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On the Forced Vibrations of the Shaft Carrying an Unsymmetrical Rotating Body*

(Response Curves of the Shaft at the Major Critical Speeds)

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A shaft carrying an unsymmetrical rotating body is statically unstable near its major critical speeds, and the unstable regions become smaller as damping forces are larger, and they vanish when the damping coefficients reach critical values. These critical damping coefficients c_{c1} , c_{c2} are given in the present paper. Then, the amplitudes and phase differences of forced lateral vibrations are studied; they are found to depend very much upon angular positions of static and dynamic unbalances. The characteristics of the response curves and phase differences are very different from those of a usual vibratory system. Furthermore, comparison with a flat shaft system is made.

1. Introduction

The authors named the rotor having two unequal principal moments of inertia I_1 , I_2 except the polar moment of inertia I_p , "unsymmetrical rotor", and they have studied analytically and experimentally the equations of motion, amplitude of forced vibrations, unstable vibrations of the whirling motions and unstable regions, natural frequencies of the system and amplitude ratios of free vibrations⁽¹⁾. In this paper, the response curves and phase differences of the system with viscous damping are calculated, and the effects of magnitudes of damping coefficients and of angular position between static and dynamic unbalance are studied.

2. Response curves and phase differences at the major critical speeds ω_c

The equations of motion of the shaft having an unsymmetrical rotor shown in Fig.1 are as follows;

$$\left. \begin{aligned} m\ddot{x} + c_1\dot{x} + \alpha x + \gamma\theta_x &= me\omega^2 \cos(\omega t + \xi) \\ m\ddot{y} + c_2\dot{y} + \alpha y + \gamma\theta_y &= me\omega^2 \sin(\omega t + \xi) \\ I\ddot{\theta}_x + I_p\omega\dot{\theta}_y + c_2\dot{\theta}_x + \gamma x + \delta\theta_x - \Delta I\{(\dot{\theta}_x \cos 2\omega t + \dot{\theta}_y \sin 2\omega t) - 2\omega(\dot{\theta}_x \sin 2\omega t - \dot{\theta}_y \cos 2\omega t)\} \\ &= \tau\omega^2\{(I_p - I)\cos(\omega t + \eta) - \Delta I\cos(\omega t - \eta)\} \\ I\ddot{\theta}_y - I_p\omega\dot{\theta}_x + c_2\dot{\theta}_y + \gamma y + \delta\theta_y - \Delta I\{(\dot{\theta}_x \sin 2\omega t - \dot{\theta}_y \cos 2\omega t) + 2\omega(\dot{\theta}_x \cos 2\omega t + \dot{\theta}_y \sin 2\omega t)\} \\ &= \tau\omega^2\{(I_p - I)\sin(\omega t + \eta) - \Delta I\sin(\omega t - \eta)\} \end{aligned} \right\} \dots\dots\dots (1)$$

where x, y are the deflections of the rotating body, i. e., the fixing point M; θ_x, θ_y are components of the inclination angle θ of the rotor in x, y direction respectively; m is the mass of the rotor, with the mass of shaft neglected; I_p the polar moment of inertia about MZ_1 ; I_1, I_2 ($I_1 > I_2$) the principal moments of inertia about MY_2 and MX_2 respectively; and $I = 1/2(I_1 + I_2)$, $\Delta I = 1/2(I_1 - I_2)$, ω angular velocity of the rotating shaft; α, γ and δ the spring constants of shaft; e the eccentricity of the rotor, i. e., static unbalance; τ a small deviational angle between principal axis MZ_1 and the tangent MZ_0 of the deflection curve of the shaft at the point M, i. e., dynamic unbalance; $\xi, \eta + 180^\circ$ are preceding angles of e and τ from principal axis MY_2 respectively; c_1 and c_2 viscous damping coefficients for deflections x, y and inclination angles θ_x, θ_y respectively.

Next dimensionless quantities are introduced for simplicity of calculation.

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$$\left. \begin{aligned} I_p/I = i_p, \quad \Delta I/I = \Delta, \quad x/\sqrt{\frac{I}{m}} = x', \quad y/\sqrt{\frac{I}{m}} = y', \quad e/\sqrt{\frac{I}{m}} = e', \quad \sqrt{\frac{\alpha}{m}} t = t', \\ \omega/\sqrt{\frac{\alpha}{m}} = \omega', \quad \gamma\sqrt{\frac{m}{I}}/\alpha = \gamma', \quad m\delta/\alpha I = \delta', \quad c_1/\sqrt{m\alpha} = c_1', \quad c_2\sqrt{\frac{m}{\alpha}}/I = c_2' \end{aligned} \right\} \dots\dots\dots (2)$$

Substituting Eqs. (2) into Eqs. (1) and omitting primes on the dimensionless quantities, we have

$$\left. \begin{aligned} \ddot{x} + c_1\dot{x} + x + \gamma\theta_x &= e\omega^2 \cos(\omega t + \xi) \\ \ddot{y} + c_2\dot{y} + y + \gamma\theta_y &= e\omega^2 \sin(\omega t + \xi) \\ \ddot{\theta}_x + i_p\omega\dot{\theta}_y + c_2\dot{\theta}_x + \gamma x + \delta\theta_x - \Delta\{(\dot{\theta}_x \cos 2\omega t + \dot{\theta}_y \sin 2\omega t) - 2\omega(\dot{\theta}_x \sin 2\omega t - \dot{\theta}_y \cos 2\omega t)\} \\ &= \tau\omega^2\{(i_p-1)\cos(\omega t + \eta) - \Delta\cos(\omega t - \eta)\} \\ \ddot{\theta}_y - i_p\omega\dot{\theta}_x + c_2\dot{\theta}_y + \gamma y + \delta\theta_y - \Delta\{(\dot{\theta}_x \sin 2\omega t - \dot{\theta}_y \cos 2\omega t) + 2\omega(\dot{\theta}_x \cos 2\omega t + \dot{\theta}_y \sin 2\omega t)\} \\ &= \tau\omega^2\{(i_p-1)\sin(\omega t + \eta) - \Delta\sin(\omega t - \eta)\} \end{aligned} \right\} \dots\dots\dots (3)$$

2.1 Forced vibrations and its unstable regions

When the rotor is excited by periodic disturbing forces due to e and τ , the shaft whirls with the same angular velocity as the rotating speed ω of the shaft.

$$\left. \begin{aligned} x &= E \cos(\omega t + \beta_1) = A \cos \omega t - B \sin \omega t, & \theta_x &= F \cos(\omega t + \beta_2) = C \cos \omega t - D \sin \omega t \\ y &= E \sin(\omega t + \beta_1) = A \sin \omega t + B \cos \omega t, & \theta_y &= F \sin(\omega t + \beta_2) = C \sin \omega t + D \cos \omega t \end{aligned} \right\} \dots\dots\dots (4)$$

Substituting Eqs. (4) into Eqs. (3), we have simultaneous linear equations of A , B , C and D of amplitudes of forced vibration.

$$\left. \begin{aligned} (1-\omega^2)A - c_1\omega B + \gamma C &= e\omega^2 \cos \xi \\ c_1\omega A + (1-\omega^2)B + \gamma D &= e\omega^2 \sin \xi \\ \gamma A + \{\delta + (i_p - 1 - \Delta)\omega^2\}C - c_2\omega D &= (i_p - 1 - \Delta)\tau\omega^2 \cos \eta \\ \gamma B + c_2\omega C + \{\delta + (i_p - 1 + \Delta)\omega^2\}D &= (i_p - 1 + \Delta)\tau\omega^2 \sin \eta \end{aligned} \right\} \dots\dots\dots (5)$$

Let the determinant consisting of coefficients α_{ij} in Eqs. (5) be $K_0' = |\alpha_{ij}|$. The major critical speed $\omega = \omega_c$ is given by putting $K_0' = 0$. When K_0' is negative, motions of the shaft become unstable and amplitudes A , B , C and D build up exponentially. The motion of the shaft is in a critical condition between stable and unstable when $K_0' = 0$:

$$K_0' = \{(1-\omega^2)X_1 - \gamma^2\} \{(1-\omega^2)X_2 - \gamma^2\} + c_1^2\omega^2 X_1 X_2 + c_2^2\omega^2(1-\omega^2)^2 + 2c_1c_2\gamma^2\omega^2 + c_1^2c_2^2\omega^4 = 0 \dots\dots\dots (6)$$

where $X = \delta + (i_p - 1)\omega^2$, $X_{1,2} = \delta + (i_p - 1 \mp \Delta)\omega^2$. Eq. (6) can be rewritten as follows:

$$\omega^4 \{(1-\omega^2)^2 + c_1^2\omega^2\} \Delta^2 = \{(1-\omega^2)X - \gamma^2\}^2 + c_1^2\omega^2 X^2 + c_2^2\omega^2(1-\omega^2)^2 + 2c_1c_2\gamma^2\omega^2 + c_1^2c_2^2\omega^4 \dots\dots\dots (7)$$

which represents the relation between unsymmetry Δ and the rotating speed ω .

The relation $2 \geq i_p \geq 2\Delta$ always holds and the

Fig. 1 The system having an unsymmetrical rotor ($a < b$)

vibratory system with an unsymmetrical rotor can be classified as follows⁽¹⁾: (a) $i_p \geq 1 + \Delta$, unstable range $[\omega_{c1}, \omega_{c2}]$, (b) $1 - \Delta > i_p$, unstable ranges $[\omega_{c1}, \omega_{c2}]$ and $[\omega_{c3}, \omega_{c4}]$ and (c) $1 + \Delta > i_p \geq 1 - \Delta$, unstable ranges $[\omega_{c1}, \omega_{c2}]$ and $[\omega_{c3}, \infty]$.

The experimental data used in the Experiment II stated in Reference⁽¹⁾ were as follows:

$$\begin{aligned} I_p &= 2.390 \text{ kg cm sec}^2, & I_1 &= 1.590 \text{ kg cm sec}^2, \\ I_2 &= 0.815 \text{ kg cm sec}^2, & l &= 50.55 \text{ cm}, & a : b &= 1 : 3.960, \\ mg &= 9.746 \text{ kg}, & \alpha &= 5.375 \times 10^2 \text{ kg/cm}, \\ \gamma &= -5.047 \times 10^3 \text{ kg/rad}, & \delta &= 6.881 \times 10^4 \text{ kg cm/rad}, \\ \sqrt{\frac{\alpha}{m}} &= 2217 \text{ rpm}, & \sqrt{\frac{I}{m}} &= 11.00 \text{ cm}. \end{aligned}$$

These values can be represented by dimensionless quantities of Eqs. (2) as follows:

$$i_p = 1.987, \quad \Delta = 0.322, \quad \gamma = -0.855 \text{ and } \delta = 1.060.$$

Since i_p is larger than $1 + \Delta$, only a lower unstable region $[\omega_{c1}, \omega_{c2}]$ can appear.

The relations between the unsymmetry Δ and the unstable range are given in Fig. 2, where the line AA furnishes the boundary lines for no damping system, the line BB for the system with damping

coefficients $c_1=c_2=0.05$ and the line CC for $c_1=c_2=0.10$. Horizontal line DD, i. e., line $\Delta=0.322$ intersects the line AA at $A_1(\omega=\omega_{c10})$, $A_2(\omega=\omega_{c20})$ and the line BB at $B_1(\omega_{c1})$, $B_2(\omega_{c2})$ but it does not cross the line CC.

The major critical speeds of a damped system can be determined not analytically but by numerical calculation of Eq.(6) or graphical procedure by Fig.2, while major critical speeds $\omega_{c i_0}$ ($i=1\sim 4$) for no damping system are furnished analytically as follows:

$$\left. \begin{aligned} \omega_{c10,30}^2 &= \frac{(i_p-1-\Delta-\delta) \pm \sqrt{(i_p-1-\Delta+\delta)^2-4(i_p-1-\Delta)\gamma^2}}{2(i_p-1-\Delta)} \\ \omega_{c20,40}^2 &= \frac{(i_p-1+\Delta-\delta) \pm \sqrt{(i_p-1+\Delta+\delta)^2-4(i_p-1+\Delta)\gamma^2}}{2(i_p-1+\Delta)} \end{aligned} \right\} \dots\dots\dots (8)$$

The motions of the shaft carrying an unsymmetrical rotor become always unstable near its major critical speeds provided $\gamma \neq 0$ and no damping. Thus for the shaft to run at higher than the major critical speeds we must pass through the critical speed rapidly enough or give the system a sufficient damping force. When the damping coefficients c_1 , c_2 are sufficiently large, the unstable regions near the major critical speed vanish, that is, the relation $K_0' \geq 0$ always holds. And we call the minimum values of damping coefficients which make K_0' always positive the critical damping coefficients c_{c1} , c_{c2} , which are shown in Fig.3. Near the lower unstable region, the deflections x , y are so much larger than the inclination angles θ_x , θ_y that c_1 is more effective than c_2 . The points A, B and C in Fig.3 correspond to the lines AA, BB and CC in Fig.2 respectively. When Δ is sufficiently small, ω in Eq. (6) is approximately replaced by the value of ω_c given by Eqs. (8) putting $\Delta=0$, and then an approximate equation for critical damping coefficients are as follows:

Fig. 2 Boundaries of unstable region

$$\frac{c_{c1}^2 \gamma^4}{(1-\omega_c^2)^2} + c_{c2}^2 (1-\omega_c^2)^2 + 2c_{c1}c_{c2}\gamma^2 + c_{c1}^2 c_{c2}^2 \omega_c^2 - \Delta^2 \omega_c^2 \{ (1-\omega_c^2)^2 + c_{c1}^2 \omega_c^2 \} = 0 \dots\dots\dots (9)$$

Approximate coefficients c_{c1} , c_{c2} are shown with broken lines in Fig.3.

2.2 Amplitudes of deflection of forced vibrations induced by eccentricity e

We can solve amplitudes A , B , C and D by Eqs. (5). The amplitude E of deflection being a vector sum of A and B always coexists with the amplitude F of inclination angle consisting of C and D , the former, however, predominates over the latter at the lower unstable range $[\omega_{c1}, \omega_{c2}]$ so that the amplitude E is only considered here.

The amplitude E induced by centrifugal force $e\omega^2$ and phase difference φ are

$$\left. \begin{aligned} E/e &= \sqrt{(A/e)^2 + (B/e)^2}, \quad \varphi = \xi - \beta_1, \\ \tan \beta_1 &= B/A \end{aligned} \right\} \dots\dots\dots (10)$$

$$\left. \begin{aligned} A/e &= \frac{\omega^2}{K_0'} \left[\{ (1-\omega^2)(X_1X_2+c_2^2\omega^2) - \gamma^2 X_1 \} \cos \xi \right. \\ &\quad \left. + \{ c_1\omega(X_1X_2+c_2^2\omega^2) + \gamma^2 c_2\omega \} \sin \xi \right] \\ B/e &= \frac{\omega^2}{K_0'} \left[- \{ c_1\omega(X_1X_2+c_2^2\omega^2) + \gamma^2 c_2\omega \} \cos \xi \right. \\ &\quad \left. + \{ (1-\omega^2)(X_1X_2+c_2^2\omega^2) - \gamma^2 X_2 \} \sin \xi \right] \end{aligned} \right\} \dots\dots\dots (11)$$

Fig. 3

Response curves and phase difference curves when $c_1=c_2=0.10$ are shown in Fig. 4 (a), (b). When $c_1=c_2=0.10$ the damping forces are too large for the motions to be unstable, and the amplitude E becomes maximum near ω_c as shown in Fig. 4. The behavior of vibration is very much affected by the value of angular position ξ . For instance, when $\xi=-45^\circ$, the amplitude E is smaller than that of a symmetrical rotor. In a symmetrical rotor the phase difference φ is nearly 90° at the rotating speed ω_c where the maximum amplitude takes place but not so in the unsymmetrical rotor. When the maximum amplitude appears, the phase difference φ is very different from 90° , and it is 114° , 56° and 127° for $\xi=-45^\circ$, 0° and 90° respectively. The following Eqs. (12) are easily derived from Eqs. (10) and (11):

$$(E)_{\xi+180^\circ} = (E)_\xi, \quad (\varphi)_{\xi+180^\circ} = (\varphi)_\xi \quad \dots\dots\dots(12)$$

Fig. 4 (a) Response curves

Fig. 4 (b) Phase difference curves

Then calculations may be done only for $90^\circ \geq \xi > -90^\circ$. The relation between the maximum amplitude $(E/e)_{\max}$ and the angle ξ is shown in polar coordinates in Fig. 4 (c) which is nearly symmetrical about the axis $\xi=45^\circ$. Clearly the curve for symmetrical rotor ($\Delta=0$) is a circle which is given by a broken line in Fig. 4 (c).

The points A_1, A_2 in Fig. 4 (b) are the intersections between a horizontal line $\varphi=90^\circ$ and vertical lines $\omega_{c10}, \omega_{c20}$, and we can approximately conclude that curves φ for $\xi=90^\circ$ and 0° pass through these points A_1 and A_2 respectively. This fact is generally explained by Eqs. (10) and (11). That is, for $\xi=0^\circ$, the relation

$$\tan \beta_1 = - \frac{c_1 \omega (X_1 X_2 + c_2^2 \omega^2) + \gamma^2 c_2 \omega}{X_1 \{(1 - \omega^2) X_2 - \gamma^2\} + c_2^2 (1 - \omega^2) \omega^2} \quad \dots\dots\dots(13)$$

holds, while we have $(1 - \omega^2) X_2 - \gamma^2 = 0$ when $\omega = \omega_{c20}$ by Eq. (6). Consequently the denominator of Eq. (13) is almost zero and we get $\varphi = -\beta_1 \doteq 90^\circ$. It can be seen that $\varphi \doteq 90^\circ$ at $\omega = \omega_{c10}$ when $\xi = 90^\circ$ by the similar procedure.

In Fig. 5 (a), (b), the unstable range $[\omega_{c1}, \omega_{c2}]$ is represented by the shadowed region which is determined by the points B_1 and B_2 in Fig. 2. The amplitude E becomes larger as the rotating speed ω approaches $\omega_{c1,2}$ and no stable amplitude exists really in the unstable range. The phase difference φ is nearly zero at a sufficiently lower rotating speed than ω_c and nearly 180° at a sufficiently higher speed than ω_c for all values of ξ , similarly to the general vibratory system. But φ changes discontinuously at $\omega = \omega_{c1,2}$ and for $\xi = -45^\circ$, it may be negative near the unstable range, that is, the forced vibration is ahead of the external force as seen in a flat shaft system.

Fig. 4 (c) Polar coordinate $\{(E/e)_{\max}, \xi\}$

2.3 Amplitudes of deflection of forced vibrations by deviational angle τ

We can not perfectly eliminate e and τ in an actual shaft and the vibration that occurs is vector

Fig. 5 (a) Response curves ($c_1=c_2=0.05$)

Fig. 5 (b) Phase difference curves ($c_1=c_2=0.05$)

sum of that induced by e and that by τ . Since the gyroscopic moment induced by τ lies in the direction $\omega t + \gamma + 90^\circ$, the point M deflects in the direction $\omega t + \gamma$. Now we consider the amplitude of deflection of forced vibration by τ . From Eqs. (4) and (5), we have

Fig. 6 (a) Response curves

Fig. 6 (b) Phase difference curves

Fig. 6 (c) Polar coordinate $\{(E/\tau)_{\max}, \eta\}$

$$E/\tau = \sqrt{(A/\tau)^2 + (B/\tau)^2}, \quad \varphi = \eta - \beta_1, \quad \tan \beta_1 = B/A \quad \dots\dots\dots(14)$$

$$\left. \begin{aligned} A/\tau &= \frac{\gamma \omega^2}{K_0'} [-(i_p - 1 - \Delta) \{(1 - \omega^2)X_2 - c_1 c_2 \omega^2 - \gamma^2\} \cos \eta - (i_p - 1 + \Delta) \omega \{c_1 X_1 + c_2 (1 - \omega^2)\} \sin \eta] \\ B/\tau &= \frac{\gamma \omega^2}{K_0'} [(i_p - 1 - \Delta) \omega \{c_1 X_2 + c_2 (1 - \omega^2)\} \cos \eta - (i_p - 1 + \Delta) \{(1 - \omega^2)X_1 - c_1 c_2 \omega^2 - \gamma^2\} \sin \eta] \end{aligned} \right\} \dots\dots\dots(15)$$

$$(E)_{\eta+180^\circ} = (E)_\eta, \quad (\varphi)_{\eta+180^\circ} = (\varphi)_\eta \quad \dots\dots\dots(16)$$

Amplitude E of deflection and phase differences φ are shown in Fig. 6 (a), (b) where damping coefficients c_1 and c_2 are 0.10. The maximum amplitude $(E/e)_{\max}$ for $\xi=90^\circ$ is almost equal to that for $\xi=0^\circ$ in Fig. 4 (a), but not in Fig. 6 (a) in which the former is much larger than the latter, because the gyroscopic moments for $\eta=90^\circ$ and $\eta=0^\circ$ are $(i_p - 1 + \Delta)\tau\omega^2$ and $(i_p - 1 - \Delta)\tau\omega^2$ respectively. The relation between the maximum amplitude $(E/\tau)_{\max}$ and the angle η is shown in Fig. 6 (c) in which the symmetrical axis of the figure is approximately the line $\eta=65^\circ$. Fig. 6 (c) is somewhat different from Fig. 4 (c).

2.4 Forced vibrations of inclination angle (when $\gamma=0$)

So far we considered only the case of $i_p > 1 + \Delta$, in which there exists one unstable range $[\omega_{e1}, \omega_{e2}]$ and the forced vibrations of deflection take place in more remarkable amplitude than those of inclination angle in the unstable range $[\omega_{e1}, \omega_{e2}]$. In the case $i_p < 1 + \Delta$ treated here, however, there are two unstable regions, the lower being $[\omega_{e1}, \omega_{e2}]$, and the higher $[\omega_{e3}, \omega_{e4}]$. In the neighborhood of the unstable range $[\omega_{e3}, \omega_{e4}]$ the vibrations of inclination angle predominate over those of deflection.

Here is treated the simplest system where an unsymmetrical rotor is mounted at the middle point of the shaft, i. e., $\gamma=0$, and the deflections x, y do not couple with inclination angles θ_x, θ_y , and the major critical speeds are $\omega_{e10} = \omega_{e20} = \omega_e = 1$. This system is considered the same as a single degree of freedom system of a mass fixed to the middle of a beam with regard to deflection. The major critical speeds ω_{e3}, ω_{e4} for the system with damping coefficient c_2 are determined from Eq. (6). In Eq. (6), putting $\gamma=c_1=0$, we get

$$\frac{\omega_{e4,3}^2}{\delta} = \frac{\left\{ (1-i_p) - \frac{c_2^2}{2\delta} \right\} \pm \sqrt{\left\{ (1-i_p) - \frac{c_2^2}{2\delta} \right\}^2 - \left\{ (1-i_p)^2 - \Delta^2 \right\}}}{(1-i_p)^2 - \Delta^2} \quad \dots\dots\dots(17)$$

If the value with the radical sign of Eq. (17) vanishes or is imaginary, the unstable region $[\omega_{e3}, \omega_{e4}]$ disappears. When unsymmetry Δ is small, the critical damping coefficient c_{e2} is approximately gained by Eq. (9) or (17) as follows:

$$c_{e2} = \Delta \sqrt{\frac{\delta}{1-i_p}} \quad \dots\dots\dots(18)$$

The amplitude F of inclination angle and the phase difference φ are obtained from the 3rd and 4th equations of Eqs. (4) and (5).

$$\left. \begin{aligned} F/\tau &= \sqrt{(C/\tau)^2 + (D/\tau)^2}, \quad \varphi = \eta + 180^\circ - \beta_2, \\ \tan \beta_2 &= D/C \end{aligned} \right\} \dots\dots\dots(19)$$

$$\left. \begin{aligned} C/\tau &= \frac{\omega^2}{K_0'} [(i_p - 1 - \Delta) X_2 \cos \eta + (i_p - 1 + \Delta) c_2 \omega \sin \eta] \\ D/\tau &= \frac{\omega^2}{K_0'} [-(i_p - 1 - \Delta) c_2 \omega \cos \eta + (i_p - 1 + \Delta) X_1 \sin \eta] \end{aligned} \right\} \dots\dots\dots(20)$$

Using the same values l and $\sqrt{I/m}$ as in Experiment II described in Reference (1), dimensionless quantities $i_p=2/3$, $\Delta=0.1$, $\gamma=0$, $\delta=5.313$ and $c_2=0.7$ are given by calculation and the results are shown in Fig. 7 (a), (b).

In section 2.3, $i_p - 1 \mp \Delta$ is always positive and the positive moment $(i_p - 1 \mp \Delta)\tau\omega^2$ acting on the shaft ($a < b$) results in inclining in the direction of $\omega t + \gamma$, but in this section the moment $(i_p - 1 \mp \Delta)\tau\omega^2$ takes always a negative value, in the brackets of which the upper sign corresponds to $\gamma = 0^\circ$ and the lower sign to $\gamma = 90^\circ$ so the shaft inclines in the direction of $\omega t + \gamma + 180^\circ$. Phase difference φ for $\gamma = 90^\circ$, 0° just passes through the points A_3, A_4 which are intersections of lines $\omega = \omega_{c30}, \omega_{c40}$ and the horizontal line $\varphi = 90^\circ$.

3. Comparison with a flat shaft system

Characters peculiar to the shaft system having an unsymmetrical rotor are also seen in the flat shaft system having a symmetrical rotor. To rotate around the axis perpendicular to the plane containing the two principal axes, of unequal moments of inertia of a rotor in the former, and of unequal moments of inertia of area of a shaft in the latter, results in the similar characteristics to those mentioned above. Namely, referring to the fixed coordinate system, values of moment of inertia of a rotor in the former as well as of stiffness of the shaft in the latter periodically change with the period π/ω .

Let the spring constants of a flat shaft be $\alpha + \Delta\alpha$, $\gamma + \Delta\gamma$ and $\delta + \Delta\delta$ in the direction of MX_2 , and $\alpha - \Delta\alpha$, $\gamma - \Delta\gamma$ and $\delta - \Delta\delta$ in the direction of MY_2 in Fig. 1. For simplicity, we treat a flat shaft of uniform cross section and let moments of inertia of area about the neutral lines MY_2 and MX_2 be I_{Y_2} , I_{X_2} ($I_{Y_2} > I_{X_2}$) respectively. Flatness Δ_s of the flat shaft is

$$\Delta_s = \frac{\Delta\alpha}{\alpha} = \frac{\Delta\gamma}{\gamma} = \frac{\Delta\delta}{\delta} = \frac{I_{Y_2} - I_{X_2}}{I_{Y_2} + I_{X_2}} \quad \dots\dots\dots(21)$$

Fig. 8 Boundaries of unstable region ($r=0$)

and the equations of motion of the flat shaft carrying a symmetrical rotor are obtained as dimensionless quantities of Eqs. (2) and Eqs. (21) as follows:

$$\left. \begin{aligned} \ddot{x} + c_1 \dot{x} + x + \gamma \theta_x - \Delta_s (x \cos 2\omega t + y \sin 2\omega t) \\ - \gamma \Delta_s (\theta_x \cos 2\omega t + \theta_y \sin 2\omega t) &= e\omega^2 \cos(\omega t + \xi) \\ \ddot{y} + c_1 \dot{y} + y + \gamma \theta_y - \Delta_s (x \sin 2\omega t - y \cos 2\omega t) \\ - \gamma \Delta_s (\theta_x \sin 2\omega t - \theta_y \cos 2\omega t) &= e\omega^2 \sin(\omega t + \xi) \\ \ddot{\theta}_x + i_p \omega \dot{\theta}_y + c_2 \dot{\theta}_x + \gamma x + \delta \theta_x - \gamma \Delta_s (x \cos 2\omega t \\ + y \sin 2\omega t) - \delta \Delta_s (\theta_x \cos 2\omega t + \theta_y \sin 2\omega t) \\ &= (i_p - 1)\tau\omega^2 \cos(\omega t + \gamma) \\ \ddot{\theta}_y - i_p \omega \dot{\theta}_x + c_2 \dot{\theta}_y + \gamma y + \delta \theta_y - \gamma \Delta_s (x \sin 2\omega t \\ - y \cos 2\omega t) - \delta \Delta_s (\theta_x \sin 2\omega t - \theta_y \cos 2\omega t) \\ &= (i_p - 1)\tau\omega^2 \sin(\omega t + \gamma) \end{aligned} \right\} \quad \dots\dots\dots(22)$$

Substituting Eqs. (4) into Eqs. (22), we obtain

$$\left. \begin{aligned} (1 - \Delta_s - \omega^2)A - c_1 \omega B + \gamma(1 - \Delta_s)C &= e\omega^2 \cos \xi \\ c_1 \omega A + (1 + \Delta_s - \omega^2)B + \gamma(1 + \Delta_s)D &= e\omega^2 \sin \xi \\ \gamma(1 - \Delta_s)A + \{\delta(1 - \Delta_s) + (i_p - 1)\omega^2\}C \\ - c_2 \omega D &= (i_p - 1)\tau\omega^2 \cos \gamma \\ \gamma(1 + \Delta_s)B + c_2 \omega C + \{\delta(1 + \Delta_s) + (i_p - 1)\omega^2\}D \\ &= (i_p - 1)\tau\omega^2 \sin \gamma \end{aligned} \right\} \quad \dots\dots\dots(23)$$

For convenience of comparison with the system treated by H.D. Taylor⁽²⁾, we consider the vibrations

Fig. 9 (a) Response curves

Fig. 9 (b) Phase difference curves

Fig. 10 (a) Response curves

Fig. 10 (b) Phase difference curves

of deflection in the simplest system of $\gamma=0$.

Amplitudes A and B are obtained from Eqs. (23),

$$\left. \begin{aligned} A/e &= \frac{\omega^2}{K_0'} [(1 + \Delta_s - \omega^2) \cos \xi + c_1 \omega \sin \xi] \\ B/e &= \frac{\omega^2}{K_0'} [-c_1 \omega \cos \xi + (1 - \Delta_s - \omega^2) \sin \xi] \\ K_0' &= (1 - \Delta_s - \omega^2)(1 + \Delta_s - \omega^2) + c_1^2 \omega^2 \end{aligned} \right\} \dots\dots\dots (24)$$

and

$$c_{e1} = \sqrt{2(1 - \sqrt{1 - \Delta_s^2})} \div \Delta_s \dots\dots\dots (25)$$

Boundaries of unstable regions of the flat shaft system are shown in Fig. 8. Response curves and phase difference curves of flatness of shaft $\Delta_s=0.322$ are shown for the case of $c_1=0.5 > c_{e1}$ in Fig. 9 (a), (b) and for the case of $c_1=0.25 < c_{e1}$ in Fig. 10 (a), (b). Fig. 4 (a), (b) of an unsymmetrical rotor system are seen almost to coincide with Fig. 9 (a),

(b) of a flat shaft system, and also Fig. 5 (a), (b) with Fig. 10 (a), (b).

Figs. 9 coincide qualitatively with the results given by Taylor for $c_1=0.07$ and $\Delta_s=0.05$, and Figs. 10 coincide qualitatively with the results by Taylor for $c_1=0.07$ and $\Delta_s=0.09$ but phase differences in the unstable region $[\omega_{c1}, \omega_{c2}]$ do not agree.

References

- (1) T. Yamamoto and H. Ōta: *Trans. Japan Soc. Mech. Engrs.*, Vol. 28, No. 188 (1962), p. 475.
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Discussion

A. TAMURA: Consider a system, the rotor of which is mounted at the middle point of the shaft, and $\Delta \neq 0$. Assuming that there is no damping, we obtain $\omega_s = \sqrt{\delta/(1-i_p)}$ from Eq. (17). On the other hand we consider the fundamental mode of vibration to incline the rotor which has no rotation. What relation is there? It seems that both the angular frequencies coincide with each other if the gyroscopic

effect is neglected.

Authors' closure

(1) Since vibrations of deflection do not couple with vibrations of inclination angle in the system questioned, the system is considered as one degree of freedom system, in so far as vibrations of deflection are considered. Accordingly, its natural frequencies

p_2 and p_3 are $\pm\sqrt{\alpha/m}$, i. e., ± 1 expressed by dimensionless quantity. Meanwhile, there are vibrations in regard to inclination angle. Its natural frequencies p_1 and p_4 are functions of the rotating speed ω because of existence of a gyroscopic moment, provided i_p does not vanish.

The supplementary figure shows the relation between natural frequency and rotating speed, where p coexists with $\bar{p}$ because of unsymmetry of the rotor. The major critical speeds are obtained as the cross points at which natural frequency p intersects the line $p=\omega$, and is generally larger than the natural frequency p_{i_0} of a nonrotating shaft. The major critical speed ω_c coincides with p_{i_0} only when there is no gyroscopic effect, i. e., $i_p=0$.

(2) Major critical speeds ω_{c30} , ω_{c40} obtained by Eqs. (8) putting $\gamma=0$ coincide with ω_{c3} , ω_{c4} given by Eqs. (17) putting $c_2=0$, because Eqs. (8) are the solutions of Eq. (6) putting $c_1=c_2=0$, and Eqs. (17) are also the solutions of Eq. (6) putting $\gamma=0$, $c_1=0$.

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On the Response Curves and the Stability of "Summed and Differential Harmonic" Oscillations*

By Toshio YAMAMOTO** and Satoru HAYASHI***

In the present paper, the authors obtain theoretically the response curves of summed and differential harmonic oscillations induced by symmetrical and unsymmetrical nonlinearities of spring forces, and they investigate the influences of nonlinearities and damping force of those vibrations. Further the stability criteria of vibrations are analytically studied. The conclusions obtained are as follows:

(1) The characteristics of summed and differential harmonic oscillations are analogous to those of subharmonic oscillations in the order of $1/2$ and $1/3$.

(2) There are the critical values of damping coefficients, and summed and differential harmonic oscillations cannot take place, when the magnitudes of damping are larger than these values.

(3) The loci of vertical tangent of the response curves divide the response curves into two parts, and the upper parts of the loci constitute a stable range, the lower ones an unstable range.

1. Introduction

Summed and differential harmonic oscillations

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induced by nonlinearities of spring force have been investigated experimentally and theoretically⁽¹⁾⁽²⁾.

In the present paper, the authors calculate the response curves and the frequencies of $[p_1+p_2]$, $[2p_1+p_2]$ and $[p_1+2p_2]$ modes of vibrations in a two degrees-of-freedom system having nonlinearities of second and third order, and they examine the