1), we have the following equation of heat conduction based on the moving coordinate system,

$$\frac{\partial^2 \theta}{\partial w^2} + \beta \frac{\partial \theta}{\partial w} + \frac{\partial^2 \theta}{\partial y^2} = 0 \quad (3)$$

where $\beta = v/a$.

Now, we assume the solution of Eq.(3) is as follows,

$$\theta = Y \cdot e^{j\lambda w} \quad (4)$$

where Y is a function of y only and $j = \sqrt{-1}$. In this case, Y must satisfy the following ordinary differential equation,

$$\frac{d^2 Y}{dy^2} - \lambda(\lambda - j\beta) Y = 0 \quad (5)$$

To satisfy the boundary condition, we select the following particular solution in the following two ranges $\lambda \geq 0$

$$\left. \begin{aligned} Y &= A e^{-\sqrt{\lambda(\lambda - j\beta)} \cdot y}, & \text{for } \lambda \geq 0 \\ Y &= A e^{\sqrt{\lambda(\lambda - j\beta)} \cdot y}, & \text{for } \lambda \leq 0 \end{aligned} \right\} \quad (6)$$

The boundary condition, that is, Eq.(2), in

terms of Fourier's integral becomes

$$-\left(\frac{\partial \theta}{\partial y}\right)_{y=0} + h\theta_{y=0} = h\theta(w) = \frac{h}{2\pi} \int_{-\infty}^{\infty} e^{j\lambda w} \cdot d\lambda \int_{-\infty}^{\infty} \Theta(\xi) e^{-j\lambda \xi} \cdot d\xi \quad (7)$$

If the coefficient A of Eq.(6) is so determined that Eq.(4) may satisfy Eq.(7), the temperature of a semi-infinite solid, θ , and the heat flux across the boundary, $q = -k(\partial\theta/\partial y)_0$, are given respectively by

$$\theta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{h}{h - f(\lambda, \beta)} e^{f(\lambda, \beta)y} \cdot e^{j\lambda w} d\lambda \int_{-\infty}^{\infty} \Theta(\xi) e^{-j\lambda \xi} \cdot d\xi \quad (8)$$

$$q = \frac{\alpha}{2\pi} \int_{-\infty}^{\infty} \frac{f(\lambda, \beta)}{f(\lambda, \beta) - h} e^{j\lambda w} \cdot d\lambda \int_{-\infty}^{\infty} \Theta(\xi) e^{-j\lambda \xi} \cdot d\xi \quad (9)$$

where $f(\lambda, \beta)$ implies the following:

$$\left. \begin{aligned} f(\lambda, \beta) &= -\sqrt{\lambda(\lambda - j\beta)} = \sqrt{\frac{\lambda}{2}} \left\{ \sqrt{\lambda^2 + \beta^2} + \lambda - j\sqrt{\lambda^2 + \beta^2 - \lambda} \right\} \quad \text{for } \lambda > 0 \\ f(\lambda, \beta) &= +\sqrt{\lambda(\lambda - j\beta)} = -\sqrt{\frac{\lambda}{2}} \left\{ \sqrt{\lambda^2 + \beta^2} + \lambda + j\sqrt{\lambda^2 + \beta^2 - \lambda} \right\} \quad \text{for } \lambda < 0 \end{aligned} \right\} \quad (10)$$

Substitutions of Eq.(10) into Eq.(8) and Eq.(9) yield the following Fourier's integrals of real variables, which give the temperature distribution in the semi-infinite solid and the heat flux across the boundary surfaces, respectively:

$$\begin{aligned} \theta(w, y) &= \frac{h}{\pi} \int_0^{\infty} \frac{e^{-\sqrt{\lambda/2} \sqrt{\lambda^2 + \beta^2} \cdot y}}{\lambda \sqrt{\lambda^2 + \beta^2} + \sqrt{2} h \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2} + \lambda + h^2} \\ &\quad \times \left[\left(\sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} + \lambda + h \right) \int_{-\infty}^{\infty} \Theta(\xi) \cos \left\{ \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} \cdot y + \lambda(w - \xi) \right\} d\xi \right. \\ &\quad \left. - \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} - \lambda \cdot \int_{-\infty}^{\infty} \Theta(\xi) \sin \left\{ \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} \cdot y + \lambda(w - \xi) \right\} d\xi \right] d\lambda \quad (11) \end{aligned}$$

$$\begin{aligned} q(w) &= \frac{\alpha}{\pi} \int_0^{\infty} \frac{\lambda \sqrt{\lambda^2 + \beta^2} + h \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2}}{\lambda \sqrt{\lambda^2 + \beta^2} + \sqrt{2} h \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2} + \lambda + h^2} \left\{ \int_{-\infty}^{\infty} \Theta(\xi) \cos \lambda(w - \xi) d\xi \right\} d\lambda \\ &\quad + \frac{\alpha}{\pi} \int_0^{\infty} \frac{h \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2}}{\lambda \sqrt{\lambda^2 + \beta^2} + \sqrt{2} h \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2} + \lambda + h^2} \left\{ \int_{-\infty}^{\infty} \Theta(\xi) \sin \lambda(w - \xi) d\xi \right\} d\lambda \quad (12) \end{aligned}$$

Example:

On the proximity surfaces of the mating teeth of gear, it can be considered that the heat is generated in the lubricating oil film, and the temperature distribution curve of the oil film along the tooth surface is

Fig. 2 The temperature distribution of heat source along the tooth surface

$$\begin{aligned} \Theta &= 1, hl = 0.167, \beta l = 10, s/l = 1/4 \\ \text{For } -(l-s) < w < (l+s), \\ \Theta(w) &= \Theta \left\{ 1 - \frac{1}{l^2} (w-s)^2 \right\} \\ \text{For } w < -(l+s), w > (l+s), \\ \Theta(w) &= 0 \end{aligned}$$

Curves are obtained by the accelerated Liebmann's method. Small circles show calculated values by Fourier's integral

Fig. 3 Isotherms in a semi-infinite solid

assumed to be approximately parabola, and its heat source moves along the tooth surface as shown in Fig. 2.

We assume the temperature distribution for $-(l-s) < w < (l+s)$,

$$\Theta(w) = \Theta \left\{ 1 - \frac{1}{l^2} (w-s)^2 \right\}$$

and for $w < -(l-s)$ or $w > (l+s)$,

$$\Theta(w) = 0$$

Then, Eq.(11) will be

$$\begin{aligned} \theta(w, y) = & \Theta \frac{4h}{\pi l} \int_0^\infty \left(\frac{\sin \lambda l}{\lambda l} - \cos \lambda l \right) e^{-\sqrt{\lambda/2} \sqrt{\lambda^2 + \beta^2} + \lambda \cdot y} \\ & \times \frac{\left[\left(\sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} + h \right) \cos \left\{ \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} - \lambda \cdot y + \lambda(w-s) \right\} \right.}{\lambda^2 (\lambda \sqrt{\lambda^2 + \beta^2} + \sqrt{2} h \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2} + \lambda + h^2)} \\ & \left. - \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} - \lambda \sin \left\{ \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} - \lambda \cdot y + \lambda(w-s) \right\} \right] d\lambda \dots\dots\dots (a) \end{aligned}$$

and Eq.(12) will be

$$\begin{aligned} q(w) = & \Theta \frac{4\alpha}{\pi l} \int_0^\infty \left(\frac{\sin \lambda l}{\lambda l} - \cos \lambda l \right) \\ & \times \frac{\left(\lambda \sqrt{\lambda^2 + \beta^2} + h \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} + \lambda \right) \cos \lambda(w-s) + h \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} - \lambda \sin \lambda(w-s)}{\lambda^2 (\lambda \sqrt{\lambda^2 + \beta^2} + \sqrt{2} h \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2} + \lambda + h^2)} d\lambda \dots\dots (b) \end{aligned}$$

Total heat flux between a and b of Fig.2 is given by

$$Q_{ab} = \int_{-(l-s)}^{l+s} q(w) dw$$

Substituting Eqs.(a) and (b) into the above equation, we have

$$Q_{ab} = \Theta \frac{4\alpha}{\pi l} \int_0^\infty \frac{\left(\lambda \sqrt{\lambda^2 + \beta^2} + h \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} + \lambda \right) \left(\frac{1}{\lambda l} - \frac{\cos 2\lambda l}{\lambda l} - \sin 2\lambda l \right)}{\lambda^2 (\lambda \sqrt{\lambda^2 + \beta^2} + \sqrt{2} h \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2} + \lambda + h^2)} d\lambda \dots\dots\dots (c)$$

An example of the results calculated by Eq.(a) is shown in Fig.3. This result is obtained by numerical integration with a digital computer HITAC 502.

3. In the case where a distribution of the heat source of $Q \text{ cal} \cdot \text{sec}^{-1} \cdot \text{cm}^{-1}$ per unit width of the boundary is assumed, and the back side of heat source is insulated thermally

In the case where a heat source of $Q \text{ cal} \cdot \text{sec}^{-1} \cdot \text{cm}^{-1}$ per unit width exists at the boundary as shown in Fig.4, if it is assumed that the heat generated is absorbed in the semi-infinite solid entirely, the boundary conditions are given by the following equation:

$$-k \left(\frac{\partial \theta}{\partial y} \right)_{y=0} = Q(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\lambda w} d\lambda \int_{-\infty}^{\infty} Q(\xi) e^{-j\lambda \xi} d\xi \dots\dots\dots (13)$$

If Eq.(3) is solved with this boundary condition, the temperature distribution inside the semi-infinite solid and the temperature of the heat source itself are given, respectively, by

Fig. 4 In the case where generated heat is defined

Fig. 5 Cylinder heated by moving heat source

$$\theta(w, y) = \frac{1}{\sqrt{2} \pi k} \int_0^\infty d\lambda \int_{-\infty}^\infty \frac{e^{-\sqrt{\lambda/2} \sqrt{\lambda^2 + \beta^2} \cdot y}}{\sqrt{\lambda} \sqrt{\lambda^2 + \beta^2}} \times Q(\xi) \left\{ \begin{aligned} &\sqrt{\lambda^2 + \beta^2} \cos \left\{ \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} \cdot y + \lambda(w - \xi) \right\} \\ &- \sqrt{\lambda^2 + \beta^2} \sin \left\{ \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} \cdot y + \lambda(w - \xi) \right\} \end{aligned} \right\} d\xi \quad (14)$$

$$\Theta(w) = \frac{1}{\alpha \pi} \int_0^\infty d\lambda \int_{-\infty}^\infty Q(\xi) \left\{ \left(1 + \frac{h \sqrt{\lambda^2 + \beta^2} \cdot y}{\sqrt{2} \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2}} \right) \cos \lambda(w - \xi) - \frac{h \sqrt{\lambda^2 + \beta^2} \cdot y}{\sqrt{2} \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2}} \sin \lambda(w - \xi) \right\} d\xi \quad (15)$$

4. In the case where a heat source moves axially with a constant velocity on the surface of a cylinder of infinite length

When the heat source moves with a constant velocity on the surface of an infinite cylinder as shown in Fig. 5, we use the following equation of heat conduction with respect to the cylindrical coordinate system,

$$a \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2} \right\} \theta = \frac{\partial \theta}{\partial t}$$

The coordinate transformation $w = x - vt$ yields the following equation based on the moving coordinate system.

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial w^2} + \beta \frac{\partial \theta}{\partial w} = 0 \quad (16)$$

When the boundary condition at the outside radius of R is taken as

$$\left(\frac{\partial \theta}{\partial r} \right)_R + h \theta_{r=R} = \Theta(w) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{j\lambda w} d\lambda \int_{-\infty}^\infty \Theta(\xi) e^{-j\lambda \xi} d\xi \quad (17)$$

the solution of differential equation (16) is given by

$$\theta(w, r) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{J_0 \{ j f(\lambda, \beta) r \} e^{j\lambda w}}{h J_0 \{ j f(\lambda, \beta) R \} - j f(\lambda, \beta) J_1 \{ j f(\lambda, \beta) R \}} \int_{-\infty}^\infty \Theta(\xi) e^{-j\lambda \xi} d\xi \quad (18)$$

Now, we define the Bessel's functions of complex argument as

$$J_0(\rho \cdot e^{j\phi}) = cer_0(\rho, \phi) + jcei_0(\rho, \phi)$$

where

$$cer_0(\rho, \phi) = 1 - \left(\frac{\rho}{2} \right)^2 \frac{\cos 2\phi}{(1!)^2} + \left(\frac{\rho}{2} \right)^4 \frac{\cos 4\phi}{(2!)^2} - \left(\frac{\rho}{2} \right)^6 \frac{\cos 6\phi}{(3!)^2} + \dots$$

$$cei_0(\rho, \phi) = \left(\frac{\rho}{2} \right)^2 \frac{\sin 2\phi}{(1!)^2} - \left(\frac{\rho}{2} \right)^4 \frac{\sin 4\phi}{(2!)^2} + \left(\frac{\rho}{2} \right)^6 \frac{\sin 6\phi}{(3!)^2} - \dots$$

and

$$J_1(\rho \cdot e^{j\phi}) = cer_1(\rho, \phi) + jcei_1(\rho, \phi)$$

where

$$cer_1(\rho, \phi) = \left(\frac{\rho}{2} \right) \cos \phi - \left(\frac{\rho}{2} \right)^3 \frac{\cos 3\phi}{1!2!} + \left(\frac{\rho}{2} \right)^5 \frac{\cos 5\phi}{2!3!} - \left(\frac{\rho}{2} \right)^7 \frac{\cos 7\phi}{3!4!} + \dots$$

$$cei_1(\rho, \phi) = \left(\frac{\rho}{2} \right) \sin \phi - \left(\frac{\rho}{2} \right)^3 \frac{\sin 3\phi}{1!2!} + \left(\frac{\rho}{2} \right)^5 \frac{\sin 5\phi}{2!3!} - \left(\frac{\rho}{2} \right)^7 \frac{\sin 7\phi}{3!4!} + \dots \quad (19)^{(4)}$$

By the use of Eq.(19), Eq.(18) can be rewritten as

$$\theta = \frac{1}{\pi} \int_{-\infty}^\infty \left[\frac{\{ U(\lambda, \beta, R) cer_0(r \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2}, \phi) + V(\lambda, \beta, R) cei_0(r \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2}, \phi) \} \cos \lambda(w - \xi)}{U(\lambda, \beta, R)^2 + V(\lambda, \beta, R)^2} \right. \\ \left. - \{ V(\lambda, \beta, R) cer_0(r \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2}, \phi) + U(\lambda, \beta, R) cei_0(r \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2}, \phi) \} \sin \lambda(w - \xi) \right] \\ \times d\lambda \int_{-\infty}^\infty \Theta(\xi) d\xi \quad (20)$$

where

$$U(\lambda, \beta, R) = h cer_0(R \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2}, \phi) - \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} \cdot cer_1(R \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2}, \phi) \\ + \sqrt{\frac{\lambda}{2}} \sqrt{\lambda^2 + \beta^2} \cdot cei_1(R \sqrt{\lambda} \sqrt{\lambda^2 + \beta^2}, \phi)$$

$$V(\lambda, \beta, R) = hcei_0(R\sqrt{\lambda}\sqrt{\lambda^2+\beta^2}, \phi) - \sqrt{\frac{\lambda}{2}}\sqrt{\lambda^2+\beta^2} + \lambda \cdot \text{cer}_1(R\sqrt{\lambda}\sqrt{\lambda^2+\beta^2}, \phi) \\ - \sqrt{\frac{\lambda}{2}}\sqrt{\lambda^2+\beta^2} - \lambda \cdot \text{cei}_1(R\sqrt{\lambda}\sqrt{\lambda^2+\beta^2}, \phi) \\ \tan \phi = \frac{\sqrt{\lambda^2+\beta^2} + \lambda}{\sqrt{\lambda^2+\beta^2} - \lambda}, \quad \cos \phi = \frac{\sqrt{\lambda^2+\beta^2} - \lambda}{\sqrt{2}\sqrt{\lambda^2+\beta^2}}, \quad \sin \phi = \frac{\sqrt{\lambda^2+\beta^2} + \lambda}{\sqrt{2}\sqrt{\lambda^2+\beta^2}}, \\ \cos 2\phi = \frac{\lambda}{\sqrt{\lambda^2+\beta^2}}, \quad \sin 2\phi = \frac{\beta}{\sqrt{\lambda^2+\beta^2}}.$$

Example:

In Fig. 6, a wire is moved to the left with a constant velocity of v through a furnace having the length of c and the constant temperature of θ . In order to obtain the temperature of the center of the wire, θ_0 , approximate calculation is made on the assumption that R is a small quantity. Then Eq.(20) becomes

$$\theta_0 = \theta \frac{\mu^2}{2\pi\nu} \left[\int_0^\infty \frac{\left(\nu - \frac{\beta}{2}\right) \{\sin \lambda(c-w) + \sin \lambda w\}}{\lambda \left\{ \left(\nu - \frac{\beta}{2}\right)^2 + \lambda^2 \right\}} d\lambda - \int_0^\infty \frac{\left(\nu + \frac{\beta}{2}\right) \{\sin \lambda(c-w) + \sin \lambda w\}}{\lambda \left\{ \left(\nu + \frac{\beta}{2}\right)^2 + \lambda^2 \right\}} d\lambda \right. \\ \left. + \int_0^\infty \frac{\cos \lambda(c-w) - \cos \lambda w}{\left\{ \left(\nu - \frac{\beta}{2}\right)^2 + \lambda^2 \right\}} d\lambda - \int_0^\infty \frac{\cos \lambda(c-w) - \cos \lambda w}{\left\{ \left(\nu + \frac{\beta}{2}\right)^2 + \lambda^2 \right\}} d\lambda \right] \quad (21)$$

where μ and ν are respectively

$$\mu = \sqrt{\frac{2h}{R}}, \quad \nu = \sqrt{\left(\frac{\beta}{2}\right)^2 + \mu^2}$$

By the following evaluation of the infinite integrals:

$$\int_0^\infty \frac{\cos \lambda x}{p^2 + \lambda^2} d\lambda = \frac{\pi}{2p} e^{-px}, \quad 0 < x \\ \int_0^\infty \frac{\sin \lambda x}{\lambda(p^2 + \lambda^2)} d\lambda = \frac{\pi}{2p^2} (1 - e^{-px}), \quad 0 < x; \\ \int_0^\infty \frac{\cos \lambda x}{p^2 + \lambda^2} d\lambda = -\frac{\pi}{2p} e^{px}, \quad x < 0 \\ \int_0^\infty \frac{\sin \lambda x}{\lambda(p^2 + \lambda^2)} d\lambda = -\frac{\pi}{2p^2} (1 - e^{px}), \quad x < 0.$$

Eq.(21) can be transformed as follows. For 0-1 region in Fig. 6,

$$\theta_{01} = \theta \frac{h}{R} \frac{1}{\nu \left(\nu - \frac{\beta}{2}\right)} \{1 - e^{-(\nu - \beta/2)c}\} e^{(\nu - \beta/2)w} \quad (21.1)$$

For 1-2 region,

$$\theta_{12} = \theta \frac{h}{R\nu} \left[\frac{1}{\left(\nu - \frac{\beta}{2}\right)} \{1 - e^{-(\nu - \beta/2)(c-w)}\} + \frac{1}{\left(\nu + \frac{\beta}{2}\right)} \{1 - e^{-(\nu + \beta/2)w}\} \right] \quad (21.2)$$

For 2-3 region,

$$\theta_{23} = \theta \frac{h}{R} \frac{1}{\nu \left(\nu + \frac{\beta}{2}\right)} \{e^{(\nu + \beta/2)c} - 1\} e^{-(\nu + \beta/2)w} \quad (21.3)$$

Temperature distribution of wire center is as shown by the fine line of Fig. 6. Maximum temperature occurs in the 1-2 region, and its position and magnitude are given as follows.

$$w_m = \frac{c}{2} \left\{ 1 - \frac{1}{\sqrt{1 + 4\frac{\alpha^2 \mu^2}{v^2}}} \right\}, \quad \theta_{0, \max} = \theta \left\{ 1 - e^{-\frac{\mu^2 c}{(2\sqrt{1 + 4\frac{\alpha^2 \mu^2}{v^2}} + \mu^2)w}} \right\} \quad (22)$$

Fig. 6 Temperature of moving wire center

The position which occurs the maximum temperature moves from the center of the furnace towards the outlet as the velocity v increases.

5. Numerical integration of

$$\frac{\partial^2 \theta}{\partial w^2} + \beta \frac{\partial \theta}{\partial w} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

by digital computer

Eq.(3) based on the moving coordinate system can be

solved directly by the digital computer. This equation is an elliptic differential equation, and when the equation is Laplace's or Poisson's equation in the category of the elliptic differential equation, accelerated Liebmann's method is effective for numerical integration by a digital computer⁽⁵⁾. Since Eq.(3) is a partial differential equation in which the first derivative with respect to w is added merely to Laplace's equation, it is considered that this equation can easily be solved also by the accelerated Liebmann's method.

Now, we divided the semi-infinite solid into regular square nets with the mesh size $\delta \times \delta$ as shown in Fig. 7, and we shall use the symbol $\theta_{i,j}$ for the temperature at the nodal point (i,j) . Then we can use the following expressions as approximate values of the partial derivatives of θ at the nodal point

$$\left. \begin{aligned} \frac{\partial \theta}{\partial w} &\approx \frac{1}{2} \left(\frac{\theta_{i+1,j} - \theta_{i,j}}{\delta} + \frac{\theta_{i,j} - \theta_{i-1,j}}{\delta} \right) \\ \frac{\partial^2 \theta}{\partial w^2} &\approx \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{\delta^2} \\ \frac{\partial^2 \theta}{\partial y^2} &\approx \frac{\theta_{i,j+1} - 2\theta_{i,j} + \theta_{i,j-1}}{\delta^2} \end{aligned} \right\} \dots (23)$$

Using Eq.(23) we can transform Eq.(3) into a finite-difference equation as follows:

$$\frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{\delta^2} + \frac{\beta}{2} \left(\frac{\theta_{i+1,j} - \theta_{i,j}}{\delta} + \frac{\theta_{i,j} - \theta_{i-1,j}}{\delta} \right) + \frac{\theta_{i,j+1} - 2\theta_{i,j} + \theta_{i,j-1}}{\delta^2} = 0$$

or

$$\theta_{i,j} = \frac{1}{4} \left\{ \theta_{i+1,j} + \theta_{i-1,j} + \theta_{i,j+1} + \theta_{i,j-1} + \frac{\beta \delta}{2} (\theta_{i+1,j} - \theta_{i-1,j}) \right\} \dots (24)$$

Eq.(24) can be applied to the nodal points in the interior of the semi-infinite solid, however at the boundary further careful considerations are neces-

sary. So, first, we recall the physical meaning of Eq.(24) as follows.

In our previous discussion a square net was used as shown in Fig. 8, so it is considered that the temperature of the interior of the elementary square net, shown by dotted line, is represented by the temperature of the nodal point. This figure is based on the moving coordinate system (w, y) fixed to the heat source, so the nodal point fixed to the semi-infinite solid moves to the left at the constant velocity v . Thus, the element [1234] was at [ABCD] before the lapse of time $t = \delta/(2v)$, and will be at [CDEF] after the lapse of time $t = \delta/(2v)$, and during this displacement, the temperature of the element changes from $(\theta_{i,j} + \theta_{i+1,j})/2$ to $(\theta_{i-1,j} + \theta_{i,j})/2$. Now we assume that the average heat flux across the four planes of the element during this displacement is given by the heat flux through the four planes, $\overline{12}$, $\overline{24}$, $\overline{34}$ and $\overline{13}$. If q_{12} , q_{24} , q_{34} , q_{13} denote the heat flux across the four planes, $\overline{12}$, $\overline{24}$, $\overline{34}$, $\overline{13}$ of the element per unit time, the following equation of continuity for heat is obtained.

$$(q_{34} + q_{13} - q_{12} - q_{24}) \frac{\delta}{v} = \rho c \delta^2 \left(\frac{\theta_{i-1,j} + \theta_{i,j}}{2} - \frac{\theta_{i,j} + \theta_{i+1,j}}{2} \right) \dots (25)$$

where

$$\left. \begin{aligned} q_{34} &= -\delta k \frac{\theta_{i,j} - \theta_{i-1,j}}{\delta} \\ q_{13} &= -\delta k \frac{\theta_{i,j} - \theta_{i,j-1}}{\delta} \\ q_{12} &= -\delta k \frac{\theta_{i+1,j} - \theta_{i,j}}{\delta} \\ q_{24} &= -\delta k \frac{\theta_{i,j+1} - \theta_{i,j}}{\delta} \end{aligned} \right\} \dots (26)$$

Using the relation $\frac{v \rho c}{k} = \beta$, and substituting Eq.(26) into Eq.(25), we obtain

Fig. 7 Semi-infinite solid divided into regular square net

Fig. 8 The nodal points of interior of the semi-infinite solid

$$\theta_{i,j} = \frac{1}{4} \left\{ \theta_{i+1,j} + \theta_{i,j+1} + \theta_{i-1,j} + \theta_{i,j-1} + \frac{\beta \delta}{2} (\theta_{i+1,j} - \theta_{i-1,j}) \right\} \quad (27)$$

This equation is quite identical to Eq.(24), and the physical meaning of Eq.(24) has been clarified.

At the boundary, the relations for heat flux are different. Either the boundary contains the nodal points as shown in Fig.9 or the boundary passes an intermediate position between the nodal points. In this paper, the boundary is assumed to contain the nodal points as shown in the figure. In this case, the element [1234] is only a lower half and the plane 13 must be considered to be a boundary. If we consider the heat flux like the elements of the interior of the semi-infinite solid, the following equation of continuity for heat is obtained.

$$(q_{34} + q_{13} - q_{12} - q_{24}) \frac{\delta}{v} = \rho c \frac{\delta^2}{2} \left(\frac{\theta_{i-1,0} + \theta_{i,0}}{2} - \frac{\theta_{i,0} + \theta_{i+1,0}}{2} \right) \quad (28)$$

where

$$\left. \begin{aligned} q_{34} &= -\frac{\delta}{2} k \frac{\theta_{i,0} - \theta_{i-1,0}}{\delta} \\ q_{13} &= -\alpha \delta (\theta_i - \theta_{i,0}) \\ q_{12} &= -\frac{\delta}{2} k \frac{\theta_{i+1,0} - \theta_{i,0}}{\delta} \\ q_{24} &= -\delta k \frac{\theta_{i,1} - \theta_{i,0}}{\delta} \end{aligned} \right\} \quad (29)$$

Assuming

$$\frac{1}{2} (\theta_{i+1,0} + \theta_{i-1,0}) \doteq \theta_{i,0}$$

and substituting Eq.(29) into Eq.(28), we obtain

$$\theta_{i,0} = \frac{h \delta \theta_i + \theta_{i,1} + \frac{\beta \delta}{2} \frac{\theta_{i+1,0} - \theta_{i-1,0}}{2}}{1 + h \delta} \quad (30)$$

If the moving velocity of heat source is zero, that is $\beta = 0$, Eq.(30) reduces to

$$\theta_{i,0} = \frac{h \delta \theta_i + \theta_{i,1}}{1 + h \delta} \quad (31)$$

If the coefficient of heat transfer at the boundary is infinite,

$$\theta_{i,0} = \theta_i \quad (32)$$

Eq.(30) can be changed to

$$-\frac{\theta_{i,1} - \theta_{i,0}}{\delta} = h (\theta_i - \theta_{i,0}) + \frac{\beta}{2} \left(\frac{\theta_{i+1,0} + \theta_{i,0}}{2} - \frac{\theta_{i,0} + \theta_{i-1,0}}{2} \right)$$

If δ is a small quantity, this equation gives

$$-\left(\frac{\partial \theta}{\partial y} \right)_{y=\delta/2} = h \left\{ \theta(w) - \theta_{y=0} \right\} + \frac{\beta}{2} \left\{ \theta_{y=0} \left(w + \frac{\delta}{2} \right) - \theta_{y=0} \left(w - \frac{\delta}{2} \right) \right\}$$

In the limit $\delta \rightarrow 0$, the above equation becomes

$$-\left(\frac{\partial \theta}{\partial y} \right)_{y=0} = h \{ \theta(w) - \theta_{y=0} \} \quad (33)$$

Eq.(33) coincides with Eq.(2) which gives the heat flux at the boundary. When we integrate Eq.(3) by numerical methods, dividing the semi-infinite

Fig. 9 The nodal points at the boundary

Fig. 10 Flow chart for the numerical calculation of Eq. (3) by the accelerated Liebmann's method

solid into the regular square net with the mesh size δ, δ is generally selected to have a finite value, and so, the temperature at the boundary should be assumed to be given by Eq.(30).

In this way a heat transfer problem reduces to the finding of a set of numerical values for the temperature θ which satisfy Eq.(24) at every nodal point inside the semi-infinite solid and satisfy Eq.(30) at the boundary. Although a large number of algebraic equations are involved, individual equations are simple and suited to be solved numerically by the repetition methods with a digital computer.

We used the accelerated Liebmann's method for this purpose. Namely, if we use the superscript, n , which is the number of iteration stages already

carried out, and the acceleration factor A , we have

$$\theta_{i,j}^{(n+1)} = \theta_{i,j}^{(n)} + AR_{i,j}^{(n)} \quad \dots\dots\dots(34)$$

where $R_{i,j}^{(n)}$ denotes the difference between $\theta_{i,j}^{(n+1)}$ and $\theta_{i,j}^{(n)}$, that is

$$R_{i,j}^{(n)} = \frac{1}{4} \left\{ \theta_{i+1,j}^{(n+1)} + \theta_{i,j+1}^{(n+1)} + \theta_{i-1,j}^{(n)} + \theta_{i,j-1}^{(n)} + \frac{\beta\delta}{2} (\theta_{i+1,j}^{(n+1)} - \theta_{i-1,j}^{(n)}) \right\} - \theta_{i,j}^{(n)} \quad \dots\dots\dots(35)$$

A similar method was used for the calculation of Eq.(30) for the boundary.

Optimum value of A depends upon the nature of the problem involved. However, for Laplace's or Poisson's equation, it falls in the range $1 \leq A \leq 2$, and $A=1$ corresponds to the ordinary Liebmann's

$$\begin{aligned} &\Theta=1, \quad hl=\infty, \quad \beta l=0, \quad s/l=0 \\ &\text{For } -(l-s) < w < (l+s) \\ &\quad \Theta(w) = \frac{\Theta}{2} \left\{ \cos \pi \left(\frac{w-s}{l} \right) + 1 \right\} \\ &\text{For } w < -(l-s), \quad w > (l+s), \\ &\quad \Theta(w) = 0 \end{aligned}$$

Fig. 11 Isotherms by the accelerated Liebmann's method (HIPAC 101)

$$\begin{aligned} &\Theta=1, \quad hl=\infty, \quad \beta l=10, \quad s/l=0 \\ &\text{For } -(l-s) < w < (l+s), \\ &\quad \Theta(w) = \frac{\Theta}{2} \left\{ \cos \pi \left(\frac{w-s}{l} \right) + 1 \right\} \\ &\text{For } w < -(l-s), \quad w > (l+s), \\ &\quad \Theta(w) = 0 \end{aligned}$$

Fig. 13 Isotherms by the accelerated Liebmann's method (HIPAC 101)

$$\begin{aligned} &\Theta=1, \quad hl=\infty, \quad \beta l=4, \quad s/l=0 \\ &\text{For } -(l-s) < w < (l+s), \\ &\quad \Theta(w) = \frac{\Theta}{2} \left\{ \cos \pi \left(\frac{w-s}{l} \right) + 1 \right\} \\ &\text{For } w < -(l-s), \quad w > (l+s), \\ &\quad \Theta(w) = 0 \end{aligned}$$

Fig. 12 Isotherms by the accelerated Liebmann's method (HIPAC 101)

$$\begin{aligned} &\Theta=1, \quad hl=\infty, \quad \beta l=40, \quad s/l=0 \\ &\text{For } -(l-s) < w < (l+s), \\ &\quad \Theta(w) = \frac{\Theta}{2} \left\{ \cos \pi \left(\frac{w-s}{l} \right) + 1 \right\} \\ &\text{For } w < -(l-s), \quad w > (l+s), \\ &\quad \Theta(w) = 0 \end{aligned}$$

Fig. 14 Isotherms by the accelerated Liebmann's method (HIPAC 101)

Fig. 15 Isotherms by the accelerated Liebmann's method (HIPAC 101)

method. The nearer A is to 2, the more effective the accelerated Liebmann's method is, but, if A is taken too large, the solution is divergent.

Fig. 10 shows an example of flow charts for the numerical calculation of Eq.(3) by the accelerated Liebmann's method with a digital computer. This chart shows the case of $h = \infty$. Calculation is repeated until residue $R_{i,j}^{(n)}$ at any nodal point is less than a certain value ϵ , and then the calculated results are printed automatically by the typewriter. Since the temperature of the boundary is given by Eq.(30), Eq.(31), or Eq.(32) depending on h and β , different flow charts exist for different h and β .

Fig. 11 to Fig. 15 show the isotherms inside the semi-infinite solid as calculated by the accelerated Liebmann's method. These calculations were carried out by a parametron digital computer, HIPAC 101, and the time required for these calculations was about 10 minutes for each case.

6. Conclusion

1. When a heat source travels along the surface of a semi-infinite solid with a constant velocity v , the temperature θ in a semi-infinite solid is given by

$$\frac{\partial^2 \theta}{\partial w^2} + \beta \frac{\partial \theta}{\partial w} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

where $w = x - vt$, $\beta = v/a$, and the temperature distribution of the heat source can be expressed as $\theta(w)$.

2. Though the eigen-values of this partial differential equation are complex numbers, the solution can be obtained in the form of a real Fourier's integral. Thus the temperature distribution in a semi-infinite solid and heat transmission across the surface can be calculated.

3. This Fourier's integral was numerically calculable by a digital computer.

4. However, it was much easier to solve directly the above partial differential equation by the accelerated Liebmann's method using a digital computer than to solve it by the Fourier's integral method, and the result obtained by the former method coincided with the result obtained by the latter in a fairly good approximation.

References

- (1) H.S. Carslaw and J.S. Jaeger: *Conduction of Heat in Solids*, (1959), p. 266, Oxford at the Clarendon Press.
- (2) D. Rosenthal: *Trans. ASME*, Vol. 68, (1946), p. 849.
- (3) P.J. Schneider: *Conduction Heat Transfer*, (1955), p. 282, Addison-Wesley Publishing Co. Inc.
- (4) Ataka and Monji: *Entōkansu-hyō* (in Japanese), (1948), p. 56, Maruzen.
- (5) Moriguchi and Takada: *Suuchi-keisan-hō II* (Iwanami Kōza, Gendai-Ōyo-Sūgaku), (in Japanese), (1958), Iwanami.

Discussion

M. OGASAWARA: (1) What magnitude of δ , i.e., a pitch of network is used in calculation? And, how is it selected?

I. MABUCHI: (2) I would like to congratulate the authors on their valuable contribution to the heat conduction problem with a moving heat source.

I believe that it would be of interest to further discuss the non-linear case, having the thermal properties changing with the temperature.

I want to ask the authors about the existence of systematic law for the accelerated constant A used in the examples.

Authors' closure

(1) In the numerical calculation, a large value was taken for δ for the first approximation, and then, on the basis of the result of this first approximation, some smaller value was adopted for δ in the proximity of the heat source. When the difference between solutions for, say, $\delta=l/5$ and $\delta=l/10$ was less than 1% of the maximum temperature of the heat source, we discontinued further reduction of δ . In this report, the smallest value for δ was $l/20$.

(2) For application to the non-linear case

where the thermal properties of a semi-infinite solid change with the temperature, our numerical methods must be modified. But this can be done easily by a digital computer.

Optimum acceleration factor A depends on the values, δ and β , and the definite systematic law for the acceleration factor could not be found. In the examples presented here, the acceleration factor was varied from 1.00 to 1.50. Generally speaking, for a given δ , the larger the value of β , the smaller A must be, otherwise the solution will be divergent. The numerical calculation was started first on the condition that $\theta_{i,j}=0$ at any nodal point in the semi-infinite solid. Empirically, if the value of $R_{i,j}$ was larger than 0.1% of the maximum temperature ($\theta=1$) of the heat source even after 30 repetitions of numerical calculation, the acceleration factor was considered to be too small.

662.612.32:536.46

Combustion Oscillation in Oil Mist Firing*

By Takashi SATO**, Ryūichi MATSUMOTO***
and Akira YAMAMOTO****

This paper deals with the unstable combustion of liquid fuel in a combustor. Owing to the increase of air-fuel ratio in the range larger than stoichiometric one, combustion becomes unstable, the noise increases and also the pressure and the temperature in the combustor oscillate at a frequency of 20 to 50 cps. Moreover, blow-off phenomenon results from this unstable combustion.

The results obtained are as follows: (1) The frequency of combustion oscillation changes with air-fuel ratio, fuel flow rate and mean drop size of fuel. (2) The amplitudes of pressure and temperature change become larger with the increase of air-fuel ratio and fuel flow rate. (3) The flow rates of fuel and atomizing air oscillate in the case of combustion oscillation. (4) The mechanism of combustion oscillation is made clear by high speed photography.

1. Introduction

Owing to the increase of air-fuel ratio, unstable combustion occurs in the combustor of a liquid fuel. Namely, combustion noise increases and the pressure and temperature in the combustor oscillate

at a frequency of 20 to 40 cps in the range larger than stoichiometric value. Moreover, these phenomena become more intensive with the increase of air-fuel ratio and blow-off phenomenon is caused by this unstable combustion.

Therefore, in order to keep a stable combustion in the large mixing ratio and to enlarge the range of stable combustion, it is important to clarify the mechanism of combustion oscillation in this kind of combustor. The purposes of this paper are to analyze the cause of such oscillation and to make clear the mechanism of combustion oscillation with

* Received 19th December, 1961.

** Professor, Faculty of Engineering, Kyoto University, Sakyo-ku, Kyoto.

*** Assistant Professor, Faculty of Engineering, Kobe University.

**** Post Graduate Student, Faculty of Engineering, Kyoto University.