section 5-1.

(4) In order to reduce the weight of hoisting drum to a minimum, the number of stiffening rings should be 1 to 3 usually.

Acknowledgement

The author wishes to express heartfelt thanks to Professor S. Taira of Kyoto University and to former Dept. Manager T. Kobori, Sect. Chief H. Miyamoto and Sect. Chief T. Egawa, of Kameari Works, Hitachi, Ltd., for their kind guidances and encouragements.

539.384.6

Strength and Deformation of a Cylindrical Shell Fixed at One End and Supported at the Other under the Weight of Lining*

By Koki MIZOGUCHI** and Keinosuke INOUE***

In this paper, the strength and deformation of a cylindrical shell fixed at one end and supported at the other under the weight of lining is discussed, utilizing the fundamental differential equation introduced by Dr. K. Mizoguchi.

The method of solving and the results here obtained can be applied to the design of a rotary kiln.

1. Solution of problem

Referring to Fig. 1, for the sake of simplicity, the following assumptions are made, i.e., one end of a thin-walled pipe is fixed and the other end of it is supported and fitted with a thin plate, the extensional and the flexural rigidities of which are sufficiently large and small respectively; furthermore the lining, the flexural rigidities and resistances of which are neglected for safety's sake, is completely

Fig. 1

* Received 7th April, 1962.

** Professor, Department of Mechanical Engineering, University of Osaka Prefecture, Sakai.

*** Graduate Student, University of Osaka Prefecture.

References

- (1) G. McCulloch and T.C. Futers: *Winding Engines and Winding Appliances*, (1924), p.290, Edward Arnold & Co.
- (2) H.H. Broughton: *Electric Winders*, (1948), p.273, E. & F.N. Spon Ltd.
- (3) N. Sag and A.C. Briggs: *Mach. Design*, Vol. 25, No.4 (1953-4), p. 265.
- (4) T. Egawa and M. Taneda: *Bulletin of JSME*, Vol.1, No. 2 (1958), p.133.
- (5) M. Taneda: *Bulletin of JSME*, Vol. 4, No. 13 (1961), p. 26.
- (6) P.G. Hodge: *Jour. Appl. Mech.*, Vol.21, No.4 (1954-12), p.336.
- (7) R.G. Sturm: *Bull. Univ. Illinois' Engng. Exp. Station*, No.329 (1941-11).
- (8) S. Timoshenko: *Theory of Plates and Shells*, (1940), p. 403, McGraw-Hill.

fixed on the inner surface of the pipe.

The fundamental differential equation in a single displacement under radial and circumferential loads introduced by Dr. K. Mizoguchi is as follows⁽¹⁾:-

$$\begin{aligned}
 & A \frac{\partial^8 w}{\partial x^8} + \frac{4B}{a^2} \frac{\partial^8 w}{\partial x^6 \partial \theta^2} + \frac{6C}{a^4} \frac{\partial^8 w}{\partial x^4 \partial \theta^4} \\
 & + \frac{8-2\nu^2}{a^4} \frac{\partial^6 w}{\partial x^4 \partial \theta^2} + \frac{6L}{a^4} \frac{\partial^4 w}{\partial x^4} \\
 & + \frac{4}{a^6} \left[\frac{\partial^8 w}{\partial x^2 \partial \theta^6} + 2 \frac{\partial^6 w}{\partial x^2 \partial \theta^4} + \frac{\partial^4 w}{\partial x^2 \partial \theta^2} \right] \\
 & + \frac{1}{a^8} \left[\frac{\partial^8 w}{\partial \theta^8} + 2 \frac{\partial^6 w}{\partial \theta^6} + \frac{\partial^4 w}{\partial \theta^4} \right] \\
 & = \frac{1}{D} \left[A \frac{\partial^4 p}{\partial x^4} + \frac{2}{a^2} U \frac{\partial^4 p}{\partial x^2 \partial \theta^2} + \frac{B}{a^4} \frac{\partial^4 p}{\partial \theta^4} \right] \\
 & - \frac{1}{D} \left[\frac{(2-\nu)}{6(1-\nu)} \left(\frac{h}{a} \right)^2 \frac{\partial^5 q}{\partial x^4 \partial \theta} - \frac{(2+\nu)}{a^2} \frac{\partial^3 q}{\partial x^2 \partial \theta} \right. \\
 & + \frac{4-3\nu+\nu^2}{12(1-\nu)a^2} \left(\frac{h}{a} \right)^2 \frac{\partial^5 q}{\partial x^2 \partial \theta^3} + \frac{1}{12a^4} \left(\frac{h}{a} \right)^2 \frac{\partial^5 q}{\partial \theta^5} \\
 & \left. - \frac{1}{a^4} \frac{\partial^3 q}{\partial \theta^3} \right] \dots \dots \dots (1)
 \end{aligned}$$

where w is the radial displacement, and D =flexural rigidity of plate, ν =Poisson's ratio. $A=1+(h/a)^2/3$, $B=1+(h/a)^2/12$, $C=1+(1-\nu^2)(h/a)^2/72$, $L=2(1-\nu^2)(a/h)^2A$, $U=1+[1+(1-\nu^2)(h/a)^2/12(1-\nu)]$, and p , q are the radial, circumferential surface forces per unit area of the shell.

Denote by γ , $f(x)=\alpha_0x+\beta_0$ the weight per unit volume of the lining and the variation of thickness of it, then the radial, circumferential forces per unit area of the wall of the pipe are expressed as follows :-

$$\left. \begin{aligned} p &= -\gamma(\alpha_0x+\beta_0) \cos \theta \\ q &= \gamma(\alpha_0x+\beta_0) \sin \theta \end{aligned} \right\} \dots\dots\dots (2)$$

Referring to Figs.1 and 2, the boundary conditions are expressed as follows :-

$$\left. \begin{aligned} 1) & \text{ for } x=0, w=0, u=0, v=0, \partial w/\partial x=0 \\ 2) & \text{ for } x=l, w=0, v=0, T_x=0, M_x=0 \end{aligned} \right\} \dots\dots\dots (3)$$

The solution of Eq.(1) in connection with the conditions (3) can be proved to be unique by means of the strain energy⁽¹⁾.

Substituting the first and second of the expressions (2) in the right hand side of Eq.(1), we can immediately obtain the particular solutions for p and q denoted by w_{pp} , w_{pq} , which take equal values in this case, and can be expressed as follows :-

$$\left. \begin{aligned} w_{pp} &= w_{pq} = -kx^5 \cos \theta - c_0x^4 \cos \theta \\ \text{where } k &= \frac{\gamma\alpha_0(h/a)^2}{1440(1-\nu^2)BD}, \quad c_0 = \frac{\gamma\beta_0(h/a)^2}{288(1-\nu^2)BD} \end{aligned} \right\} \dots\dots\dots (4)$$

$$\left. \begin{aligned} A\lambda^4 + \frac{4B}{a^2}\lambda^2 + \frac{12(1-\nu^2)B^2}{a^2h^2} &= 0, \text{ or } \lambda = \alpha + i\beta \\ \alpha^2 &= \frac{B}{Aa^2} \left[\frac{a}{h} \sqrt{3(1-\nu^2)A} - 1 \right], \quad \beta^2 = \frac{B}{Aa^2} \left[\frac{a}{h} \sqrt{3(1-\nu^2)A} + 1 \right] \end{aligned} \right\} \dots\dots\dots (7)$$

The expressions of u_{pp} , v_{pp} corresponding to w_{pp} and of u_{pq} , v_{pq} corresponding to w_{pq} of Eq.(4) can be obtained by the $u \sim w$ and $v \sim w$ relations which do not contain p , q respectively⁽¹⁾⁽⁸⁾.

Next the corresponding displacements u_c , v_c to w_c of Eq.(5), which can be determined by either relation mentioned above, because w_c is for $p=q=f=0$, are determined by the former in this case.

Thus, the expressions of u , v corresponding to $w=w_{pp}+w_{pq}+w_c$ take the following forms :

$$\left. \begin{aligned} u &= u_{pp} + u_{pq} + u_c \\ &= 5ak \left[2x^4 + 24(2+\nu)a^2x^2 + \frac{24a^4}{B} \left\{ 2(2+\nu)U - A + (2+\nu)^2 + \frac{5+\nu-\nu^2+\nu^3}{12(1-\nu)} \left(\frac{h}{a} \right)^2 \right\} \right] \cos \theta \\ &\quad + 8ac_0[x^3 + 6(2+\nu)a^2x] \cos \theta - 3c_1a[x^2 + 2(2+\nu)a^2] \cos \theta - 2ac_2x \cos \theta \\ &\quad - c_3a \cos \theta + a\Re\varphi \{ \alpha_1 \cos \lambda(x-l) - \alpha_2 \sin \lambda x \} \cos \theta \\ v &= v_{pp} + v_{pq} + v_c \\ &= k \left[2x^5 - 40\nu a^2x^3 - \frac{5a^4}{B} (G_p + G_q)x \right] \sin \theta + c_0 \left[2x^4 - 24\nu a^2x^2 - \frac{a^4}{B} (G_p + G_q) \right] \sin \theta \\ &\quad - c_1(x^3 - 6\nu a^2x) \sin \theta - c_2(x^2 - 2\nu a^2) \sin \theta - c_3x \sin \theta \\ &\quad - c_4 \sin \theta - \Re\eta \{ \alpha_1 \sin \lambda(x-l) + \alpha_2 \cos \lambda x \} \sin \theta \end{aligned} \right\} \dots\dots\dots (8)$$

Fig. 2

For the complementary solution w_c of Eq.(1), we take the following expression :

$$\left. \begin{aligned} w_c &= c_1x^3 \cos \theta + c_2x^2 \cos \theta + c_3x \cos \theta + c_4 \cos \theta \\ &\quad + \Re \xi \{ \alpha_1 \sin \lambda(x-l) + \alpha_2 \cos \lambda x \} \cos \theta \end{aligned} \right\} \dots\dots\dots (5)$$

$$\xi \equiv A\lambda^4 + \frac{2}{a^2}U\lambda^2 + \frac{B}{a^4}$$

The general solution of Eq.(1) is represented in the form of the sum of them as follows :

$$w = w_{pp} + w_{pq} + w_c \dots\dots\dots (6)$$

In the expression (5), c_1 , c_2 , c_3 , c_4 , α_1 and α_2 are arbitrary constants which must be determined by the boundary conditions (3) and α_1 , α_2 are complex numbers and $\Re$ means to take the real part of the function following it, λ being a complex number defined by the following equations.

where

$$\left. \begin{aligned} G_p &\equiv 4 \left\{ 6(1+2\nu) + \nu(1-\nu) \left(\frac{h}{a} \right)^2 \right\}, \quad G_q \equiv 2 \left\{ 12(2+\nu)\nu - (1-\nu)^2 \left(\frac{h}{a} \right)^2 \right\} \\ \varphi &\equiv \frac{\lambda}{a^2} \left[\left\{ \nu - \frac{2-3\nu+3\nu^2}{12(1-\nu)} \left(\frac{h}{a} \right)^2 \right\} \lambda^2 - \frac{B}{a^2} \right] \\ \eta &\equiv \frac{(2-\nu)}{6(1-\nu)} \left(\frac{h}{a} \right)^2 \lambda^4 + \frac{1}{a^2} \left\{ (2+\nu) + \frac{4-3\nu+\nu^2}{12(1-\nu)} \left(\frac{h}{a} \right)^2 \right\} \lambda^2 + \frac{B}{a^4} \end{aligned} \right\} \dots\dots\dots (9)$$

and G_p coincides with G in the previous paper⁽¹⁾ by one of the authors.

Substituting the expressions (4), (5) and (8) in the relations⁽¹⁾ between stresses and strains, the expressions of stress resultants and couples are obtained as follows :

$$\left. \begin{aligned} T_x &= \frac{12D}{h^2} \left[20akx \left[2(1-\nu^2)x^2 + 3 \left\{ 4(2+\nu) - \frac{\nu G}{12B} \right\} a^2 \right] + 12c_0a \left[2(1-\nu^2)x^2 + \left\{ 4(2+\nu) - \frac{\nu G}{12B} \right\} a^2 \right] \right. \\ &\quad \left. - 2a(1-\nu^2)c_2 - 6c_1a(1-\nu^2)x + a\Re\zeta \{ \alpha_1 \sin \lambda(x-l) + \alpha_2 \cos \lambda x \} \right] \cos \theta \\ M_x &= D \left[-5kx \left\{ 8(1-\nu^2)x^2 - \frac{\nu G}{B} a^2 \right\} + 6c_1(1-\nu^2)x - 24(1-\nu^2)c_0x^2 \right. \\ &\quad \left. + \left\{ 2(1-\nu^2)c_2 + \frac{\nu G}{B} c_0a^2 \right\} - \Re\rho \{ \alpha_1 \sin \lambda(x-l) + \alpha_2 \cos \lambda x \} \right] \cos \theta \\ T_\theta &= \frac{12D}{h^2} \left[5a^3kx \left[48\nu(2+\nu) - \frac{G}{B} \right] + 12c_0a^3 \left[4\nu(2+\nu) - \frac{G}{12B} \right] \right. \\ &\quad \left. + \Re \frac{1}{a} \left\{ (1-\nu^2) - \frac{\nu^2}{4} \left(\frac{h}{a} \right)^2 \right\} \lambda^4 \{ \alpha_1 \sin \lambda(x-l) + \alpha_2 \cos \lambda x \} \right] \cos \theta \\ S &= -\frac{30(1-\nu)D}{h^2} a^2k \left[48(1+\nu)x^2 + \frac{a^2}{B} \left[G + 24 \left\{ 2(2+\nu)U - A + (2+\nu)^2 + \frac{5+\nu-\nu^2+\nu^3}{12(1-\nu)} \left(\frac{h}{a} \right)^2 \right\} \right] \right] \sin \theta \\ &\quad + \frac{72(1-\nu^2)a^2D}{h^2} (c_1 - 8c_0x) \sin \theta - \frac{6(1-\nu)D}{h^2} \Re\lambda \left\{ \frac{(2-\nu)}{6(1-\nu)} \left(\frac{h}{a} \right)^2 \lambda^4 \right. \\ &\quad \left. + \frac{2(1+\nu)B}{a^2} \lambda^2 \right\} \{ \alpha_1 \cos \lambda(x-l) - \alpha_2 \sin \lambda x \} \sin \theta \\ M_\theta &= D \left[\frac{5a^2G}{B} kx + \frac{G}{B} c_0a^2 - \Re\lambda^2 \left\{ \nu A\lambda^4 + \frac{1}{a^2} (2\nu U + V) \lambda^2 \right\} \{ \alpha_1 \sin \lambda(x-l) + \alpha_2 \cos \lambda x \} \right] \cos \theta \\ \Re &= 5(1-\nu)Dak \left[24\nu x^2 + \frac{G}{B} a^2 \right] \sin \theta + 6\nu(1-\nu)aD(8c_0x - c_1) \sin \theta \\ &\quad - \frac{(1-\nu)D}{a} \Re\lambda \left\{ V\lambda^4 - \frac{\nu B}{a^2} \lambda^2 \right\} \{ \alpha_1 \cos \lambda(x-l) - \alpha_2 \sin \lambda x \} \sin \theta \\ S' - \frac{\Re}{a} &= -\frac{30(1-\nu)Dka^2}{h^2} \left[48(1+\nu)Ix^2 + \frac{AG}{B} a^2 + \frac{24}{B} a^2 \left[2(2+\nu)U - A + (2+\nu)^2 \right. \right. \\ &\quad \left. \left. + \frac{5+\nu-\nu^2+\nu^3}{12(1-\nu)} \left(\frac{h}{a} \right)^2 \right] \right] \sin \theta + \frac{72(1-\nu^2)a^2ID}{h^2} (c_1 - 8c_0x) \sin \theta \\ &\quad - D\Re\varphi' \{ \alpha_1 \cos \lambda(x-l) - \alpha_2 \sin \lambda x \} \sin \theta \\ Nx + \frac{1}{a} \frac{\partial \Re}{\partial \theta} &= 5kD \left[-24(1-\nu)^2x^2 + \frac{(2-\nu)G}{B} a^2 \right] \cos \theta \\ &\quad + 6(c_1 - 8c_0x)(1-\nu)^2D \cos \theta - D\Re\psi \{ \alpha_1 \cos \lambda(x-l) - \alpha_2 \sin \lambda x \} \cos \theta \end{aligned} \right\} \dots\dots\dots (10)$$

where

$$\left. \begin{aligned} V &\equiv 1 - \frac{\nu}{6(1-\nu)} \left(\frac{h}{a} \right)^2, \quad I \equiv 1 + \frac{\nu}{6(1+\nu)} \left(\frac{h}{a} \right)^2, \quad G = G_p + G_q \\ \zeta &\equiv \frac{1}{a^2} \left[\frac{(2-\nu)}{12} \left(\frac{h}{a} \right)^2 \lambda^4 + \frac{B}{a^2} (1-\nu^2) \lambda^2 \right] \\ \rho &\equiv \lambda^2 \left[A\lambda^4 + \frac{1}{a^2} (2U + \nu V) \lambda^2 + \frac{(1-\nu^2)B}{a^4} \right] \\ \varphi' &\equiv \lambda \left[\frac{\nu A}{a^2} \lambda^4 + \frac{12(1-\nu^2)BI}{a^2h^2} \lambda^2 \right] \\ \psi &\equiv \lambda^3 \left[A\lambda^4 + \frac{1}{a^2} \left\{ 4-\nu + \frac{1-\nu}{3} \left(\frac{h}{a} \right)^2 \right\} \lambda^2 + \frac{(1-\nu)^2B}{a^4} \right] \end{aligned} \right\} \dots\dots\dots (11)$$

Substituting the expressions (4), (5), (8) and (10) in the boundary conditions (3), we have

$$\left. \begin{aligned}
c_4 + \Re \xi (\alpha_2 - \alpha_1 \sin \lambda l) &= 0 \\
\frac{120ka^4}{B} \left\{ 2(2+\nu)U - A + (2+\nu)^2 + \frac{5+\nu-\nu^2+\nu^3}{12(1-\nu)} \left(\frac{h}{a}\right)^2 \right\} - 6(2+\nu)a^2c_1 - c_3 + \Re \alpha_1 \varphi \cos \lambda l &= 0 \\
\frac{G}{B} a^4 c_0 - 2\nu a^2 c_2 + c_4 + \Re \gamma (\alpha_2 - \alpha_1 \sin \lambda l) &= 0 \\
c_3 + \Re \alpha_1 \xi \lambda \cos \lambda l &= 0 \\
-2kl^5 - 2c_0 l^4 + c_1 l^3 + c_2 l^2 + c_3 l + c_4 + \Re \alpha_2 \xi \cos \lambda l &= 0 \\
k \left[2l^5 - 40\nu a^2 l^3 - \frac{5a^4 G}{B} l \right] + c_0 \left[2l^4 - 24\nu a^2 l^2 - \frac{G}{B} a^4 \right] \\
- c_1 (l^3 - 6\nu a^2 l) - c_2 (l^2 - 2\nu a^2) - c_3 l - c_4 - \Re \alpha_2 \gamma \cos \lambda l &= 0 \\
20kl \left[2(1-\nu^2)l^2 + 3 \left\{ 4(2+\nu) - \frac{\nu G}{12B} \right\} a^2 \right] + 12c_0 \left[2(1-\nu^2)l^2 + \left\{ 4(2+\nu) - \frac{\nu G}{12B} \right\} a^2 \right] \\
- 2(1-\nu^2)c_2 - 6(1-\nu^2)c_1 l + \Re \alpha_2 \xi \cos \lambda l &= 0 \\
5kl \left[8(1-\nu^2)l^2 - \frac{\nu G}{B} a^2 \right] + 24(1-\nu^2)c_0 l^2 - 6(1-\nu^2)c_1 l \\
- \left\{ 2(1-\nu^2)c_2 + \frac{\nu G}{B} c_0 a^2 \right\} + \Re \alpha_2 \rho \cos \lambda l &= 0
\end{aligned} \right\} \dots\dots\dots (12)$$

Taking the difference of the first and third, and the sum of the second and fourth of Eqs. (12), we have the following expressions :

$$\left. \begin{aligned}
c_2 &= \frac{G}{2\nu B} c_0 a^2 + \Re \frac{(\xi - \eta)}{2\nu a^2} (\alpha_1 \sin \lambda l - \alpha_2) \\
c_1 &= \frac{20ka^2}{B} \left\{ 2U - \frac{A}{(2+\nu)} + (2+\nu) + \frac{5+\nu-\nu^2+\nu^3}{12(1-\nu)(2+\nu)} \left(\frac{h}{a}\right)^2 \right\} + \Re \alpha_1 \frac{(\varphi + \xi \lambda)}{6(2+\nu)a^2} \cos \lambda l
\end{aligned} \right\} \dots\dots\dots (13)$$

From the fourth and first of Eqs. (12), we have

$$c_3 = -\Re \alpha_1 \xi \lambda \cos \lambda l, \quad c_4 = \Re \xi (\alpha_1 \sin \lambda l - \alpha_2) \dots\dots\dots (14)$$

Substituting Eqs. (13) and (14) in the fifth of Eqs. (12), we get

$$\begin{aligned}
&\Re \alpha_1 \left[\nu l \left\{ 2 \left(\frac{a}{l}\right)^2 \xi \lambda - \frac{(\varphi + \xi \lambda)}{3(2+\nu)} \right\} \cos \lambda l - \left\{ 2\nu \left(\frac{a}{l}\right)^2 \xi + (\xi - \eta) \right\} \sin \lambda l \right] + \Re \alpha_2 \left[\left\{ 2\nu \left(\frac{a}{l}\right)^2 \xi + (\xi - \eta) \right\} \right. \\
&\quad \left. - 2\nu \left(\frac{a}{l}\right)^2 \xi \cos \lambda l \right] = -2\nu k l a^4 \left[2 \left(\frac{l}{a}\right)^2 - \frac{20}{B} \left\{ 2U - \frac{A}{(2+\nu)} + (2+\nu) + \frac{5+\nu-\nu^2+\nu^3}{12(1-\nu)(2+\nu)} \left(\frac{h}{a}\right)^2 \right\} \right] \\
&\quad + \left[\frac{G}{B} - 4\nu \left(\frac{l}{a}\right)^2 \right] c_0 a^4 \dots\dots\dots (15)
\end{aligned}$$

Summing up the fifth and sixth of Eqs. (12) and substituting Eqs. (13), we have

$$\begin{aligned}
&\Re \alpha_1 \left[\frac{\nu(\varphi + \xi \lambda)l}{(2+\nu)} \cos \lambda l + (\xi - \eta) \sin \lambda l \right] + \Re \alpha_2 [(\xi - \eta)(\cos \lambda l - 1)] \\
&= 5kl a^4 \left[8\nu \left(\frac{l}{a}\right)^2 + \frac{1}{B} \left[G - 24\nu \left\{ 2U - \frac{A}{(2+\nu)} + (2+\nu) + \frac{5+\nu-\nu^2+\nu^3}{12(1-\nu)(2+\nu)} \left(\frac{h}{a}\right)^2 \right\} \right] \right] + 24\nu \left(\frac{l}{a}\right)^2 c_0 a^4 \dots\dots\dots (16)
\end{aligned}$$

The difference in the seventh and the eighth of Eqs. (12) is

$$\Re \alpha_2 [(\rho - \zeta) a^2 \cos \lambda l] = 48(2+\nu)(5kl + c_0) a^4 \dots\dots\dots (17)$$

Adding Eq. (16) to the eighth of Eqs. (12) multiplied by $\nu a^2/(1-\nu^2)$, we obtain

$$\Re \alpha_2 \left[(\xi - \eta) + \frac{\nu \rho a^2}{(1-\nu^2)} \right] \cos \lambda l = \frac{G}{(1-\nu^2)B} (5kl + c_0) a^4 \dots\dots\dots (18)$$

We can determine α_2 from Eqs. (17) and (18) by Cramer's rule, utilizing the properties of complex numbers, which leads to the following compact forms :

$$\left. \begin{aligned}
\alpha_2 &= A_2 / \Delta_0, \quad \bar{\alpha}_2 = \bar{A}_2 / \bar{\Delta}_0 \\
\Delta_0 &= 2i \vartheta \omega_5 \bar{\omega}_6 \\
\Delta_2 &= 2(f_c \bar{\omega}_6 - f_d \bar{\omega}_5) \\
\text{where } \omega_5 &\equiv (\rho - \zeta) a^2 \cos \lambda l \\
\omega_6 &\equiv \left[(\xi - \eta) + \frac{\nu \rho a^2}{(1-\nu^2)} \right] \cos \lambda l \\
f_c &\equiv 48(2+\nu)(5kl + c_0) a^4 \\
f_d &\equiv \frac{G}{(1-\nu^2)B} (5kl + c_0) a^4
\end{aligned} \right\} \dots\dots\dots (19)$$

In the expressions (19), $\mathcal{J}$ means to take the imaginary part of the function following it and i means $\sqrt{-1}$, and $\bar{\alpha}_2, \bar{l}_0, \bar{l}_2, \dots$ mean complex conjugates of $\alpha_2, l_0, l_2, \dots$.

Substituting the value of α_2 obtained in Eqs. (19), into Eqs. (15) and (16), we can determine the arbitrary constant α_1 by means of the similar procedure to that in the case of α_2 , and similar compact forms can be obtained.

$$\alpha_1 = A_1/\bar{A}, \quad \bar{\alpha}_1 = \bar{A}_1/\bar{A}$$

$$A = 2i\mathcal{J}\omega_1\bar{\omega}_3$$

$$A_1 = f_e \cdot \bar{\omega}_3 - f_f \cdot \bar{\omega}_1$$

where

$$f_e = 2f_a - (\alpha_2\omega_2 + \bar{\alpha}_2\bar{\omega}_2)$$

$$f_f = 2f_b - (\alpha_2\omega_4 + \bar{\alpha}_2\bar{\omega}_4)$$

$$f_a \equiv -2\nu k l a^4 \left[2\left(\frac{l}{a}\right)^2 - \frac{20}{B} \left\{ 2U - \frac{A}{(2+\nu)} + (2+\nu) + \frac{5+\nu-\nu^2+\nu^3}{12(1-\nu)(2+\nu)} \left(\frac{h}{a}\right)^2 \right\} \right] + \left[\frac{G}{B} - 4\nu \left(\frac{l}{a}\right)^2 \right] c_0 a^4$$

$$f_b \equiv 5k l a^4 \left[8\nu \left(\frac{l}{a}\right)^2 + \frac{1}{B} \left\{ G - 24\nu \left\{ 2U - \frac{A}{(2+\nu)} + (2+\nu) + \frac{5+\nu-\nu^2+\nu^3}{12(1-\nu)(2+\nu)} \left(\frac{h}{a}\right)^2 \right\} \right\} \right] + 24\nu \left(\frac{l}{a}\right)^2 c_0 a^4$$

$$\omega_1 \equiv \left\{ 2\nu \left(\frac{a}{l}\right)^2 \xi \lambda l - \frac{\nu(\varphi + \xi \lambda)l}{3(2+\nu)} \right\} \cos \lambda l - \left\{ 2\nu \left(\frac{a}{l}\right)^2 \xi + (\xi - \eta) \right\} \sin \lambda l$$

$$\omega_2 \equiv \left\{ 2\nu \left(\frac{a}{l}\right)^2 \xi + (\xi - \eta) \right\} - 2\nu \left(\frac{a}{l}\right)^2 \xi \cos \lambda l$$

$$\omega_3 \equiv \frac{\nu(\varphi + \xi \lambda)l}{(2+\nu)} \cos \lambda l + (\xi - \eta) \sin \lambda l$$

$$\omega_4 \equiv (\xi - \eta)(\cos \lambda l - 1)$$

.....(20)

In the expressions (20), the meaning of $\mathcal{J}, i, \bar{\alpha}_1, \bar{A}, \bar{l}_1, \dots$ is the same as in the case of α_2 .

Substituting the values of α_1, α_2 above obtained in Eqs. (13) and (14), the constants c_1, c_2, c_3 and c_4 can be obtained, i.e. all the arbitrary constants have been determined and the problem has been solved.

2. Deflection

Substitute the values of $c_1, c_2, c_3, c_4, \alpha_1$ and α_2 above obtained in Eqs. (5) and (6), and neglect the

value $(h/a)^2$ compared with unity and also neglect $\cos 2\lambda l$ compared with $\cosh 2\beta l$ putting $\cosh 2\beta l \cong \sinh 2\beta l \cong e^{2\beta l}/2$ under the constraints of $l/a > 1, a/h > 30$, then we obtain an approximation with sufficient accuracy for w .

The numerical calculation is worked for several values of $m = (\alpha_0 l + \beta_0)/\beta_0$ which is the ratio of the thickness of lining at the supported end to that at the fixed end.

The full lines in Figs. 3 and 4 represent curves

Fig. 3 Variations of displacements along generatrix

Fig. 4 Variations of displacements along generatrix

of $w/2c_0l^4$ with respect to the axis for each $m=1, 0.8, 0.6, 0.4$, where $\nu=0.3$, $a/h=50$, $l/a=3$ and $\nu=0.3$, $a/h=70$, $l/a=12$ respectively, and $2c_0l^4$ is the maximum deflection obtained by the simple beam theory when the thickness of lining takes a constant value β_0 along the generatrix.

We try to compare these curves with those computed by the modified beam theory as in the case of the previous paper⁽²⁾ by one of the authors.

For the beam fixed at $x=0$ and supported at $x=l$ as shown in Fig.5, we have the following expression according to the modified beam theory.

$$\frac{d^2y}{dx^2} = -\frac{1}{EI} \left(M + \frac{\kappa EI}{AG} q \right) \quad (21)$$

where

$$\left. \begin{aligned} \kappa &= 2, \quad I = \pi a^3 h, \quad A = 2\pi a h, \quad E/G = 2(1+\nu) \\ q &= w_0(\alpha_0 x + \beta_0), \quad w_0 = 2\pi a \gamma \end{aligned} \right\} \quad (22)$$

y being the deflection obtained by the modified beam theory.

Referring to Fig.5 the equations of equilibrium of forces and moments become as follows :-

$$\left. \begin{aligned} R_A + R_B &= w_0 l \left(\frac{\alpha_0 l}{2} + \beta_0 \right) \\ M_A + R_B l - \frac{w_0 l^2}{6} (2\alpha_0 l + 3\beta_0) &= 0 \end{aligned} \right\} \quad (23)$$

and the boundary conditions are expressed as follows :-

$$\left. \begin{aligned} 1) \text{ for } x=0, \quad \frac{dy}{dx} &= \frac{\kappa R_A}{AG}, \quad y=0 \\ 2) \text{ for } x=l, \quad y &= 0 \end{aligned} \right\} \quad (24)$$

For the bending moment in the cross section

$$EI \frac{d^2y}{dx^2} = M_A - R_A x + \frac{w_0 x^2}{6} (\alpha_0 x + 3\beta_0) - 2(1+\nu) a^2 w_0 (\alpha_0 x + \beta_0) \quad (26)$$

By applying the conditions (24) to the expression (26), we obtain the deflection y as follows :-

$$y = \frac{w_0 l^4 \beta_0}{240 EI} \left\{ 2 \left\{ (m-1) \left(\frac{x}{l} \right) + 5 \right\} \left(\frac{x}{l} \right)^4 - 20 \left(\frac{x}{l} \right)^2 \left[(m+2) + 4(1+\nu) \left(\frac{a}{l} \right)^2 \left\{ (m-1) \left(\frac{x}{l} \right) + 3 \right\} \right] \right. \right. \\ \left. \left. + \frac{(m-1) \{ 9 + 40(1+\nu) (a/l)^2 \} + 5 \{ 5 + 24(1+\nu) (a/l)^2 \}}{1 + 6(1+\nu) (a/l)^2} \left\{ 12(1+\nu) \left(\frac{a}{l} \right)^2 \left(\frac{x}{l} \right) + 3 \left(\frac{x}{l} \right)^2 - \left(\frac{x}{l} \right)^3 \right\} \right\} \quad (27)$$

The dotted lines in Fig.3 and Fig.4 represent the curves of $y/2c_0l^4$.

It is recognized from the behaviours of the full and dotted lines that the full lines satisfy the condition of $\partial w / \partial x = 0$ at the fixed end, representing fairly wavy characters near there, while the dotted lines have some gradients of $\kappa R_A / AG$ at the fixed end, being nearly straight, and fairly different from the full lines near there.

When l/a is small, the quantities of dotted lines are smaller than those of the full lines, especially for smaller x , while, when l/a is great, the former is larger than the latter for larger x , and yet the difference between them is small for smaller x .

Generally speaking, both coincides with each other when l/a is great.

In Fig.3 and Fig.4, the chain lines give the values computed by the simple beam theory for $m=1$, and are so different from those of full lines that they are out of the question especially when l/a is small.

3. Stress

All the stress-resultants and couples are obtained by the same way as in the case of the deflection w .

Fig. 5

Fig. 6 Variations of stress resultants along generatrix

at x distance from the fixed end, we have

$$M = -M_A + R_A x - \frac{w_0 x^2}{6} (\alpha_0 x + 3\beta_0) \quad (25)$$

Substitute the expressions (22), (25) in Eq. (21), using the relations (23), we have

Fig. 6 to Fig. 9 represent the curves of the stress-resultants and couples along the generatrix where $\nu=0.3$, $a/h=50$, $l/a=3$ or $\nu=0.3$, $a/h=70$, $l/a=12$.

In these diagrams, the values of t_x , t_θ , m_x , m_θ , s' and n_x are defined by the following expressions :-

$$\left. \begin{aligned} t_x &\equiv \frac{T_x}{2\beta_0 \gamma l}, & t_\theta &\equiv \frac{T_\theta}{2\beta_0 \gamma l} \\ m_x &\equiv \frac{6M_x}{h} / 2\beta_0 \gamma l, & m_\theta &\equiv \frac{6M_\theta}{h} / 2\beta_0 \gamma l \\ s' &\equiv \left[S' - \frac{\mathcal{M}}{a} \right] / 2\beta_0 \gamma l, & n_x &\equiv \left[N_x + \frac{\partial \mathcal{M}}{a \partial \theta} \right] / 2\beta_0 \gamma l \end{aligned} \right\} \dots\dots\dots (28)$$

where the values of $S' - (\mathcal{M}/a)$ and $N_x + (\partial \mathcal{M}/a \partial \theta)$ are the effective circumferential and transverse shearing force respectively. S is very close to $S' - (\mathcal{M}/a)$ and $\mathcal{M}$ is very small, so the diagrams for them are omitted.

Fig. 6 and Fig. 7 show the curves of T_x , T_θ , $S' - (\mathcal{M}/a)$ and $N_x + (\partial \mathcal{M}/a \partial \theta)$.

The curves of T_x are flowing on the whole, and in reality they represent wave characters near the ends, but the amplitudes of them are so small that they cannot be represented in the diagrams. When l/a is great, the behaviours of T_x coincide with those computed by the simple beam theory.

The values of T_θ are compressive for smaller x and tensile for larger x , taking maximum values near the ends, and are nearly straight in the middle.

Fig. 7 Variations of stress resultants along generatrix

As l/a increases, the values of t_θ decrease for all values of x and the positions at which they take maximum values relatively approach the ends.

The behaviours of $S' - (\mathcal{M}/a)$ and $N_x + (\partial \mathcal{M}/a \partial \theta)$ resemble those of the previous paper⁽²⁾ by one of the authors: i.e., at the fixed end, the load is balanced by $S' - (\mathcal{M}/a)$ when l/a is small and by $N_x + (\partial \mathcal{M}/a \partial \theta)$ when l/a is great, while at a little distance from the fixed end the load is always balanced by $S' - (\mathcal{M}/a)$ for all values of l/a .

Fig. 8 and Fig. 9 show the curves of stress-couples; they are strongly wavy near the ends, and all these quantities and characters cannot be predicted by the simple beam theory.

4. Conclusion

At the fixed end, all the stress-resultants and couples take significant values because the edge of the pipe is constrained, and all of them except the axial one represent wave characters near the ends,

Fig. 8 Variations of stress couples along generatrix

Fig. 9 Variations of stress couples along generatrix

but at some distance from the ends the amplitudes of these waves are so small that they cannot be represented in the diagrams. When l/a becomes greater, these wave characters occur relatively nearer to the fixed end.

As mentioned above, the behaviours of the stress-resultants and couples are complicated, and cannot be predicted by the simple beam theory or the membrane theory.

The curves of the deflection w also represent

wave characters near the fixed end, still as a whole they are fairly close to those computed by the modified beam theory, but quite different from those computed by the simple beam theory.

References

- (1) K. Mizoguchi: *Trans. Japan Soc. Mech. Engrs.*, Vol. 20, No. 95 (1954), p. 484.
- (2) K. Mizoguchi: *Trans. Japan Soc. Mech. Engrs.*, Vol. 24, No. 139 (1958), p. 145.
- (3) K. Mizoguchi: *Trans. Japan Soc. Mech. Engrs.*, Vol. 27, No. 176 (1961), p. 494.

621.8.031.6:621-822

Theoretical and Experimental Analysis of Stick-Slip in Hydraulic Driving Mechanisms*

By Atsushi MATSUZAKI** and Seiya HASHIMOTO**

The discontinuous motion of a body sliding at low velocity is known as stick-slip. In this paper, stick-slip in hydraulic driving mechanisms is analyzed quantitatively on the bases of the characteristic of flow control valve, the equation of continuity for oil flow and the equation of motion for the sliding element under the assumptions that the characteristic of flow control valve is linear and the friction between sliding surfaces is composed of static, Coulomb and viscous friction. An experimental verification of the theoretical analysis is made using the model table which is the same size as a table of No. 2 $\frac{1}{2}$ milling machine. Then it is shown that the motion of a table in stick-slip can be described by the pressure-flow characteristic of flow control valve, the friction-velocity relation between sliding surfaces, the compressibility of working oil and others.

1. Introduction

A phenomenon of stick-slip is often observed when a body slides over the other at low speed. When it occurs between machine tool components, it deteriorates the accuracy of finishing and shortens the tool life. Besides it may disturb the stability of a servo-system in an automatic control machine tool.

The phenomenon of stick-slip is closely related to the friction-velocity relation between the sliding surfaces. In a mechanical driving system, great efforts have been made in the past to make analysis of stick-slip. But in a hydraulic driving mechanism, little work has been done on quantitative analysis, though stick-slip is liable to occur in the case of hydraulic drive because of low rigidity compared with the case of mechanical one. This paper deals

primarily with the theoretical and experimental analysis of the motion of a sliding element under the condition of stick-slip in the hydraulic driving mechanism.

2. Nomenclature

- A_p : cross-sectional area of piston
 A_f : cross-sectional area of pipe
 b_e : equivalent coefficient of viscous friction of flow in pipe regarding that between sliding surface
 b_T : coefficient of viscous friction between sliding surfaces
 F_s : static friction force
 F_d : Coulomb friction force
 $f_T(v)$: friction force
 g : acceleration of gravity
 $k_x = \frac{\partial q_m}{\partial x}$: flow gain of flow control valve
 $k_p = -\frac{\partial q_m}{\partial p_m}$: decrease of flow rate per unit change

* Received 29th November, 1961.

** Research Fellow, Central Research Laboratory, Hitachi Ltd., Kokubunji, Tokyo.