Practical Technique for Forming an Analog Computer Model of Process Element*

By Kazuyuki Hotta**

Forming a simple but sufficiently good analog computer model for each element of a process plant is a technique of great practical import in designing and analyzing a large system. A novel way to the solution of this problem was suggested by Paynter and Takahashi in 1956. In practice along this principle, however, one faces the following two immediate questions:

i) Given a particular process element, how can one select the most appropriate type of transfer function to simulate it?

ii) How can one conveniently determine the numerical values of the parameters when the number of such parameters is more than three?

The technique developed here will nicely answer the questions and allow one to make a unique choice for the required model as the result of a straightforward process of model-forming.

1. Introduction

It is an accepted fact that an analog computer as a simulator is one of the most powerful tools for designing and analyzing a larger system. Prior to taking to the computer, however, it becomes necessary to select an adequate dynamic model for each element of the system. While the accuracy is of the primary importance for the model, simplicity is also desirable because the size of the computer available is usually limited. In many cases, a theoretical model based on physical laws assumes a very complicated form and is not easily amenable to the computer handling, whereas there is no definite rule to extract a simple model from observed response data of a system element.

The objective of the study reported here is to establish a routine technique through which one can always reach a unique linear low frequency model of a given system element. Fig. 1 sketches the outline of the model forming process. This technique owes its fundamental principle to H. M. Paynter(1) and much elaboration is added to it for easier application of the principle to engineering practice.

2. Characteristic vector

In order to be able to construct a simple dynamic model, one must first obtain some information concerning the dynamics of the real system. This information may be given in the form of observed dynamic responses of the real system in one case, and a sophisticated theoretical transfer function in the other. It would be therefore very desirable to have some characteristic value which is able to uniquely characterize the real system and obtainable from the information of any kind. As one of such, we here propose a characteristic vector defined in the following manner.

Suppose the real system can be represented by a certain transfer function \( G(s) \) and neither \( G(s) \) nor its logarithm \( \log G(s) \) has a pole or poles at \( s = 0 \). It is then possible to expand these two functions into Maclaurin's series and we write them in the forms due to Paynter:

\[
G(s) = \sum_{k=0}^{\infty} (-s)^k a_k/k! \tag{1}
\]

\[
\log G(s) = \sum_{k=0}^{\infty} (-s)^k c_k/k! \tag{2}
\]

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Fig. 1 Outline of model forming process
Now, an infinite dimensional vector with $a_0$ as its $k$th component is capable of defining the system uniquely and hereafter called the characteristic vector of type $A$ and denoted by $A$. Another infinite dimensional vector $C$ is similarly defined with $a_0$'s replaced by $a_0$'s and called the characteristic vector of type $C$. It may appear meaningless to talk about infinite dimensional vectors, but this is justified by the considerations which follow.

If the lower components of the characteristic vector of one system are equal or nearly equal to those of another system, their behavior should be similar to each other at least in the lower frequency range. For the purpose of constructing a low frequency model, it is therefore only necessary to know a couple of lower components of the characteristic vector.

As is easily seen from the definition of the vectors, addition of two characteristic vectors $A_1$ and $A_2$, with usual rule of the vectorial addition observed, corresponds to the physical configuration of a parallel connection of two elements of which characteristic vectors are $A_1$ and $A_2$ respectively, and that of the two characteristic vectors $C_1$ and $C_2$ corresponds to a series connection.

Characteristic vectors of type $A$ and type $C$ of a system are of course not independent; that is; the $a_0$'s and $c_0$'s are inter-related to each other by the following equations.

\[
\begin{align*}
\theta_0 &= \log_a a_0 \\
\theta_1 &= a_0/a_0 \\
\theta_2 &= (a_0/a_0)-(a_0/a_0)^2 \\
\theta_3 &= (a_0/a_0)-3(a_0/a_0)^2+2(a_0/a_0)^3 \\
\theta_4 &= (a_0/a_0)-4(a_0/a_0)^2+6(a_0/a_0)^3 \\
\theta_5 &= -3(a_0/a_0)^2+6(a_0/a_0)^3 \\
\theta_6 &= a_0 \theta_0 \\
\theta_7 &= a_0 \\
\theta_8 &= a_0+c_0 \\
\theta_9 &= a_0+3c_0c_1+c_0^2 \\
\theta_{10} &= a_0+4c_0c_1+6c_0^2c_1^2+3c_0^3+c_1^4 \\
\end{align*}
\]

It is therefore easy to see that a characteristic vector of a compound system can be synthesized from those of the simpler subsystems.

What is then the way to obtain the characteristic vectors of simpler systems from varied kinds of informations shown in Fig. 1? The answer to this question is quite obvious when the dynamics of the real system is given in terms of a transfer function.

When the available information is in the form of an observed transient response, we can evaluate the characteristic vector of type $A$ by making use of the following relations between the impulse response $g(t)$ of the real system and $a_0$'s.

\[
a_k = \int_0^\infty g(t)dt = \int_{0}^{\infty} \mathcal{L}(g)(s)ds 
\]

where $f(t)$ denotes the step response of the system.

Eq. (5) is true provided $g(t)$ is piecewise continuous and there exist positive numbers $M$ and $\varepsilon$ such that they satisfy the following relation.

\[
|g(t)| < Me^{-\varepsilon t} 
\]

Finally, observed frequency response data can be used to evaluate the characteristic vector of type $C$ in the following manner.

Combining the general relation $\log G(s) = \log |G(s)|+j[\angle G(s)]$ with Eq. (2), we have the following approximate relations in the lower frequency region.

\[
\xi \equiv -\angle G(j\omega)/\omega \equiv c_1-c_0(\omega)^4, \quad \eta \equiv \text{ln} |G(j\omega)|=\text{ln} |G(0)| \\
\xi \equiv -c_0(\omega)^2+2c_0(\omega)^4/24 \\
\]

Examination of Eq. (7) will show that the graphical method as sketched in Fig.2 will yield $c_1, c_2, c_3$ and $c_4$ is obtained by the following formula.

\[
c_4 = 12(\eta^2-2\xi)/\xi^2
\]

3. Construction of model

Upon obtaining the lower components of the characteristic vector of the real system, one can now construct a simple low frequency model of that system by adequately selecting a simple transfer function and by adjusting the parameters of the transfer function so that the lower components of the characteristic vector of this transfer function coincide with those of the characteristic vector of the real system.

But here we have no established policy in selecting the most adequate model transfer function for a particular system, nor is it convenient to solve a higher order algebraic equation, which is usually encountered in the process of parameter adjustment.
To cope with this situation, we first set a limit to the scope of our model selection and use only those transfer functions that have less than three degrees of freedom. After establishing a routine technique of model forming within this limit, we will extend the technique to the models having more degrees of freedom than three.

Here as a convention, we mean by the degree of freedom the number of the adjustable parameters and by the order, the total number of the poles and zeros which the model transfer function contains.

3-1 Selection of model transfer function—t-plane technique First problem one faces when construction of a simple model of a given real system is intended, is how to single the most adequate transfer function out of the multitude of possible alternatives.

In the field of process control, a simple lag combined with a pure delay has been widely accepted as a convenient model, and this is not without reason because of the amazing versatility this model exhibits in spite of its extreme simplicity.

Pure delay is, however, not convenient to be handled on an analog computer and with this fact in mind we can sort out two fundamental transfer functions, stable and easily amenable to analog computer handling. They are

1 degree of freedom
\[ G_1 = \frac{1}{1 + Ts} \] (8)

2 degrees of freedom
\[ G_2 = \frac{1}{1 + 2\xi^2/s + s^2/\omega_n^2} \] (9)
\[ 0 < \xi < 1 \]

Since we avoid for the present the transfer functions having more degrees of freedom than three, only four possible transfer functions can be constructed out of the above two fundamental transfer functions. They are listed in Table 1 and referred to hereafter as the standard models.

Now, how can one pick up the best model out of these four transfer functions to simulate a given real system?

A diagram, called t-plane, is very useful to answer this question. As is shown in Fig. 3, the abscissa of t-plane corresponds to a non-dimensional quantity \( t^* = c_1t \), and the ordinate, to another non-dimensional quantity \( t^* = c_1t \). Every characteristic vector of type C is represented by a single point on this plane, whereas a point on this plane represents infinitely many characteristic vectors.

A transfer function of one degree of freedom is represented by a single point irrespective of the magnitude of its parameter. For instance, all simple lags are represented by a point \( P_n \), and all pure delays \( G_d = e^{-Ts} \) by a point \( O \) of Fig. 3. Once \( n \) is fixed, the transfer function
\[ [G_n]^n = \frac{1}{(1 + Ts)^n} \] (10)
also represents a system of one degree of freedom and \( P_1, P_2, P_3 \) in Fig. 3 correspond to the system with \( n = 1, 2, 3 \) respectively.

The representative point of a system with two degrees of freedom moves along a certain curve as the parameters change their values relatively to each other. For instance, the representative point of the system
\[ G_{\alpha} = \frac{1}{(1 + Ts)(1 + Ts)} \]
\[ T_1 > T_2 \]

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### Table 1 Standard third-order models with three degrees of freedom

<table>
<thead>
<tr>
<th>Notation</th>
<th>Transfer function</th>
<th>( c_1 )</th>
<th>( t^* )</th>
<th>( t^*/2 )</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1^2 )</td>
<td>( 1/(1 + Ts_1)(1 + Ts_2)(1 + Ts_3) )</td>
<td>( T_1 + T_2 + T_3 )</td>
<td>( t_1^* + t_2^* + t_3^* )</td>
<td>( t_1^* + t_2^* + t_3^* )</td>
<td>( t_1^* + t_2^* + t_3^* )</td>
</tr>
<tr>
<td>( G_2^2 )</td>
<td>( 1/(1 + Ts_1)(1 + Ts_2)(1 + Ts_3) )</td>
<td>( 2c_1^2 + s^2/\omega_n^2 )</td>
<td>( t_1^* + t_2^* + t_3^* )</td>
<td>( t_1^* + t_2^* + t_3^* )</td>
<td>( t_1^* + t_2^* + t_3^* )</td>
</tr>
<tr>
<td>( G_1^2 )</td>
<td>( 1/(1 + Ts_1)(1 + Ts_2)(1 + Ts_3) )</td>
<td>( T_1 + T_2 + T_3 )</td>
<td>( t_1^* + t_2^* + t_3^* )</td>
<td>( t_1^* + t_2^* + t_3^* )</td>
<td>( t_1^* + t_2^* + t_3^* )</td>
</tr>
</tbody>
</table>

\[ t_i = T_i/c_1, \quad t = T/c_1, \quad \xi = 2\xi/c_1\omega_n, \quad \phi = 1/c_1\omega_n \]
falls on a segment $P_1P_2$ in Fig. 3 and points $P_a, L_2, L_3, P_i$ correspond to the systems with $T_i/T_L=1, 2, 5$ and $\infty$.

The oscillatory second-order systems defined by Eq. (9) have their representative points on a half line $P_iQ_i\sqrt{\zeta}$ and points $P_a, Q_i\sqrt{\zeta}, Q_i\sqrt{\zeta}$ correspond to the systems with $\zeta=1, \sqrt{3}/2, 1/\sqrt{3}$ respectively. Finally the curved full line $P_iO$ is the trace of the representative point of the familiar lag delay system

$$G_{dl}=e^{-t_r}/(1+Ts)$$

and points $O, D_1, D_2, P_i$ on this curve represent the systems with $T/L=0, 1, 4$ and $\infty$.

For a system of three degrees of freedom the representative point distributes over an open region, or a territory. The territories for the transfer functions listed in Table 1 are shown in Fig. 4. Contours of a new non-dimensional quantity

$$t^*_{\text{rel}}=c_1\sqrt{t^*_{\text{rel}}}$$

are drawn in each territory for later use.

From these diagrams it is seen that the representative points of the simpler transfer functions cited above mainly distribute over the part of $t$-plane where $t^*_{\text{rel}}<1$ and $t^*_{\text{rel}}<2$. What sort of systems are then represented by the points outside this part? A qualitative account for this point is provided through the following observation.

Suppose the area bounded by the impulse response and the time axis has uniform areal density of unit magnitude, positive for the parts above the time axis and negative for the parts below the same axis. Then $c_1$ gives the abscissa of the center of gravity of this area and $c_2$ is equal to the square of the radius of gyration of the area around the vertical axis through the center of gravity; $c_1$ is related to the skewness of the area and positive if the area trails off toward the direction of increasing time axis and zero if the area had a vertical axis of symmetry.

In view of these facts, we can see that the schematic responses drawn in the ovals in Fig. 3 characterize the system which is represented by the point on $t$-plane in the neighbourhood of respective ovals.

Now, if a transfer function is to be a low frequency model of a given real system, the territory of the model on $t$-plane must contain within it the characteristic point of the real system. Conversely, once we locate the characteristic point of the real system on $t$-plane, we can immediately predict which one of the transfer functions is best as the low frequency model of the system.

Should the characteristic point belong to more than one territories, we would still be able to single out the best model by evaluating $t^*_{\text{rel}}$ of the real system through Eq. (13) and by rejecting all other possible models but the one which has the nearest $t^*_{\text{rel}}$ value at the characteristic point.

We reserve the discussion of the cases in which the characteristic point of the real system does not belong to any territories shown in Fig. 4.

### 3-2 Adjustment of parameters

Now that

![Fig. 4 (a) Territories of standard models. without zero](image-url)

![Fig. 4 (b) Territories of standard models. with a zero](image-url)
the best type of transfer function as a low frequency model of a given real system is decided, we have reached the stage to determine the numerical values of the parameters. In principle this process presents no difficulty. All we need to do is to derive the general expression of \( c_1, c_2, c_3 \), etc of the model in terms of the parameters it contains and equalize them to those of the real system.

In practice this process calls for some ingenuity, because it becomes necessary to solve a cubic equation if directly attacked.

We are going to show, in the following, a short-cut to the solution of this problem for each transfer function in Table 1.

i) Non-oscillatory third order model without zero

\[ G_3^1 = \frac{1}{(1+T_{s1})(1+T_{s2})(1+T_{ss})} \]

Parameters are determined by the following equations.

\[
\begin{align*}
t_1 + t_2 + t_3 &= 1 \\
t_1^2 + t_2^2 + t_3^2 &= t_3^3 \\
t_1^3 + t_2^3 + t_3^3 &= t_3^4/2
\end{align*}
\]

(14)

Here the non-dimensionalized parameters are defined as \( t_i = T_i / C_i \geq 0 \).

The sum of three distances from an interior or peripheral point of an equilateral triangle of unit height to its three sides is always equal to unity. There exists, therefore, one to one correspondence between these points and all possible combinations of \( t_1, t_2 \), and \( t_3 \) as is shown in Fig. 5. Moreover since we can arbitrarily assume that \( t_1 \leq t_2 \leq t_3 \), we only need one sixth part of the triangle. Now covering this triangle with contours for \( t_1^3 \) and \( t_3^3 \), we get a handy diagram like the one in Fig. 5 for evaluating the parameters. All we have to do is to calculate \( t_1^3 \) and \( t_3^3 \) of the real system and locate the intersection of the corresponding contours on the triangular diagram. Then the three distances shown in Fig. 5 give the numerical values of three non-dimensional parameters.

ii) Oscillatory third order model without a zero

\[ G_3^1 = \frac{1}{(1+T_{s1})(1+2Z_s/\omega_n+\epsilon^2/\omega_n^2)} \]

Equations for determining the non-dimensional parameters are in this case as follows:

\[
\begin{align*}
\xi + \tau &= 1 \\
\xi^2 + \tau^2 - 2\phi^2 &= t_1^3 \\
\xi^3 + \tau^3 - 3\xi\phi^2 &= t_3^3/2
\end{align*}
\]

(15)

where \( \tau = T_1 / C_1 \), \( \xi = 2Z_s / C_1 \omega_n \), \( \phi = 1 / C_1 \omega_n \). A similar diagrammatic technique to the previous one is employed again. This time one to one correspondence is established between the point in a semi-infinite strip of unit width and all possible combinations of the three non-dimensional parameters as shown in Fig. 6. Numerical values of the parameters can be obtained by using two sets of contours as was done with Fig. 5.

iii) Non-oscillatory third order model with a zero

\[ G_3^{1-1} = \frac{1}{(1+T_{s1})(1+T_{s2})(1+T_{ss})} \]

This transfer function can be also expressed in the following alternative forms.

\[
\begin{align*}
G_3^{1-1} &= \frac{1}{(1+T_{s1})(1+T_{s2})(1+T_{ss})} \\
&= 1 - (1 - \epsilon)T_{s1}/(1+T_{s1}) \\
&\quad - \epsilon T_{s2}/(1+T_{s2}) \\
&\quad + (1 - \epsilon)/(1+T_{ss}) + \epsilon/(1+T_{ss})
\end{align*}
\]

(16)

where \( T = t_1 + (1 - \epsilon)T_{s2} \).

Non-dimensional parameters must satisfy the follow-
ing equations.

\begin{align*}
  t_1 + t_2 = & \tau = 1, \\
  t_1^2 + t_2^2 = & \tau^2/4, \\
  t_1^3 + t_2^3 = & \tau^3/2 \\
\end{align*}

Introduction of new notations \( \gamma, x, y \), defined by Eqs. (19) permits us to get the solution of these simultaneous equations without solving a cubic equation.

\begin{align*}
  \gamma = & t_1 - 1, \quad x = t_2^2 - 1, \quad y = t_3^3 - 2 \\
\end{align*}

Then

\begin{align*}
  \gamma = & (y/6x - 1) + \sqrt{(1 - y/6x)^2 + x^2} \\
  x = & y/(x + 2\gamma^5) \\
  t_1 = & (1 - \epsilon t_2)/(1 - \epsilon) \\
\end{align*}

Here we excluded from consideration the cases where \( t_1 = 1 \), or \( t_2 = 1 \), the cases in which this model reduces to a simple lag.

iv) Oscillatory third order model with a zero

\[
  G_s(\tau) = (1 + T s)/(1 + 2\zeta s/\omega_n + s^2/\omega_n^2)
\]

The equations to be solved to get the parameters are

\begin{align*}
  \xi - & \tau = 1 \\
  \xi^2 - & \tau^2 - 2\phi = t_2 \\
  \xi^3 - & 3\xi^2\phi = t_4/2 \\
\end{align*}

Solving a cubic equation is again avoided and we have

\begin{align*}
  \xi = & y/3x \\
  \tau = & \xi - 1 \\
  \phi = & \sqrt{\tau - x^2/2} \\
\end{align*}

where \( x \) and \( y \) are defined in Eq. (19).

4. Higher order model—tractor function

In the previous discussion we have seen that a third order low frequency model can be readily constructed for all systems of which the characteristic point on \( t \)-plane belongs to at least one of the territories of the standard models shown in Table 1. It is, however, quite possible that the characteristic point of the real system does not belong to any of these territories, or even if it does, a more sophisticated model than third order may be desirable on some occasions.

We introduce a new technique, a tractor function technique as it is termed here, in order to extend the technique developed in earlier sections to the higher order model.

A tractor function is a transfer function which, being connected in series to another system, has the effect to shift the characteristic point of the system on \( t \)-plane.

Let \( G_p \) denote the transfer function, if any, of the real system and \( H \), that of the tractor function. Then the ways in which the tractor function is of use are stated as follows:

i) When the characteristic point on \( t \)-plane of \( G_p \) does not belong to any territory of the standard third order models, an adequate tractor function is so selected that the characteristic point of the overall system \( H \cdot G_p \) is within one of the territories of the standard models. We can construct a third order model of \( H \cdot G_p \) at this shifted point and let it be denoted by \( G_m \). Then we can obtain a model for \( G_p \) as

\[
  G_p = G_m = G_m' / H
\]

As is easily seen, there are in general infinitely many tractor functions that can shift the characteristic point of \( G_p \) into the territory of a standard model. Criterion for deciding the best \( H \) is agreement of the \( t_i \) value between \( H \cdot G_p \) and \( G_m' \).

ii) Second case is one where the characteristic point of \( G_p \) belongs to a territory of a standard model and a third order model \( G_m \) can be constructed at this point, but it is desirable to have a better model than \( G_m' \). If this situation is encountered, we choose a tractor function \( H \) in such manner that after shifting the characteristic point, the agreement of the \( t_i \) value between \( H \cdot G_p \) and the third order model \( G_m' \) at this new point may be improved over that between \( G_p \) and \( G_m' \). Then an improved model \( G_m \) is obtained as

\[
  G_m = G_m' / H
\]

Now, according to the definition of the tractor function given above, almost all types of transfer function can be a tractor function. But since we are here in this paper concerned with constructing a simple model, the tractor function itself must be simple in its form. In addition, \( 1/H \) should represent a stable system. With these restrictions in mind we are rather limited in our choice of the tractor functions. Typical examples of the tractor functions as well as their behavior are presented in what follows.

\[
  H_t(\lambda, \mu) = (1 + \lambda \omega_n)/(1 + \lambda \mu \omega_n)
\]

Here \( c_t \) is the second component of the characteristic vector of type \( C \) for \( G_t \) and from stability requirement \( \mu \geq 0 \). In special cases in which \( \lambda = 0 \) or \( \mu = 0 \), \( H_t \) introduces respectively an additional pole or zero to the standard third order model and when \( \lambda \neq 0, \mu \neq 0 \), \( H_t \) causes no change in the order of contact of the transient response curve to the time axis at \( t = 0 \).

The manner in which \( H_t \) shifts the characteristic point of \( G_t \) on \( t \)-plane varies depending on the starting position. Let the characteristic values of \( G_p \) be denoted by \( c_t, t_1^t, t_2^t, t_3^t \) and those of \( G_p H \) by \( c_t', t_1'^t, t_2'^t, t_3'^t \) then

\begin{align*}
  c_t' = & (1 + \lambda - \mu)c_t \\
  t_1'^t = & (t_1^t + \lambda^2 - \mu^2)(1 + \lambda - \mu)^2 \\
  t_2'^t = & (t_2^t + 2\lambda^2 - 2\mu^2)(1 + \lambda - \mu)^4 \\
  t_3'^t = & (t_3^t + 4\lambda^2 - 6\mu^2)(1 + \lambda - \mu)^6
\end{align*}
As we increase \( \lambda \) starting from \( \lambda = \mu \), or equivalently from \( H_1 = 1 \), while keeping \( \mu \) constant, the characteristic point of \( G_0H \) moves along a smooth curve. In order to know the direction of the curve at its starting point, the following formulae are of help.

\[
\begin{align*}
[\partial t_s^{n}/\partial t_s^{n}](t_1^{*}) & = 3\left(2\mu - t_4^{*}\right)/2\left(\mu - t_3^{*}\right) \\
[\partial t_s^{n}/\partial \lambda](t_1^{*}) & = 2\left(\mu - t_3^{*}\right) \\
[\partial t_s^{n}/\partial \mu](t_1^{*}) & = 3\left(2\mu - t_4^{*}\right) \\
[\partial t_s^{n}/\partial \lambda](t_2^{*}) & = 4\left(\mu - t_3^{*}\right)
\end{align*}
\]

(31)

For two typical cases where \( t_s^{*} > 1, t_s^{*} > 2 \) and \( t_s^{*} < 1, t_s^{*} < 2 \), the manner in which \( H_1 \) shifts the characteristic point of \( G_p \) is shown schematically in Fig. 7.

ii) \( H_1(\alpha, \beta) = 1/(1 + \alpha c_s + \beta c_s^2) \)

Here \( c_1 \) has the same meaning as that in \( H_1 \), and \( \alpha, \beta \) are two arbitrary constants. Shifting of the characteristic point of \( G_p \) by \( H_1 \) is governed by the following equations.

\[
\begin{align*}
c_1' &= (1 + \alpha \alpha c_1) \\
t_1' &= (t_1 + \alpha^2 - 2\beta)/(1 + \alpha)^2 \\
t_2' &= (t_2 + 2\alpha^2 - 6\alpha \beta)/(1 + \alpha)^3 \\
t_3' &= (t_3 + 6\alpha^2 - 24\alpha^2 \beta + 12\beta^2)/(1 + \alpha)^4
\end{align*}
\]

(33)

iii) \( H_1(\gamma, \beta) = (1 + \gamma c_s)/(1 + \gamma c_s + \delta c_s^2) \)

\( c_1 \) is again the same as that in the previous cases, and \( \gamma \geq 0 \). The increment of the characteristic values are

\[
\begin{align*}
\delta c_1 &= c_1' - c_1 = 0 \\
\delta t_1 &= t_1' - t_1 = -2\beta \\
\delta t_2 &= t_2' - t_2 = -6\gamma \beta \\
\delta t_3 &= t_3' - t_3 = 12\beta(\beta - 2\gamma^2)
\end{align*}
\]

(35)

Since these increments, therefore the direction and the magnitude of the shift on \( t \)-plane, are independent of the starting point of the shift, \( H_1 \) can be used very conveniently at the cost of slight increase in complexity of the resulting model.

As we have seen in above discussion, the tractor function enables us to extend the technique developed solely for the construction of a third order model to that of higher order models. But since matching a tractor function with the real system is, so to speak, looking at the system through a colored filter, should the color be too strong, it would obscure the fine details of the system.

In order to avoid this, the absolute values of the \( \lambda, \mu, \alpha, \ldots \ldots \ldots \) etc. must be kept sufficiently smaller than unity, which is tantamount to introducing no dominant poles nor zeros into the real system.

### 5. Sample application

As an example of practical application of the technique developed above, we construct a couple of models for a pure delay system. Although this is rather trivial as an example, it would suffice to show the outline of the process for model forming.

For a pure delay \( G_d = e^{-ts} \), all the components but \( c_1 = L \) of the characteristic vector of type \( C \) are zero. A glance at Fig. 4 therefore tells us that among the standard third order models, either \( G_{d1} \) or \( G_{d1}^{-1} \) can be a low frequency model for \( G_d \). It is also expected that \( G_{d1}^{-1} \) can make a better model than \( G_{d1} \) because of the better agreement of \( t_s \) with its ideal value zero.

Now, we can fix the numerical values of these models by Fig. 6 and Eqs. (23)~(26) respectively as

![Fig. 7 Shift of characteristic point by \( H_1(\lambda, \mu) \)](image)

![Fig. 8 Frequency response of various models for a pure delay system \( e^{-ts} \)](image)
\[ G_{d1} = \frac{1}{1 + 0.3735Ls + 0.2660L^2s^2} \left( 1 + 0.6265Ls \right) \]

\[ G_{d1}^{-1} = \frac{1}{(1 - Ls/3)(1 + 2Ls/3 + L^2s^2/6)} \]

Sketches of the frequency response of these models are shown in Fig. 8 in contrast with that of the real system.

Let us now try to improve these models employing the tractor function technique. As the simplest example we take up \( H_i(\lambda, \mu) \) with \( \mu = 0 \). For this particular case Eq. (30) becomes

\[ c_i = (1 + \lambda)L, \quad t_i^2 = \lambda^2/(1 + \lambda)^2, \quad t_i^4 = 2\lambda^4/(1 + \lambda)^4 \]

If \( \lambda \) is decreased gradually starting from \( \lambda = 0 \), the characteristic point of \( G_{d2} \) moves in the right downward direction on the plane. At the same time the value of \( t_i^4 \) of \( G_{d2} \) increases from \( t_i^4 = 0 \) up. On the other hand we notice in Fig. 4(a) that the \( t_i^4 \) value of \( G_{d1} \) decreases from \( t_i^4 = 1 \) down as the characteristic point moves in the right downward direction. Therefore we can expect that the \( t_i^4 \) values of \( G_{d2} \) and \( G_{d1} \) coincide to each other at a certain value of \( \lambda < 0 \).

Evaluating \( \lambda \) for this point on Fig. 4 or through a little trial calculation, we obtain \( \lambda = -0.251 \).

Since for this point the characteristic values for \( G_{d2} \) are

\[ c_i = 0.749L, \quad t_i^2 = 0.1123, \quad t_i^4 = -0.07527 \]

we can construct \( G_{d1} \) model at this point and dividing it by \( H \), we obtain an improved model for \( G_{d2} \) as

\[ G_{d2} \cong G_{d1} = \frac{1 - 0.251Ls}{(1 + 0.321Ls)(1 + 0.427Ls + 0.1116L^2s^2)} \]

From the frequency response diagrams of this model shown in Fig. 8, improvement is seen very clearly.

6. Discussion

From above consideration it follows that we can construct a low frequency model for the most of linear real systems unless the real system is of extreme nature.

What is then the goodness of the model thus constructed?

It seems hard to establish a general criterion for this point, since the required quality of the model varies depending on the cases and moreover in many cases we do not know to the full the nature of the real system with which the model should be compared.

It is, however, always possible to compare the response of the model with that of the real system, and this gives a rough measure of the goodness of the model.

Furthermore even if such comparison proved that the model was not fully satisfactory, we would still claim that the model be the best possible model within the given simplicity. For the standard third order models listed in Table 1 exhaust all possible transfer functions of less than three degrees of freedom which can be expressed by the least number of ordinary analog computer elements and of which the step response does not show a jump at \( t = 0 \), and we picked up the best one from among them.

If, however, we do not restrict ourselves to the third order models, many other transfer functions of three degrees of freedom can be considered. For instance Eq. (39) represents a higher order model with three degrees of freedom.

\[ G(z) = [1 + f_1z + f_2z^2]G_{d1} \]

Here \( f_1 \) and \( f_2 \) are respectively a linear and a quadratic homogeneous functions of \( T \) and \( 1/(\omega_s) \). Although this type of model can be constructed through the tractor function technique from \( G_{d1} \), it would be more convenient to have the diagram like Fig. 4 for this type of models.

Appendix

In practice it may be required to re-plot the diagrams shown in Fig. 4 in workable scale, so the formulae for plotting the curves are listed below.

Here we use the following notations.

\[ y = t_i^4 - 2, \quad x = t_i^2 - 1 \]

i) The boundary curves of \( G_{d_1}^{-1} \) and \( G_{d_1}^{1/2} \) in Fig. 4(b) are given by

\[ t_i^4 = 3(t_i^2)^2/2 + 1/2 \]

\[ t_i^4 = 6(t_i^2 - 1)(1 + \sqrt{1 - x}/2) + 2 \]

ii) \( t_i^4 \) values for \( G_{\text{d}_1}^{-1} \) and \( G_{\text{d}_1}^{1/2} \) are evaluated by

\[ t_i^4 = 6 + 4(y - 3x)(y - 3x)/3x + 1 + 3x(x + 4) \]

iii) \( t_i^4 \) values for \( G_{\text{d}_2} \) and \( G_{\text{d}_1} \) in Fig. 4(a) are given by

\[ t_i^4 = 3(2\gamma^2\gamma - 1) - 4x\gamma + x^3 + 4x + 2 \]

\[ y = 3(\gamma - 1)(2\gamma^2 - x) \]

where \( \gamma \) is a parameter.

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Reference

(1) H.M. Paynter, Regelungstechnik, Moderne Theorien u. ihre Verwendbarkeit, (1957), 243, Verlag Oldenburg, Munich.