621-503, 5:519, 283

Time-Dependent Probability Distribution of the Non-Stationary Response of Nonlinear Control Systems*

By Yoshikazu SAWARAGI**, Yoshifumi SUNAHARA***
and Takashi SOEDA****

In this paper, to evaluate the time-dependent probability density function of the response of nonlinear control systems subjected to a suddenly applied Gaussian random input, an approximate method of solving the combined Fokker-Planck equation with the equation of a control system is developed.

The basic idea presented here is as follows. The nonlinear transfer characteristic contained in the system is replaced by an equivalent linear one with piecewise linear segments, in the sense of the least square value. The form of probability density function is also replaced by the form of connected Gaussian type probability density function. Under these conceptions, a numerical calculation is carried out. Detailed illustrations are shown by several examples.

List of principal symbols

- t : time variable
 s : operator with respect to the Laplace transformation
 τ_1 : the time at which the impulse is applied to the system
 $v(t)$: desired signal
 $u(t)$: stationary Gaussian random disturbance
 $x(t)$: controlled variable
 $z(t)$ and $y(t)$: input and output of a nonlinear element respectively
 $f(z)$: transfer characteristic of a nonlinear element of zero-memory type
 m_u and ψ_u : the mean value and the mean squared value of the signal $u(t)$ respectively
 m_v and ψ_v : the mean value and the mean squared value of $v(t)$ respectively
 $R_u(t_1, t_2)$: the auto-correlation function of $u(t)$ depending on time instants t_1 and t_2
 $R_v(t_1, t_2)$: the auto-correlation function of $v(t)$ depending on time instants t_1 and t_2
 k and T : time-invariant parameters
 $f_p(z, t)$: the time-dependent probability density

function of the signal $z(t)$

$F_p(z, t)$: the time-dependent cumulative distribution function of the signal $z(t)$

1. Introduction

Two aspects of finding the response probability distribution of nonlinear control systems subjected to stationary Gaussian random inputs were developed by the authors. The first is a method of finding higher order moments of the response⁽¹⁾ and the second is that of solving the Fokker-Planck equation by introducing the conception of Markov processes in statistics⁽²⁾⁽³⁾.

When a time-invariant nonlinear control system is excited by a suddenly applied stationary Gaussian random input, the response becomes a non-stationary, non-Gaussian random time series; and the probability density function depends on time. Recognizing the above, we must solve the Fokker-Planck equation with the term depending on time.

We consider a control system containing a nonlinear element of zero-memory type as shown in Fig. 1. A nonlinear function of the form $y=f(z)$ is used to express the symmetric relationship between the instantaneous input, $z(t)$, and the instantaneous output, $y(t)$. For the convenience of mathematical calculation, we consider the case of $G_2(s)=k/Ts$. We have

$$T \frac{dz(t)}{dt} + kf[z(t)] = ku(t) \dots\dots\dots (1)$$

* Received 16th December, 1961.

** Professor, Faculty of Engineering, Kyoto University.

*** Assistant Professor, Faculty of Engineering, Kyoto University.

**** Assistant Professor, Faculty of Engineering, Tokushima University, Tokushima.

Let the stationary Gaussian random input suddenly applied to the system be a white noise with

$$R_u(t_1, t_2) = 2\Phi_u \delta(t_1 - t_2) \quad (2)$$

as its auto-correlation function, where δ is Dirac's Delta function and Φ_u is a constant. Let t be increased to $t + \Delta t$ in Eq. (1), where Δt is very small compared with the time constant of the system. Then we have

$$\Delta z = -\frac{kf(z)}{T} \Delta t + \frac{k}{T} \int_t^{t+\Delta t} u(t) dt \quad (3)$$

From Eq. (3), the following Fokker-Planck equation can easily be derived⁽²⁾

$$\beta \frac{\partial^2 f_p(z, t)}{\partial z^2} + \frac{\partial}{\partial z} [\alpha(z) f_p(z, t)] = -\frac{\partial f_p(z, t)}{\partial t} \quad (4)$$

$$G_n(z, z', t) = \sum_i \frac{\exp(-\tilde{\lambda}_i^2)}{\sqrt{\tilde{\lambda}_i^2 - \beta U(z)}} \exp\left\{\pm \sqrt{\frac{-1}{\beta}} \int_0^z \sqrt{\tilde{\lambda}_i^2 - \beta U(z)} dz\right\} \quad (8)_1$$

$$U(z) = \{\alpha(z)^2 - 2\beta\alpha'(z)\} / 4\beta^2 \quad (8)_2$$

and $\tilde{K}$ and $\tilde{\lambda}_i$ are a normalized factor and an eigenvalue respectively⁽⁴⁾. Furthermore, $z'(t)$ and $\alpha'(z)$ are derivatives of $z(t)$ and $\alpha(z)$ respectively. Although the non-Gaussian probability density function, $f_p(z, t)$, of $z(t)$ is determined by Eq. (7) and is plotted as shown in Fig. 2, it is very difficult to evaluate this by Eq. (7). Therefore, as shown in Fig. 2, we represent the original probability density function, $f_p(z, t)$, by the connection of segments of Gaussian type. Recognizing that this original curve is symmetric with respect to the axis of $z=0$, the above consideration is expressed as

$$f_{p1}(z, t) = K_1 \exp\left\{-\frac{z^2}{2\sigma_1(t)}\right\} \quad \text{for } -l < z < l \quad (9)$$

$$f_{p2}(z, t) = K_2 \exp\left\{-\frac{z^2 + 2m(t)z}{2\sigma_2(t)}\right\} \quad \text{for } l < z < \infty \quad (10)_1$$

and

$$f_{p3}(z, t) = K_2 \exp\left\{-\frac{z^2 - 2m(t)z}{2\sigma_2(t)}\right\} \quad \text{for } -\infty < z < -l \quad (10)_2$$

where $\sigma_i(t)$ ($i=1, 2$) and $m(t)$ are the variance and the mean value respectively. The functions, $f_{p1}(z, t)$, $f_{p2}(z, t)$ and $f_{p3}(z, t)$, show Gaussian probability density functions in the interval between $-l < z < l$, $l < z < \infty$ and $-\infty < z < -l$ respectively. The constants, K_1 and K_2 , in Eqs. (9) and (10) are unknown and these are determined so that they may satisfy the following conditions;

$$\int_0^l f_{p1}(z, t) dz + \int_l^\infty f_{p2}(z, t) dz = \frac{1}{2} \quad (11)$$

and

$$f_{p1}(l, t) = f_{p2}(l, t) \quad (12)$$

By solving Eqs. (9), (10)₁, (11) and (12) simultaneously, we have

Fig. 1 Block diagram of a nonlinear control system subjected to a suddenly applied stationary Gaussian random input

where

$$\alpha(z) = \lim_{\Delta t \rightarrow 0} \frac{A_1(t, \Delta t, z)}{\Delta t} = -\frac{kf(z)}{T} \quad (5)$$

$$\beta = \lim_{\Delta t \rightarrow 0} \frac{B_1(t, \Delta t, z)}{2\Delta t} = \frac{k^2}{T^2} \Phi_u \quad (6)$$

and $A_1(t, \Delta t, z)$ and $B_1(t, \Delta t, z)$ denote the conditional mean values of Δz and $(\Delta z)^2$ respectively.

If we assume that $\beta \ll 1$, the solution of Eq. (4) is approximately given by

$$f_p(z, t) = \tilde{K} \exp\left\{-\frac{1}{2} \int \frac{\alpha(z)}{\beta} dz\right\} \cdot G_n(z, z', t) \quad (7)$$

where

Fig. 2 The original and the approximate probability density functions of the random signal, $z(t)$

$$K_1 = K_2 \exp \left\{ -\frac{l^2 + 2m(t)l}{2\sigma_2(t)} \right\} \exp \left\{ \frac{l^2}{2\sigma_1(t)} \right\} \dots\dots\dots(13)$$

and

$$K_2 = 1/2\sqrt{2\pi} \left[\sqrt{\sigma_1(t)} \Phi \left\{ \frac{l}{\sqrt{\sigma_1(t)}} \right\} \exp \left\{ -\frac{l^2 + 2m(t)l}{2\sigma_2(t)} + \frac{l^2}{2\sigma_1(t)} \right\} \right. \\ \left. + \sqrt{\sigma_2(t)} \exp \left\{ \frac{m(t)^2}{2\sigma_1(t)} \right\} \left\{ \frac{1}{2} - \Phi \left(\frac{l+m(t)}{\sqrt{\sigma_2(t)}} \right) \right\} \right] \dots\dots\dots(14)$$

where

$$\Phi(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^\zeta \exp \left(-\frac{u^2}{2} \right) du \dots\dots\dots(15)$$

In the y - z plane, as shown in Fig. 3, the above descriptions correspond to the following relations

$$f_1(z) = \lambda_1 z \quad \text{for } 0 < z < l \dots\dots\dots(16)$$

and

$$f_2(z) = \lambda_2 z + \mu_2 \quad \text{for } l < z < \infty \dots\dots\dots(17)$$

where

$$\mu_2 = l(\lambda_1 - \lambda_2), \quad (\lambda_1, \lambda_2 \neq 0) \dots\dots\dots(18)$$

and λ_i ($i=1, 2$) and μ_2 are the values of gradient of equivalent linearized segments and intersection of the vertical axis.

2. Determination of the mean value, $m(t)$, and the variance, $\sigma_i(t)$

The equation of control systems containing the transfer characteristic given by Eq. (16) is expressed as

$$\frac{dz(t)}{dt} + \frac{k}{T} \lambda_1 z(t) = \frac{k}{T} u(t) \quad \text{for } 0 < z < l \dots\dots(19)$$

By a similar consideration, it can easily be obtained that

$$\frac{dz(t)}{dt} + \frac{k}{T} \lambda_2 z(t) + \frac{k}{T} \mu_2 = \frac{k}{T} u(t) \\ \text{for } l < z < \infty \dots\dots\dots(20)$$

Therefore, from Eqs. (19) and (20), the mean value, $m(t)$, and the variance, $\sigma_i(t)$, in Eqs. (9) and (10) can easily be obtained respectively as

$$m(t) = -\frac{q_2}{p_2} \{1 - \exp(-p_2 t)\} \dots\dots\dots(21)$$

and

$$\sigma_i(t) = \frac{k^2}{T^2} \Phi_u \frac{1}{p_i} \{1 - \exp(-2p_i t)\} \dots\dots\dots(22)$$

Fig. 3 The piecewise linearization of the nonlinear characteristic

where

$$p_i = k\lambda_i/T, \quad (i=1, 2), \quad q_2 = k\mu_2/T \dots\dots\dots(23)$$

3. Determination of gradients λ_1 and λ_2

Values of λ_1 and λ_2 in Eqs. (16) and (17) are determined so that they may satisfy

$$\frac{\partial I}{\partial \lambda_1} = 0 \quad \text{and} \quad \frac{\partial I}{\partial \lambda_2} = 0 \dots\dots\dots(24)$$

from the following equation which is introduced by using the least square method

$$I = \int_0^l \{f(z) - \lambda_1 z\}^2 p_{j-1}(z) dz \\ + \int_l^\infty \{f(z) - \lambda_2 z - \mu_2\}^2 p_{j-1}(z) dz \dots\dots\dots(25)_1$$

where $p_{j-1}(z)$ is the weight function of the Gaussian type ;

$$p_{j-1}(z) = \frac{1}{\sqrt{2\pi M_{j-1}}} \exp \left(-\frac{z^2}{2M_{j-1}} \right) \dots\dots\dots(25)_2$$

and M_{j-1} is the variance of the weight function at the time $t=t_{j-1}$. From Fig. 3, it is easily shown that $\lambda_1 = f(l)/l$ $\dots\dots\dots(26)$

By substituting Eqs. (18) and (26) into Eq. (25)₁ and by differentiating partially with respect to the

Table 1 Values of gradient and division of the linearization of the nonlinear element given by $f(z) = 0.5z + 0.1z^3$

Variance of weight function M	Gradient		Division l
	λ_1	λ_2	
0.5	0.533	0.856	0.581
0.8	0.553	1.067	0.731
1.0	0.567	1.210	0.816
2.0	0.867	1.917	1.155

Fig. 4 Nonlinear characteristic given by Eq. (30)

variables l and λ_2 instead of λ_1 and λ_2 , it is possible to determine the values of l and λ_2 from the following equations ;

$$\begin{aligned} N_1 \int_0^l z f(z) p_{j-1}(z) dz - N_1 \frac{f(l)}{l} \int_0^l z^2 p_{j-1}(z) dz + N_2 \int_l^\infty f(z) p_{j-1}(z) dz \\ - \lambda_2 N_2 \int_l^\infty z p_{j-1}(z) dz + N_2 N_3 \int_l^\infty p_{j-1}(z) dz = 0 \end{aligned} \quad (27)$$

and

$$- \int_l^\infty z f(z) p_{j-1}(z) dz + \lambda_2 \int_l^\infty z^2 p_{j-1}(z) dz + l \int_l^\infty f(z) p_{j-1}(z) dz - N_4 \int_l^\infty z p_{j-1}(z) dz + l N_3 \int_l^\infty p_{j-1}(z) dz = 0 \quad (28)$$

where

$$\left. \begin{aligned} N_1 &= \{f(l) - l f'(l)\} / l^2 \\ N_2 &= \lambda_2 - f'(l) \\ N_3 &= \{l \lambda_2 - f(l)\} \\ N_4 &= \{2l \lambda_2 - f(l)\} \end{aligned} \right\} \quad (29)$$

and $f'(l) = df(l)/dl$. As an example, we calculate the values of λ_1 and λ_2 at the time $t = t_j$. In this case, as shown in Fig. 4, the nonlinear element is given by

$$f(z) = bz + cz^3 \quad (b, c > 0) \quad (30)$$

By substituting Eqs. (25)₂ and (30) into Eqs. (27) and (28), we have

$$L_1 \lambda_2^2 + L_2 \lambda_2 + L_3 = 0 \quad (31)$$

and

$$L_4 \lambda_2 + L_5 = 0 \quad (32)$$

where

$$L_1 = l \sqrt{2\pi M_{j-1}} \left\{ \frac{1}{2} - \Phi \left(\frac{l}{\sqrt{M_{j-1}}} \right) \right\} - M_{j-1} \exp \left(-\frac{l^2}{2M_{j-1}} \right) \quad (33)$$

$$L_2 = 2 \left[M_{j-1} (2cl^2 + cM_{j-1} + b) \exp \left(-\frac{l^2}{2M_{j-1}} \right) - l \sqrt{2\pi M_{j-1}} \left\{ \frac{1}{2} - \Phi \left(\frac{l}{\sqrt{M_{j-1}}} \right) \right\} (b + 2cl^2) \right] \quad (34)$$

$$\begin{aligned} L_3 = -(b + 3cl^2) \left[M_{j-1} (cl^2 + 2cM_{j-1} + b) \exp \left(-\frac{l^2}{2M_{j-1}} \right) - l \sqrt{2\pi M_{j-1}} \left\{ \frac{1}{2} - \Phi \left(\frac{l}{\sqrt{M_{j-1}}} \right) \right\} (b + 2cl^2) \right] \\ + 2c^2 l \left\{ 3l M_{j-1}^2 \exp \left(-\frac{l^2}{2M_{j-1}} \right) - M_{j-1} \sqrt{2\pi M_{j-1}} \Phi \left(\frac{l}{\sqrt{M_{j-1}}} \right) (3M_{j-1} - l^2) \right\} \end{aligned} \quad (35)$$

$$L_4 = -l M_{j-1} \exp \left(-\frac{l^2}{2M_{j-1}} \right) + (M_{j-1} + l^2) \sqrt{2\pi M_{j-1}} \left\{ \frac{1}{2} - \Phi \left(\frac{l}{\sqrt{M_{j-1}}} \right) \right\} \quad (36)$$

and

$$L_5 = l M_{j-1} (cl^2 - cM_{j-1} + b) \exp \left(-\frac{l^2}{2M_{j-1}} \right) - \sqrt{2\pi M_{j-1}} (cl^4 + bl^2 + bM_{j-1} + 3cM_{j-1}) \left\{ \frac{1}{2} - \Phi \left(\frac{l}{\sqrt{M_{j-1}}} \right) \right\} \quad (37)$$

By solving Eqs. (31) and (32) simultaneously, we can determine the values of l and λ_2 . Table 1 shows the numerical results of variance, M .

4. The time-dependent skewness, $\beta_1(t)$, and the kurtosis, $\beta_2(t)$

We introduce the time-dependent skewness, $\beta_1(t)$, and the kurtosis, $\beta_2(t)$, as the measure of discrepancy from the normal distribution. By using Eqs. (9) and (10), the μ_i th moments μ_i ($i=1, 2, 3, 4$) become as follows ;

$$\mu_1(t) = 0 \quad (38)$$

$$\begin{aligned} \mu_2(t) &= 2 \left\{ \int_0^l z^2 f_{p1}(z, t) dz + \int_l^\infty z^2 f_{p2}(z, t) dz \right\} \\ &= 2k_1 \sqrt{\sigma_1(t)} \sigma_1(t) \left\{ \sqrt{2\pi} \Phi(a_1) - a_1 \exp \left(-\frac{a_1^2}{2} \right) \right\} + 2K_2 \sqrt{\sigma_2(t)} \exp \left\{ -\frac{m(t)^2}{2\sigma_2(t)} \right\} \\ &\quad \times \left[\{b_1 \sigma_2(t) - 2\sqrt{\sigma_2(t)}\} \exp \left(-\frac{b_1^2}{2} \right) + \sqrt{2\pi} \{\sigma_2(t) + m(t)^2\} \left\{ \frac{1}{2} - \Phi(b_1) \right\} \right] \end{aligned} \quad (39)$$

$$\mu_3(t) = 0 \quad (40)$$

and

$$\begin{aligned}\mu_4(t) &= 2 \left\{ \int_0^l z^4 f_{p1}(z, t) dz + \int_l^\infty z^4 f_{p2}(z, t) dz \right\} \\ &= 2K_1 \sqrt{\sigma_1(t)} \sigma_1(t)^2 \left[3\sqrt{2\pi} \sigma_1(t) \Phi(a_1) - a_1 \{3 + \sigma_1(t) a_1^2\} \exp\left(-\frac{a_1^2}{2}\right) \right] \\ &\quad + 2K_2 \sqrt{\sigma_2(t)} \left[\{\sigma_2(t)^3 b_1^3 - 4m(t) \sqrt{\sigma_2(t)}\} b_1^2 + 3b_1 \sigma_2(t) \{b_2(t) + 2m(t)^2\} - 4m(t) \sqrt{\sigma_1(t)} \{2\sigma_2(t) + m(t)\} \right. \\ &\quad \times \exp\left(-\frac{b_1^2}{2}\right) + \sqrt{2\pi} \{3\sigma_2(t)^3 + m(t)^4 + 6m(t)^2 \sigma_2(t)\} \left\{ \frac{1}{2} - \Phi(b_1) \right\} \left. \right] \exp\left\{-\frac{m(t)^2}{2\sigma_2(t)}\right\} \dots\dots\dots (41)\end{aligned}$$

where

$$a_1 = l/\sqrt{\sigma_1(t)}, \quad b_1 = \{l + m(t)\}/\sqrt{\sigma_2(t)} \dots\dots\dots (42)$$

The skewness, $\beta_1(t)$, and the kurtosis, $\beta_2(t)$, of the time-dependent probability density function are calculated by

$$\beta_1(t) = \mu_3(t)/\mu_2(t)^{3/2} = 0 \dots\dots\dots (43)_1$$

and

$$\beta_2(t) = \mu_4(t)/\mu_2(t)^2 \dots\dots\dots (43)_2$$

5. Examples

Example-1: In Eq. (1), it is assumed as shown in Fig. 5 that the nonlinear element of zero-memory type is given by

$$f(z) = b \tanh cz \quad (b, c > 0) \dots\dots\dots (44)$$

The auto-correlation function of the stationary Gaussian random input suddenly applied to the system is given by Eq. (2). In this case, it is easily found that the response of the system subjected to a suddenly applied random input becomes a stationary random time series as the time elapses. Therefore, as shown in Fig. 6, the time $t = t_s$ at which the response approaches a steady state can

Fig. 5 Nonlinear characteristic given by Eq. (44)

Fig. 6 The method of determining the time-dependent probability density function

be determined with the help of a graphical procedure⁽³⁾. The probability density function of $z(t)$ in the range of $t > t_s$ can thus be regarded as the stationary one. We divide the time interval from t_0 to t_s into contiguous disjoint intervals of $t_{s-j+1} - t_{s-j} = \Delta_{s-j+1}$ ($j=1, 2, \dots, s$) and consider the probability density function of $z(t)$ in the interval, Δ_{s-j} , at the time $t = t_{s-j}$ ⁽³⁾.

The Fokker-Planck equation introduced from the control equation becomes

$$\begin{aligned}\frac{k^2}{T^2} \Phi_u \frac{\partial^2 f_p(z, t)}{\partial z^2} + \frac{kb}{T} \frac{\partial}{\partial z} \{f_p(z, t) \tanh cz\} \\ = \frac{\partial f_p(z, t)}{\partial t} \dots\dots\dots (45)\end{aligned}$$

The stationary probability density function of $z(t)$ satisfies the following differential equation derived from Eq. (45);

— Method of piecewise linearization
 Method of solving Fokker-Planck equation

Fig. 7 The time-dependent cumulative distribution function of the signal $z(t)$ of the system containing the nonlinear characteristic, $f(z) = b \tanh cz$, shown in example 1

$$\frac{k^2}{T^2} \Phi_u \frac{\partial f_p}{\partial z} + \frac{kb}{T} [f_p \tanh cz] = 0 \quad (46)$$

By solving Eq. (46), we have

$$f_p(z) = K / (\cosh cz)^{bT/ck\Phi_u} \quad (47)$$

where K is a normalized factor given by

$$K = \frac{c}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2} + \frac{bT}{2ck\Phi_u}\right)}{\Gamma\left(\frac{bT}{ck\Phi_u}\right)} \quad (48)$$

and $\Gamma(\zeta)$ is the Gamma function. Since random processes considered here approach the steady state as $t \rightarrow \infty$, the probability density function shown in Eq. (47) can be considered as

$$f_p(z, t)|_{t=\infty} = K / (\cosh cz)^{bT/ck\Phi_u} \quad (49)$$

Computed results of the cumulative distribution function given by Eq. (49) for the given set of parameters b, c, T and k are plotted in Fig. 7. From Eq. (49), it is easily seen that the stationary variance, $\sigma_z(t)|_{t=\infty} \equiv M_\infty$, equals to 0.72. Moreover, we can determine the time t_s as $t_s = 1.2$. We must determine the probability density function, $f_p(z, t)$, of $z(t)$ at the time $t_s = 1.2$. The procedure can be outlined in the following steps.

(1) From Eq. (24), since we can calculate the values of λ_1 and λ_2 for the case where the variance, M_∞ , of the weight function is 0.72, the equations corresponding to Eqs. (16) and (17) are written as follows;

$$f_1(z) = 1.44z \quad \text{for } 0 < z < 0.56 \quad (50)$$

and

$$f_2(z) = 0.18z + 0.706 \quad \text{for } 0.56 < z < \infty \quad (51)$$

(2) From Eqs. (21) and (22), the variance and the mean value are respectively given by

$$\sigma_1(t)|_{t=1.2} = 0.673 \quad \text{for } 0 < z < 0.56 \quad (52)$$

and

$$\sigma_2(t)|_{t=1.2} = 1.945, \quad m(t)|_{t=1.2} = 0.761 \quad \text{for } 0.56 < z < \infty \quad (53)$$

(3) Segments of Gaussian type probability density can thus be obtained as

$$f_{p1}(z, t)|_{t=1.2} = 0.346 \exp\left(-\frac{z^2}{1.346}\right) \quad \text{for } 0 < z < 0.56 \quad (54)$$

and

$$f_{p2}(z, t)|_{t=1.2} = 0.401 \exp\left(-\frac{z^2 + 1.522z}{3.89}\right) \quad \text{for } 0.56 < z < \infty \quad (55)$$

By a similar method, we can successively evaluate the probability density function of the response at the time $t = t_j$. Solid curves in Fig. 7 show the computing results of the time dependent cumulative distribution function. Fig. 8 shows the numerical results of the kurtosis, $\beta_2(t)$, for the distribution

Fig. 8 The kurtosis, $\beta_2(t)$, for the time-dependent probability density function of the signal $z(t)$

Fig. 9 Block diagram of the nonlinear control system in example 2

function given by Fig. 7.

Example-2: We consider the nonlinear control system as shown in Fig. 9. The input signal $v(t)$ and the response signal $z(t)$ of the system are related as

$$T \frac{dz(t)}{dt} + z(t) + kf[z(t)] = T \frac{dv(t)}{dt} + v(t) \quad (56)$$

We assume that the stationary Gaussian random input with the following statistical characteristics is suddenly applied to the system at the time $t=0$.

(1) the mean value;

$$m_v = 0 \quad (57)$$

(2) the auto-correlation function;

$$R_v(t_1, t_2) = \psi_v \exp(-\beta|t_2 - t_1|), \quad (\beta > 0) \quad (58)$$

(3) the spectral density:

$$S_v(\omega) = 2\psi_v\beta/(\beta^2 + \omega^2) \quad (59)$$

Since the nonlinear characteristic can be expressed by the connection of segments as shown in section 1, Eq. (56) is expressed as

$$T \frac{dz(t)}{dt} + (1 + k\lambda_1)z(t) = T \frac{dv(t)}{dt} + v(t) \quad \text{for } 0 < z < l \quad (60)$$

and

$$T \frac{dz(t)}{dt} + (1 + k\lambda_2)z + k\mu_2 = T \frac{dv(t)}{dt} + v(t) \quad \text{for } l < z < \infty \quad (61)$$

The mean value, $m(t)$, and the variance, $\sigma_i(t)$, are respectively determined as

$$m(t) = -\frac{k\mu_1}{1 + k\lambda_2} \left\{ 1 - \exp\left(-\frac{1 + k\mu_1}{T} t\right) \right\} \quad (62)$$

and

$$\frac{\sigma_i(t)}{\psi_0} = \frac{1+\beta T(1+k\lambda_i)}{(1+k\lambda_i)(1+k\lambda_i+\beta T)} + \frac{2k\lambda_i(\beta T-1)}{(1+k\lambda_i+\beta T)(\beta T-1-k\lambda_i)} \exp[-\{\beta+(1+k\lambda_i)/T\}t] \\ + \frac{(k\lambda_i)^2}{(1+k\lambda_i)(1+k\lambda_i-\beta T)} \exp[-2\{(1+k\lambda_i)/T\}t] \quad \dots\dots\dots (63)$$

Figs. (10) and (11) show the time-dependent cumulative distribution function of the system response $z(t)$.

6. Conclusions

In this paper, a method of obtaining the non-Gaussian probability distribution function of the response of nonlinear control systems subjected to a suddenly applied stationary Gaussian random input is described. The basic conceptions are as follows :

(1) We represent the nonlinear characteristic by equivalent straight-line segments in the sense of the least square method.

(2) By using the step (1), we determine the probability density functions of approximate Gaussian type as shown in Eqs. (9) and (10).

(3) We connect the curves in such a manner as they may satisfy the conditions given by Eqs. (11) and (12).

The analytical procedure presented here is a generalization of equivalent linearization technique of a nonlinear characteristic. To justify the results of analytical studies presented here, we compared the stationary probability density curves obtained by this method with the one obtained by means of

solving the Fokker-Planck equation. It is emphasized that the present analysis can be applied to time-variant nonlinear control systems.

Appendix : The solution of Fokker-Planck equation⁽⁴⁾

In order to obtain the solution of Eq. (4), we assume

$$f_p(z, t) = M(z)N(t) \quad \dots\dots\dots (A-1)$$

where M and N are functions of z and t respectively. Substituting Eq. (A-1) into Eq. (4), we have

$$\frac{1}{N} \cdot \frac{dN}{dt} = -\frac{\beta}{M} \frac{d^2M}{dz^2} + \alpha'(z) + \frac{\alpha(z)}{M} \frac{dM}{dz} \quad \dots\dots\dots (A-2)$$

where $\alpha'(z)$ expresses the derivative of $\alpha(z)$ with respect to z . By using the constant $\tilde{\lambda}^2$, Eq. (A-2) becomes

$$\frac{1}{N} \frac{dN}{dt} = -\tilde{\lambda}^2 \quad \dots\dots\dots (A-3)$$

and

$$\frac{d^2M}{dz^2} + \frac{\alpha(z)}{\beta} \frac{dM}{dz} + \frac{\alpha'(z) + \tilde{\lambda}^2}{\beta} M = 0 \quad \dots\dots\dots (A-4)$$

From Eq. (A-3), we have

$$N(t) = \exp(-\tilde{\lambda}^2 t) \quad \dots\dots\dots (A-5)$$

— Method of piecewise linearization
 Method of solving Fokker-Planck equation

Fig.10 The time-dependent cumulative distribution function of the signal $z(t)$ for system containing the nonlinear characteristic, $f(z) = bz + cz^3$, shown in example 2

— Method of piecewise linearization
 Method of solving Fokker-Planck equation

Fig.11 The time-dependent cumulative distribution function of the signal $z(t)$ for the system containing the nonlinear characteristic, $f(z) = b \tanh cz$

By setting

$$M(z) = \tilde{W} \exp \left\{ -\frac{1}{2} \int \frac{\alpha(z)}{\beta} dz \right\} \dots\dots\dots (A-6)$$

Eq. (A-4) becomes

$$\frac{d^2 \tilde{W}}{dz^2} + \left\{ \frac{\tilde{\lambda}^2}{\beta} - U(z) \right\} \tilde{W} = 0 \dots\dots\dots (A-7)$$

where

$$U(z) = \frac{1}{4\beta^2} \{ \alpha(z)^2 - 2\alpha'(z)\beta \} \dots\dots\dots (A-8)$$

Furthermore, if we assume that $\beta \ll 1$, by setting

$$\rho = \frac{\tilde{\lambda}^2}{\beta} - U(z) \dots\dots\dots (A-9)$$

and by introducing the Liouville transformation,

$$g = \sqrt[4]{\rho}, \quad \tilde{W} = \psi/g, \quad \theta = \int_0^z \sqrt{\rho} dz \dots\dots\dots (A-10)$$

Eq. (A-7) yields,

$$\frac{d^2 \psi}{d\theta^2} + \psi = 0 \dots\dots\dots (A-11)$$

From Eqs. (A-9), (A-10) and (A-11), we have

$$\tilde{W}(z) = \frac{\text{const}}{2 \sqrt[4]{\tilde{\lambda}^2 - \beta U(z)}} \exp \left\{ \pm \sqrt{\frac{-1}{\beta}} \times \int_0^z \sqrt{\tilde{\lambda}^2 - \beta U(z)} dz \right\} \dots\dots\dots (A-12)$$

The probability density function, $f_p(z, t)$, is approximately determined as shown in Eq. (7).

References

- (1) Y. Sawaragi, Y. Sunahara and T. Nakamizo: *Tech. Reps. Engng. Res. Inst., Kyoto Univ.*, X-1, Report No. 78 (1961).
- (2) Y. Sawaragi, Y. Sunahara and T. Soeda: *Tech. Reps. Engng. Res. Inst., Kyoto Univ.*, X-2, Report No. 79 (1961).
- (3) Y. Sawaragi, Y. Sunahara and T. Soeda: *Tech. Reps. Engng. Res. Inst., Kyoto Univ.*, XI-1, Report No. 91 (1962).
- (4) A.T. Bharucha-Reid: *Elements of the Theory of Markov Processes and their Applications* (1960), p. 130, McGraw-Hill.

532.526:532.582.8

Experiments on the Boundary Layer of an Oscillating Circular Cylinder*

By Ei-ichi HORI**

The mean and fluctuating velocities in the boundary layer of a circular cylinder, oscillating in the direction of the wind tunnel stream, have been measured using the hot-wire anemometer. The results have been compared with theoretical predictions.

(1) The effect of the velocity fluctuation on the mean velocity in the boundary layer is found to be negligible when the ratio of the amplitude of fluctuation of the outer flow to the mean velocity is less than 0.3. This agrees with theoretical predictions.

(2) Upstream of the point of minimum pressure, the effect of the frequency of oscillation on the boundary layer is found to be negligible when the frequency parameter is less than 0.3. The results of calculation, on the assumption of quasisteady flows, agree well with experimental velocity distributions.

(3) When the separation point is approached, the discrepancy between the theoretical and experimental results increases. Although the discrepancy is mainly attributable to the premature truncation of the series solution used in the theoretical calculation, other causes are left to be explored.

1. Introduction

In many engineering fluid flow problems, such as those of turbomachinery, boundary layer flows are

nearly but not quite steady. Fluid velocity fluctuates with small amplitude about a certain mean value.

Lighthill⁽¹⁾ was the first who proposed a method of theoretical calculation of such laminar fluctuating boundary layer by linearizing the differential equation on the assumption that the fluctuation velocity was small compared to the mean velocity. When the

* Received 30th November, 1961.

** Chief Research Fellow, Central Research Laboratory, Hitachi Ltd., Kokubunji, Tokyo.